

Multiple Objective Programming Involving Differentiable (H_p, r) -invex Functions

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ABSTRACT

In this paper, we introduce new types of generalized convex functions which include locally (H_p, r) -pre-invex functions and (H_p, r) -invex functions. Relationship between these two new classes of functions are established. We also present the conditions for optimality in differentiable mathematical programming problems where the functions considered are (H_p, r) -invex functions introduced in this paper.

RESUMEN

Este trabajo, establece nuevos tipos de funciones convexas generalizadas que incluyen localmente funciones (H_p, r) de pre-invex y funciones (H_p, r) -invex. La relación entre

estas dos nuevas clases de funciones estn establecidas. Tambin se presentan las condiciones de optimalidad en diferenciables problemas de programacin matemtica, donde las funciones consideradas en este artculo son funciones (H_p, r) -invex.

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1 Introduction

Convexity plays a central role in many aspects of mathematical programming (see [22, 5, 8]) including analysis of stability [9, 16], sufficient optimality conditions and duality [12, 15, 11]. Based on convexity assumptions, nonlinear programming problems can be solved efficiently. There have been many attempts to weaken the convexity assumptions in order to treat many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1, 2, 19, 27, 28, 25]. One of these concepts, invexity, was introduced by Hanson in [11]. Hanson has shown that invexity has a common property in mathematical programming with convexity that Karush Kuhn Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [7] introduced the concept of pre-invex functions which is a special case of invexity.

Following [11] and [7], many authors have introduced concepts of generalized invexity and pre-invexity including strictly pseudoinvex functions and quasiinvex functions [13], prepseudoinvex and prequasiinvex functions [21] and r -pre-pseudoinvex functions [1]. The relationships between some of these generalized invex functions were studied in [20, 21].

Recently, Antczak [3] introduced new definitions of p -invex sets and (p, r) -invex functions which can be seen as generalization of invex functions. He also discussed nonlinear programming problems involving the (p, r) -invexity-type functions in [2, 4].

On the other hand, Kaul et al. [14] introduced the classes of locally connected sets which generalizes the arcwise connected sets [6] and locally star-shaped sets [10]. Based on the new class of sets, they [14, 15] introduced a new class of functions called locally connected functions. They [15] also defined the directional derivative (with respect to a vector function) of a real valued function, and also defined locally P -connected functions in terms of its right differential.

Motivated by [7, 14, 1, 2, 3, 4, 15, 18], Yuan et al. introduced definition of a new class of sets, locally H_p -invex sets, and definitions of classes of generalized convex functions called locally (H_p, r, α) -pre-invex functions. And we give the concept of locally differentiable (H_p, r) -invex functions, discuss the relationship between (H_p, r) -invexity and locally $(H_p, r, 1)$ -pre-invexity. Based on the definition of locally differentiable (H_p, r) -invexity, we have managed to deal with nonlinear programming problems under some assumptions.

The rest of the paper is organized as follows: In Section 2, we give some preliminary concepts

regarding locally H_p -convex sets, locally (H_p, r, α) -preinvex function, and differentiable (H_p, r) -invex function, discuss the relationship between (H_p, r) -invexity and locally $(H_p, r, 1)$ -pre-invexity. In Section 3, we present the conditions for optimality in differentiable mathematical programming problems in which the functions considered belong to the classes of functions introduced in section. In Section 4, we present the conditions of optimality for the following nonlinear differentiable fractional programming problems (MFP) with the same convexity assumption.

2 Differentiable (H_p, r) -invex Functions

Let R^n be the n -dimensional Euclidean space, $R_+^n = \{x \in R^n | x \geq 0\}$ and $\dot{R}_+^n = \{x \in R^n | x > 0\}$. In this section, we give definitions of locally H_p -invex set.

Definition 1. [3] Let $a_1, a_2 > 0$, $\lambda \in (0, 1)$ and $r \in \mathbb{R}$. Then the weighted r -mean of a_1 and a_2 is given by

$$M_r(a_1, a_2; \lambda) := \begin{cases} (\lambda a_1^r + (1 - \lambda)a_2^r)^{\frac{1}{r}} & \text{for } r \neq 0, \\ a_1^\lambda a_2^{1-\lambda} & \text{for } r = 0. \end{cases}$$

Definition 2. [29] $S \subset \mathbb{R}^n$ is a locally H_p -invex set if and only if, for any $x, u \in S$, there exist a maximum positive number $a(x, u) \leq 1$ and a vector function $H_p : S \times S \times [0, 1] \rightarrow \mathbb{R}^n$, such that

$$H_p(x, u; 0) = e^u, \quad H_p(x, u; \lambda) \in \dot{\mathbb{R}}_+^n, \\ \ln(H_p(x, u; \lambda)) \in S, \quad \forall 0 < \lambda < a(x, u) \text{ for } p \in \mathbb{R}.$$

and $H_p(x, u; \lambda)$ is continuous on the interval $(0, a(x, u))$, where the logarithm and the exponentials appearing in the relation are understood to be taken componentwise.

Definition 3. [29] A function $f : S \rightarrow \mathbb{R}$ defined on a locally H_p -invex set $S \subset \mathbb{R}^n$ is said to be locally (H_p, r) -pre-invex on S if, for any $x, u \in S$, there exists a maximum positive number $a(x, u) \leq 1$ such that

$$f(\ln(H_p(x, u; \lambda))) \leq \ln(M_r(e^{f(x)}, e^{f(u)}; \lambda^\alpha)), \quad \forall 0 < \lambda < a(x, u) \text{ for } p \in \mathbb{R}$$

where the logarithm and the exponentials appearing on the left-hand side of the inequality are understood to be taken componentwise. If u is fixed, then f is said to be locally (H_p, r) -pre-invex at u . Correspondingly, if the direction of above inequality is changed to the opposite one, then f is said to (H_p, r) -pre-incave on S or at u .

Now, we introduce the classes of differentiable (H_p, r) -invex functions. For convenience, we assume that S be a H_p -invex set, H_p is right differentiable at 0 with respect to variable λ for each given pair $x, u \in S$, and $f : S \rightarrow \mathbb{R}$ is differentiable on S . The symbol $H'_p(x, u; 0+) \triangleq (H'_{p1}(x, u; 0+), \dots, H'_{pn}(x, u; 0+))^T$ denotes the right derivative of H_p at 0 with respect to variable λ for each given pair $x, u \in S$; $\nabla f(x) \triangleq (\nabla_1 f(x), \dots, \nabla_n f(x))^T$ denotes the differential of f at x , where $\partial_i f(x)$ is partial differential of f with respect to the i -th componentwise; $\frac{\nabla f(u)}{e^u}$ denotes $\left(\frac{\nabla_1 f(u)}{e^{u_1}}, \dots, \frac{\nabla_n f(u)}{e^{u_n}}\right)^T$.

Definition 4. Let S be a H_p -invex set, H_p is right differentiable at 0 with respect to variable λ for each given pair $x, u \in S$, and $f : S \rightarrow \mathbb{R}$ is differentiable on S . If for all $x \in S$, one of the relations

$$\begin{aligned} \frac{1}{r} e^{rf(x)} &\geq \frac{1}{r} e^{rf(u)} \left[1 + r \frac{\nabla f(u)^T}{e^u} H'_p(x, u; 0+) \right] & (>) \quad \text{for } r \neq 0, \\ f(x) - f(u) &\geq \frac{\nabla f(u)^T}{e^u} H'_p(x, u; 0+) & (>) \quad \text{for } r = 0, \end{aligned} \quad (2.1)$$

holds, then f is said to be (H_p, r) -invex (strictly (H_p, r) -invex) at $u \in S$. If the inequalities (2.1) are satisfied at any point $u \in S$, then f is said to be (H_p, r) -invex (strictly (H_p, r) -invex) on S .

Remark 5. Any function f satisfying (2.1) is called (H_p, r) -invex (strictly (H_p, r) -invex) on S . However, if $H_p(x, u; \lambda) = M_p(e^{\eta(x,u)+u}, e^u; \lambda)$ and $a(x, u) = 1$ for all $x, u \in S$, we will say that f is (p, r) -invex (strictly (p, r) -invex) with respect to η on S ; furthermore, f is r -invex (strictly (p, r) -invex) with respect to η on S in the case $p = 0$ and f is invex (strictly invex) with respect to η on S in the case $p = 0$ and $r = 0$.

Remark 6. In order to define an analogous class of (strict) (H_p, r) -incave functions on the H_p -invex set, the direction of the inequality (2.1) in the definition of these functions should be changed to the opposite one.

Theorem 7. Let $S \subset \mathbb{R}^n$ be a H_p -invex set, H_p is right differentiable at 0 with respect to variable λ for each given pair $x, u \in S$, and $f : S \rightarrow \mathbb{R}$ is differentiable on S . If f is (H_p, r, α) -pre-invex ((H_p, r, α) -pre-incave) on S and $\alpha = 1$, then f is (H_p, r) -invex ((H_p, r) -incave) on S .

Proof. The theorem will be proved only in the case f is (H_p, r, α) -pre-invex, and we just prove that the theorem is true when $r \geq 0$ (the proof in the case when $r < 0$ is analogous to the one when $r > 0$; only the directions of the inequalities should be changed to the opposite ones).

Firstly, we prove the theorem is true in the case when $r > 0$. Since $f : S \rightarrow \mathbb{R}$ is a (H_p, r, α) -pre-invex function and $\alpha = 1$, by Definition 3, we have

$$e^{rf(u)} \left[e^{rf(\ln(H_p(x, u; \lambda))) - rf(u)} - 1 \right] \lambda^{-1} \leq e^{rf(x)} - e^{rf(u)}.$$

By letting $\lambda \rightarrow 0$, we get the inequality

$$e^{rf(x)} - e^{rf(u)} \geq r e^{rf(u)} \frac{\nabla f(u)^T}{e^u} H'_p(x, u; 0+)$$

that is

$$\frac{1}{r} e^{rf(x)} \geq \frac{1}{r} e^{rf(u)} \left[1 + r \frac{\nabla f(u)^T}{e^u} H'_p(x, u; 0+) \right]$$

Now, we prove the case $r = 0$. By Definition 3, we have

$$[f(\ln(H_p(x, u; \lambda))) - f(u)] \lambda^{-1} \leq f(x) - f(u).$$

By letting $\lambda \rightarrow 0$, we get the inequality

$$f(x) - f(u) \geq \frac{\nabla f(u)^T}{e^u} H'_p(x, u; 0+)$$

Therefore, we get the desired result. \square

3 Optimality for Multiple Objective Programming

In this section, we present the conditions for optimality in differentiable mathematical programming problems in which the functions considered belong to the classes of functions introduced earlier in this paper.

Consider the following form of optimization problem

$$(VOP) \quad \begin{aligned} & \min f(x) \\ & g(x) \leq 0, \quad x \in S, \end{aligned}$$

where $S \subset \mathbb{R}^n$, $f : S \rightarrow \mathbb{R}^q$, $g : S \rightarrow \mathbb{R}^m$.

Let us denote by E the set of feasible solutions of (VOP), i.e., the set of the form $E := \{x \in S | g(x) \leq 0\}$. From now on, we assume that H_p is right differentiable at 0 with respect to variable λ for each given pair $x, u \in S$, $f_i : S \rightarrow \mathbb{R} (i = 1, \dots, q)$, $g_j : S \rightarrow \mathbb{R} (j = 1, \dots, m)$ are differentiable on S , $f = (f_1, \dots, f_q)$, $g = (g_1, \dots, g_m)$ and S is an H_p -invex (nonempty) set.

Definition 8. $\bar{x} \in E$ is said to be an efficient solution for problem (VOP), if there exists no $x \in E$ such that $f(x) \leq f(\bar{x})$; $\bar{x} \in E$ is said to be a weak efficient solution for problem (VOP), if there exists no $x \in E$ such that $f(x) < f(\bar{x})$;

Theorem 9. Let E be a H_p -invex set with the respect to the same H_p . Assume that $\bar{x} \in S$ is feasible for problem (VOP), and there exists $\lambda \in \mathbb{R}^q$, $\mu \in \mathbb{R}^m$ such that

$$\sum_i^q \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^m \mu_j \nabla g_j(\bar{x}) = 0, \tag{3.1}$$

$$\sum_{j=1}^m \mu_j g_j(\bar{x}) = 0, \tag{3.2}$$

$$\lambda = (\lambda_1, \dots, \lambda_q) \geq 0, \sum_i^q \lambda_i = 1, \mu = (\mu_1, \dots, \mu_m) \geq 0. \tag{3.3}$$

If $f_i (i = 1, \dots, q)$ are strictly (H_p, r) -invex and $g_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S , then \bar{x} is an efficient solution of problem (VOP).

Proof. Here we prove only the cases when $r > 0$ or $r = 0$ (the proof of the case when $r < 0$ is similar to the one when $r > 0$; the only changes arise from the form of inequalities defining the class of (H_p, r) -invex functions). Assume that x is an arbitrary feasible point for problem (VOP). On the contrary to suppose that \bar{x} is not an efficient solution of problem (VOP). Thus, there exists $x \in E$ such that $f(x) \leq f(\bar{x})$.

We first consider the case when $r > 0$. Therefore,

$$\sum_i^q \lambda_i \left\{ e^{rf_i(x)} - e^{rf_i(\bar{x})} \right\} \leq 0.$$

By hypothesis, f and $g_i (i = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S ; therefore, for all $x \in S$, the

inequalities

$$\frac{1}{r} e^{rf_i(x)} > \frac{1}{r} e^{rf_i(\bar{x})} \left[1 + r \frac{\nabla f_i(\bar{x})^T}{e^{\bar{x}}} H'_p(x, u; 0+) \right], \quad i = 1, \dots, q, \quad (3.4)$$

$$\frac{1}{r} e^{rg_j(x)} \geq \frac{1}{r} e^{rg_j(\bar{x})} \left[1 + r \frac{\nabla g_j(\bar{x})^T}{e^{\bar{x}}} H'_p(x, \bar{x}; 0+) \right], \quad j = 1, \dots, m, \quad (3.5)$$

are true. Denote $I(\bar{x}) \triangleq \{j | \mu_j > 0, j = 1, \dots, m\}$. By (3.2), we have $g_j(\bar{x}) = 0$ if $j \in I(\bar{x})$ and $\mu_j = 0$ if $g_j(\bar{x}) \neq 0$, thus $g_j(x) \leq g_j(\bar{x})$ for $j \in I(\bar{x})$. Therefore, from (3.4) and (3.5), we have

$$e^{r(f_i(x) - f_i(\bar{x}))} > 1 + r \frac{\nabla f_i(\bar{x})^T}{e^{\bar{x}}} H'_p(x, u; 0+) \quad (3.6)$$

$$r \frac{\nabla g_j(\bar{x})^T}{e^{\bar{x}}} H'_p(x, \bar{x}; 0+) \leq 0, \quad j \in I(\bar{x}). \quad (3.7)$$

By (3.6), (3.7) and (3.3), we deduce that

$$\left(\frac{\sum_{i=1}^m \lambda_i \nabla f_i(\bar{x})}{e^{\bar{x}}} + \frac{\sum_{j \in I(\bar{x})} \mu_j \nabla g_j(\bar{x})}{e^{\bar{x}}} \right)^T H'_p(x, \bar{x}; 0+) < 0. \quad (3.8)$$

Notice that $\sum_{j=1}^m \mu_j \nabla g_j(\bar{x}) = \sum_{j \in I(\bar{x})} \mu_j \nabla g_j(\bar{x})$, by (3.8), we have

$$\left(\frac{\sum_{i=1}^m \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^m \mu_j \nabla g_j(\bar{x})}{e^{\bar{x}}} \right)^T H'_p(x, \bar{x}; 0+) < 0.$$

This, together with (3.1), follows a contradiction $0 < 0$.

Now, we prove the theorem is true in the case when $r = 0$. Since $f(x) \leq f(\bar{x})$, then $\sum_i^q \lambda_i (f_i(x) - f_i(\bar{x})) \leq 0$. By Definition 4, we have

$$f_i(x) - f_i(\bar{x}) > \frac{\nabla f_i(\bar{x})^T}{e^{\bar{x}}} H'_p(x, \bar{x}; 0+), \quad i = 1, \dots, q,$$

$$g_j(x) - g_j(\bar{x}) \geq \frac{\nabla g_j(\bar{x})^T}{e^{\bar{x}}} H'_p(x, \bar{x}; 0+), \quad j = 1, \dots, m,$$

On the same line as the case when $r > 0$, we have the contradiction $0 < 0$ again. Therefore, \bar{x} is an efficient solution for problem (VOP). \square

Theorem 10. Let E be a H_p -invex set with the respect to the same H_p . Assume that $\bar{x} \in S$ is feasible for problem (VOP), and there exists $\lambda \in \mathbb{R}^q$, $\mu \in \mathbb{R}^m$ such that

$$\sum_i^q \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^m \mu_j \nabla g_j(\bar{x}) = 0, \quad (3.9)$$

$$\sum_{j=1}^m \mu_j g_j(\bar{x}) = 0, \quad (3.10)$$

$$\lambda = (\lambda_1, \dots, \lambda_q) > 0, \sum_i^q \lambda_i = 1, \mu = (\mu_1, \dots, \mu_m) \geq 0. \quad (3.11)$$

If $f_i (i = 1, \dots, q)$, $g_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S , then \bar{x} is an efficient solution of problem (VOP).

Proof. The theorem can be proved on simple lines as Theorem 9. \square

The assumption on functions in Theorem 9 (or Theorem 10) could also be given in another form. It is enough to assume that the Lagrange function $f_i + \sum_{j=1}^m \mu_j g_j (i = 1, \dots, m)$ are strictly (H_p, r)-inconv (or strictly (H_p, r)-inconv) . And so, the following two theorems are true. Their proofs are on the same line as Theorem 9, therefore we delete them here.

Theorem 11. Assume that $\bar{x} \in S$ is feasible for problem (VOP), and there exists $\lambda \in \mathbb{R}^q, \mu \in \mathbb{R}^m$ satisfying (3.1), (3.2) and (3.3). If $f_i + \sum_{j=1}^m \mu_j g_j (i = 1, \dots, m)$ are strictly (H_p, r)-inconv at \bar{x} on S , then \bar{x} is an efficient solution of problem (VOP).

Theorem 12. Assume that $\bar{x} \in S$ is feasible for problem (VOP), and there exists $\lambda \in \mathbb{R}^q, \mu \in \mathbb{R}^m$ satisfying (3.9), (3.10) and (3.11). If $f_i + \sum_{j=1}^m \mu_j g_j (i = 1, \dots, m)$ are (H_p, r)-inconv at \bar{x} on S , then \bar{x} is an efficient solution of problem (VOP).

For weak efficient solution of problem (VOP), we have the following theorems.

Theorem 13. Assume that $\bar{x} \in S$ is feasible for problem (VOP), and there exists $\lambda \in \mathbb{R}^q, \mu \in \mathbb{R}^m$ satisfying (3.1), (3.2) and (3.3). If $f_i (i = 1, \dots, q), g_j (j = 1, \dots, m)$ are (H_p, r)-inconv at \bar{x} on S , then \bar{x} is a weak efficient solution of problem (VOP).

Theorem 14. Assume that $\bar{x} \in S$ is feasible for problem (VOP), and there exists $\lambda \in \mathbb{R}^q, \mu \in \mathbb{R}^m$ satisfying (3.1), (3.2) and (3.3). If $f_i + \sum_{j=1}^m \mu_j g_j (i = 1, \dots, m)$ are (H_p, r)-inconv at \bar{x} on S , then \bar{x} is a weak efficient solution of problem (VOP).

4 Multiple Objective Fractional Programming

In this section, we present the conditions of optimality for the following nonlinear differentiable fractional programming problems (MFP).

$$\begin{aligned}
 \text{(MFP)} \quad \min \quad & \frac{f(x)}{g(x)} \triangleq \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_q(x)}{g_q(x)} \right)^T \\
 \text{s.t.} \quad & h(x) = (h_1(x), h_2(x), \dots, h_m(x)) \leq 0 \\
 & x \in S,
 \end{aligned}$$

where $S \subset \mathbb{R}^n, f : S \rightarrow \mathbb{R}^q, g : S \rightarrow \mathbb{R}^q, h : S \rightarrow \mathbb{R}^m, f = (f_1, \dots, f_q), g = (g_1, \dots, g_q), h = (h_1, \dots, h_m)$, are differentiable on S and S is an H_p -inconv (nonempty) set. Moreover, for $i = 1, \dots, q, g_i(x) > 0$ for all $x \in S$.

Let us denote by E the set of feasible solutions of (MFP), i.e., the set of the form $E := \{x \in S | h(x) \leq 0\}$. We assume that H_p is right differentiable at 0 with respect to variable λ for each given pair $x, u \in S$.

Definition 15. $\bar{x} \in E$ is said to be an efficient solution for problem (MFP), if there exists no $x \in E$ such that $\frac{f(x)}{g(x)} \leq \frac{f(\bar{x})}{g(\bar{x})}$; $\bar{x} \in E$ is said to be a weak efficient solution for problem (MFP), if there exists no $x \in E$ such that $\frac{f(x)}{g(x)} < \frac{f(\bar{x})}{g(\bar{x})}$;

Theorem 16. Assume that $\bar{x} \in S$ is feasible for problem (MFP), and there exists $\lambda \in \mathbb{R}^q$, $u \in \mathbb{R}^q$, $\mu \in \mathbb{R}^m$ such that

$$\sum_{i=1}^q \lambda_i (\nabla f_i(\bar{x}) - u_i \nabla g_i(\bar{x})) + \sum_{j=1}^m \mu_j \nabla h_j(\bar{x}) = 0, \quad (4.1)$$

$$\sum_{j=1}^m \mu_j h_j(\bar{x}) = 0, \quad (4.2)$$

$$\lambda = (\lambda_1, \dots, \lambda_q) \geq 0, \sum_{i=1}^q \lambda_i = 1, \mu = (\mu_1, \dots, \mu_m) \geq 0, \quad (4.3)$$

$$u = (u_1, \dots, u_q) \geq 0, u_i = f_i(\bar{x})/g_i(\bar{x}), i = 1, \dots, q. \quad (4.4)$$

If $f_i - u_i g_i (i = 1, \dots, q)$ are strictly (H_p, r) -invex at \bar{x} on S , and $h_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S , then \bar{x} is an efficient solution of problem (MFP).

Proof. Similar to Theorem 9, here we prove only the cases when $r > 0$ or $r = 0$. Assume that x is an arbitrary feasible point for problem (MFP). On the contrary to suppose that \bar{x} is not an efficient solution of problem (MFP). Thus, there exists $x \in E$ such that $\frac{f(x)}{g(x)} \leq \frac{f(\bar{x})}{g(\bar{x})} = u$.

We first consider the case when $r > 0$. Therefore,

$$\sum_{i=1}^q \lambda_i \left\{ e^{r(f_i(x) - u_i g_i(x))} - 1 \right\} \leq 0.$$

By the convexity of $f_i - u_i g_i (i = 1, \dots, q)$ and $h_j (j = 1, \dots, m)$, we have

$$e^{r(f_i(x) - u_i g_i(x))} - 1 > r \left(\frac{\nabla f_i(\bar{x}) - u_i \nabla g_i(\bar{x})}{e^{\bar{x}}} \right)^T H'_p(x, u; 0+), i = 1, \dots, q, \quad (4.5)$$

$$\frac{1}{r} e^{r h_j(x)} \geq \frac{1}{r} e^{r h_j(\bar{x})} \left[1 + r \frac{\nabla h_j(\bar{x})^T}{e^{\bar{x}}} H'_p(x, \bar{x}; 0+) \right], j = 1, \dots, m, \quad (4.6)$$

Thus, by (4.2), (4.3), (4.5) and (4.6), we deduce that

$$\left(\frac{\sum_{i=1}^q \lambda_i (\nabla f_i(\bar{x}) - u_i \nabla g_i(\bar{x})) + \sum_{j=1}^m \mu_j \nabla h_j(\bar{x})}{e^{\bar{x}}} \right)^T H'_p(x, u; 0+) < 0$$

Therefore, we get a contradiction $0 < 0$.

Similarly, we can prove the theorem in the case when $r = 0$. □

Theorem 17. Assume that $\bar{x} \in S$ is feasible for problem (MFP), and there exists $\lambda \in \mathbb{R}^q$, $u \in \mathbb{R}^q$, $\mu \in \mathbb{R}^m$ satisfying (4.1), (4.2), (4.3) and (4.4). If one of the following holds:

1) $f_i (i = 1, \dots, q)$ are strictly (H_p, r) -invex at \bar{x} on S , $u_i g_i (i = 1, \dots, q)$ are (H_p, r) -incave at \bar{x} on S , and $h_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S ;

2) $u_i g_i (i = 1, \dots, q)$ are strictly (H_p, r) -incave at \bar{x} on S , $f_i (i = 1, \dots, q)$ and $h_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S ;

then \bar{x} is an efficient solution of problem (MFP).

Proof. In the same way as Theorem 16, here we prove only 1) for the cases when $r > 0$, 2) can be proved similar to 1). Assume that x is an arbitrary feasible point for problem (MFP). On the contrary to suppose that \bar{x} is not an efficient solution of problem (MFP). Thus, there exists $x \in E$ such that $\frac{f(x)}{g(x)} \leq \frac{f(\bar{x})}{g(\bar{x})} = u$. Therefore,

$$\sum_i^q \lambda_i \left\{ e^{rf_i(x)} - e^{ru_i g_i(x)} \right\} \leq 0. \tag{4.7}$$

1) Since $f_i (i = 1, \dots, q)$ are strictly (H_p, r) -invex at \bar{x} on S , $u_i g_i (i = 1, \dots, q)$ are (H_p, r) -incave at \bar{x} on S , and $h_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S , then

$$e^{r(f_i(x))} - e^{r(f_i(\bar{x}))} > r \left(\frac{\nabla f_i(\bar{x})}{e^{\bar{x}}} \right)^T H'_p(x, u; 0+), i = 1, \dots, q, \tag{4.8}$$

$$e^{r(u_i g_i(x))} - e^{r(u_i g_i(\bar{x}))} \leq r \left(\frac{u_i \nabla g_i(\bar{x})}{e^{\bar{x}}} \right)^T H'_p(x, u; 0+), i = 1, \dots, q, \tag{4.9}$$

$$\frac{1}{r} e^{rh_j(x)} \geq \frac{1}{r} e^{rh_j(\bar{x})} \left[1 + r \frac{\nabla h_j(\bar{x})}{e^{\bar{x}}} H'_p(x, \bar{x}; 0+) \right], j = 1, \dots, m, \tag{4.10}$$

Notice that $e^{rf_i(\bar{x})} = e^{r(u_i g_i(\bar{x}))}$, by (4.2)-(4.4), (4.7)-(4.10), we have

$$\left(\frac{\sum_{i=1}^q \lambda_i (\nabla f_i(\bar{x}) - u_i \nabla g_i(\bar{x})) + \sum_{j=1}^m \mu_j \nabla h_j(\bar{x})}{e^{\bar{x}}} \right)^T H'_p(x, u; 0+) < 0.$$

Therefore, we get a contradiction $0 < 0$. □

Theorem 18. Assume that $\bar{x} \in S$ is feasible for problem (MFP), and there exists $\lambda \in \mathbb{R}^q$, $u \in \mathbb{R}^q$, $\mu \in \mathbb{R}^m$ such that

$$\sum_i^q \lambda_i (\nabla f_i(\bar{x}) - u_i \nabla g_i(\bar{x})) + \sum_{j=1}^m \mu_j \nabla h_j(\bar{x}) = 0, \tag{4.11}$$

$$\sum_{j=1}^m \mu_j h_j(\bar{x}) = 0, \tag{4.12}$$

$$\lambda = (\lambda_1, \dots, \lambda_q) > 0, \sum_i^q \lambda_i = 1, \mu = (\mu_1, \dots, \mu_m) \geq 0, \tag{4.13}$$

$$u = (u_1, \dots, u_q) \geq 0, u_i = f_i(\bar{x})/g_i(\bar{x}), i = 1, \dots, q. \tag{4.14}$$

If $f_i - u_i g_i (i = 1, \dots, q)$ and $h_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S , then \bar{x} is an efficient solution of problem (MFP).

Proof. The theorem can be proved on simple lines as Theorem 16.

Theorem 19. Assume that $\bar{x} \in S$ is feasible for problem (MFP), and there exists $\lambda \in \mathbb{R}^q$, $u \in \mathbb{R}^q$, $\mu \in \mathbb{R}^m$ satisfying (4.11), (4.12), (4.13) and (4.14). If $u_i g_i (i = 1, \dots, q)$ are (H_p, r) -incave at \bar{x} on S , $f_i (i = 1, \dots, q)$ and $h_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S , then \bar{x} is an efficient solution of problem (MFP).

Proof. The theorem can be proved on simple lines as Theorem 17.

Theorem 20. Assume that $\bar{x} \in S$ is feasible for problem (MFP), and there exists $\lambda \in \mathbb{R}^q$, $u \in \mathbb{R}^q$, $\mu \in \mathbb{R}^m$ such that

$$\begin{aligned} \sum_i^q \lambda_i (\nabla f_i(\bar{x}) - u_i \nabla g_i(\bar{x})) + \sum_{j=1}^m \mu_j \nabla h_j(\bar{x}) &= 0, \\ \sum_{j=1}^m \mu_j h_j(\bar{x}) &= 0, \\ \lambda &= (\lambda_1, \dots, \lambda_q) \geq 0, \sum_i^q \lambda_i = 1, \mu = (\mu_1, \dots, \mu_m) \geq 0, \\ u &= (u_1, \dots, u_q) \geq 0, u_i = f_i(\bar{x})/g_i(\bar{x}), i = 1, \dots, q. \end{aligned}$$

If one of the following holds:

- 1) $f_i - u_i g_i (i = 1, \dots, q)$ and $h_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S ;
 - 2) $u_i g_i (i = 1, \dots, q)$ are (H_p, r) -incave at \bar{x} on S , $f_i (i = 1, \dots, q)$ and $h_j (j = 1, \dots, m)$ are (H_p, r) -invex at \bar{x} on S ;
- then \bar{x} is a weak efficient solution of problem (MFP).

Proof. The theorem can be proved on simple lines as Theorem 16.

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