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On Weak concircular Symmetries of Trans-Sasakian manifolds

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ABSTRACT

The object of the present paper is to study weakly concircular symmetric and weakly concircular Ricci symmetric trans-Sasakian manifolds.

RESUMEN

El objeto del presente trabajo es el estudio de variedades simétricas débilmente concirculares y variedades simétricas trans-Sasakian débilmente concircular de Ricci.

Keywords. weakly symmetric manifold, weakly concircular symmetric manifold, weakly Ricci symmetric manifold, concircular Ricci tensor, weakly concircular Ricci symmetric manifold, α-Sasakian manifold, β-Kenmotsu manifold, trans-Sasakian manifold.

Mathematics Subject Classification: 53C15, 53C25.

1 Introduction

The notion of weakly symmetric manifolds was introduced by Tamássy and Binh [9]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a weakly symmetric manifold if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) &= A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) \\ &+ H(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) \\ &+ E(V)R(Y, Z, U, X)\end{aligned} \tag{1.1}$$

for all vector fields X, Y, Z, U, $V \in \chi(M^n)$; $\chi(M)$ being the Lie algebra of smooth vector fields of M, where A, B, H, D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. In 1999 De and Bandyopadhyay [3] studied a $(WS)_n$ and proved that in such a manifold the associated 1-forms $B = H$ and $D = E$. Hence (1.1) reduces to the following:

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) &= A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) \\ &+ B(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) \\ &+ D(V)R(Y, Z, U, X).\end{aligned} \tag{1.2}$$

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [11]. The interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$, which is defined by [11]

$$\tilde{C}(Y, Z, U, V) = R(Y, Z, U, V) - \frac{r}{n(n-1)}\big[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\big], \tag{1.3}$$

where r is the scalar curvature of the manifold.

Recently Shaikh and Hui [7] introduced the notion of weakly concircular symmetric manifolds. A Riemannian manifold $(M^n, g)(n > 2)$ is called weakly concircular symmetric manifold if its concircular curvature tensor $\tilde{C}$ of type (0, 4) is not identically zero and satisfies the condition

$$\begin{aligned}(\nabla_X \tilde{C})(Y, Z, U, V) &= A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V) \\ &+ H(Z)\tilde{C}(Y, X, U, V) + D(U)\tilde{C}(Y, Z, X, V) \\ &+ E(V)\tilde{C}(Y, Z, U, X)\end{aligned} \tag{1.4}$$

for all vector fields X, Y, Z, U, $V \in \chi(M^n)$, where A, B, H, D and E are 1-forms (not simultaneously zero) an n-dimensional manifold of this kind is denoted by $(W\tilde{C}S)_n$. Also it is shown that [7], in a $(W\tilde{C}S)_n$ the associated 1-forms $B = H$ and $D = E$, and hence the defining condition (1.4) of a $(W\tilde{C}S)_n$ reduces to the following form:

$$\begin{aligned}(\nabla_X \tilde{C})(Y, Z, U, V) &= A(X)\tilde{C}(Y, Z, U, V) + B(Y)\tilde{C}(X, Z, U, V) \\ &+ B(Z)\tilde{C}(Y, X, U, V) + D(U)\tilde{C}(Y, Z, X, V) \\ &+ D(V)\tilde{C}(Y, Z, U, X),\end{aligned} \tag{1.5}$$

where A, B and D are 1-forms (not simultaneously zero).

Again Tamássy and Binh [10] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold (M^n, g) $(n > 2)$ is called weakly Ricci symmetric manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(Y, X), \tag{1.6}$$

where A, B and D are three non-zero 1-forms, called the associated 1-forms of the manifold, and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n-dimensional manifold is denoted by $(WRS)_n$.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at each point of the manifold and let

$$P(Y, V) = \sum_{i=1}^{n} \tilde{C}(Y, e_i, e_i, V), \tag{1.7}$$

then from (1.3), we get

$$P(Y, V) = S(Y, V) - \frac{r}{n} g(Y, V). \tag{1.8}$$

The tensor P is called the concircular Ricci symmetric tensor [4], which is a symmetric tensor of type (0, 2). In [4] De and Ghosh introduced the notion of weakly concircular Ricci symmetric manifolds. A Riemannian manifold $(M^n, g)(n > 2)$ is called weakly concircular Ricci symmetric manifold [4] if its concircular Ricci tensor P of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X P)(Y, Z) = A(X)P(Y, Z) + B(Y)P(X, Z) + D(Z)P(Y, X), \tag{1.9}$$

where A, B and D are three 1-forms (not simultaneously zero).

In [5] Oubiña introduced the notion of trans-Sasakian manifolds which contains both the class of Sasakian and cosympletic structures, and are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type (0, 0), $(\alpha, 0)$ and $(0, \beta)$ are the cosympletic, α-Sasakian and β-Kenmotsu manifold respectively. In particular, if $\alpha = 1, \beta = 0$; and $\alpha = 0, \beta = 1$, then a trans-Sasakian manifold reduces to a Sasakian and Kenmotsu manifold respectively. Thus trans-Sasakian structures provide a large class of generalized quasi-Sasakian structures. Tamássy and Binh [10] studied weakly symmetric and weakly Ricci symmetric Sasakian manifolds and proved that in such a manifold the sum of the associated 1-forms vanishes everywhere. Again Özgür [6] studied weakly symmetric and weakly Ricci symmetric Kenmotsu manifolds and proved that in such a manifold the sum of the associated 1-forms is zero everywhere and hence such a manifold does not exist unless the sum of the associated 1-forms is everywhere zero.

The object of the present paper is to study *weakly concircular symmetric and weakly concircular Ricci symmetric trans-Sasakian manifolds.* Section 2 deals with preliminaries of trans-Sasakian manifolds. Recently Shaikh and Hui [8] studied weakly symmetric and weakly Ricci symmetric trans-Sasakian manifolds and proved that the sum of the associated 1-forms of a weakly symmetric and also of a weakly Ricci symmetric trans-Sasakian manifold of non-vanishing ξ-sectional curvature are non-zero everywhere and hence such two structure exists, provided that the manifold is of non-vanishing ξ-sectional curvature. However, in section 3 of the paper we have obtained all the 1-forms of a weakly concircular symmetric trans-Sasakian manifold and hence such a structure exist always. Again in section 4 we study weakly concircular Ricci symmetric trans-Sasakian manifolds and obtained all the 1-forms of a weakly concircular Ricci symmetric trans-Sasakian manifold and consequently such a structure is always exist. Also it is proved that the sum of the associated 1-forms of a weakly concircular Ricci symmetric trans-Sasakian manifold is non-vanishing everywhere.

2 Trans-Sasakian manifolds

A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold [1] if it admits an (1, 1) tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g, which satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be trans-Sasakian manifold [5] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 of the Hermitian manifolds, where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J\left(Z, f\frac{d}{dt}\right) = \left(\phi Z - f\xi, \eta(Z)\frac{d}{dt}\right)$$

for any vector field Z on M and smooth function f on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be stated by the condition [2]

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.4}$$

where α, β are smooth functions on M and such a structure is said to be the trans-Sasakian structure of type (α, β). From (2.4) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta\{X - \eta(X)\xi\}, \tag{2.5}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

In a trans-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, the following relations hold:

$$\begin{aligned} R(X, Y)\xi &= (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\phi Y - (X\beta)\phi^2(Y) \\ &+ 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\alpha)\phi X + (Y\beta)\phi^2(X), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \eta(R(X,Y)Z) &= (\alpha^2-\beta^2)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] \\ &- 2\alpha\beta[g(\phi X,Z)\eta(Y)-g(\phi Y,Z)\eta(X)] \\ &- (Y\alpha)g(\phi X,Z)-(X\beta)\{g(Y,Z)-\eta(Y)\eta(Z)\} \\ &+ (X\alpha)g(\phi Y,Z)+(Y\beta)\{g(X,Z)-\eta(Z)\eta(X)\}, \end{aligned} \tag{2.8}$$

$$S(X,\xi) = [2n(\alpha^2-\beta^2)-(\xi\beta)]\eta(X)-((\phi X)\alpha)-(2n-1)(X\beta), \tag{2.9}$$

$$R(\xi,X)\xi = (\alpha^2-\beta^2-\xi\beta)[\eta(X)\xi - X], \tag{2.10}$$

$$S(\xi,\xi) = 2n(\alpha^2-\beta^2-\xi\beta), \tag{2.11}$$

$$(\xi\alpha)+2\alpha\beta = 0, \tag{2.12}$$

$$Q\xi = [2n(\alpha^2-\beta^2)-(\xi\beta)]\xi+\phi(grad\alpha)-(2n-1)(grad\beta), \tag{2.13}$$

where R is the curvature tensor of type (1, 3) of the manifold and Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$ for any vector fields X, Y on M.

3 Weakly concircular symmetric trans-sasakian manifolds

Definition 3.1. *A trans-Sasakian manifold* $(M^{2n+1},g)(n>1)$ *is said to be weakly concircular symmetric if its concircular curvature tensor* $\tilde{C}$ *of type (0, 4) satisfies (1.5).*

Setting $Y = V = e_i$ in (1.5) and taking summation over i, $1 \le i \le 2n+1$, we get

$$\begin{aligned} &(\nabla_X S)(Z,U) - \frac{dr(X)}{n}g(Z,U) \\ &= A(X)\left[S(Z,U)-\frac{r}{n}g(Z,U)\right]+B(Z)\left[S(X,U)-\frac{r}{n}g(X,U)\right] \\ &+D(U)\left[S(X,Z)-\frac{r}{n}g(X,Z)\right]+B(R(X,Z)U)+D(R(X,U)Z) \\ &-\frac{r}{n(n-1)}\Big[\{B(X)+D(X)\}g(Z,U)-B(Z)g(X,U)-D(U)g(Z,X)\Big]. \end{aligned} \tag{3.1}$$

Plugging $X = Z = U = \xi$ in (3.1) and then using (2.7) and (2.11), we obtain

$$A(\xi)+B(\xi)+D(\xi) = \frac{2n^2\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-(\xi(\xi\beta))\}-dr(\xi)}{2n^2\{\alpha^2-(\xi\beta)-\beta^2\}-r}. \tag{3.2}$$

This leads to the following:

Theorem 3.1. *In a weakly concircular symmetric trans-Sasakian manifold* (M^{2n+1},g) $(n>1)$, *the relation (3.2) holds.*

Next, substituting X and Z by ξ in (3.1) and then using (2.7) and (2.12) we obtain

$$\begin{aligned}&(\nabla_\xi S)(\xi,U)-\frac{dr(\xi)}{n}\eta(U) \\ &= \big[A(\xi)+B(\xi)\big]\Big[S(U,\xi)-\frac{r}{n}\eta(U)\Big]+D(U)\Big[(2n-1)\{\alpha^2-(\xi\beta) \\ &-\beta^2\}-\frac{n-2}{n(n-1)}r\Big]+\Big[\alpha^2-(\xi\beta)-\beta^2-\frac{r}{n(n-1)}\Big]\eta(U)D(\xi).\end{aligned} \tag{3.3}$$

From (2.9), we have

$$\begin{aligned}(\nabla_\xi S)(\xi,U) &= \nabla_\xi S(\xi,U)-S(\nabla_\xi\xi,U)-S(\xi,\nabla_\xi U) \\ &= \nabla_\xi S(\xi,U)-S(\xi,\nabla_\xi U) \\ &= [2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-(\xi(\xi\beta))]\eta(U) \\ &\quad -(2n-1)(U(\xi\beta))-(\phi U(\xi\alpha)).\end{aligned} \tag{3.4}$$

By virtue of (3.3) and (3.4) we obtain from (3.2) that

$$\begin{aligned}D(U) &= \frac{[2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-(\xi(\xi\beta))-\frac{dr(\xi)}{n}]\eta(U)}{(2n-1)[\alpha^2-(\xi\beta)-\beta^2]-\frac{n-2}{n(n-1)}r} \\ &- \frac{(2n-1)(U(\xi\beta))+(\phi U(\xi\alpha))}{(2n-1)[\alpha^2-(\xi\beta)-\beta^2]-\frac{n-2}{n(n-1)}r} \\ &+ D(\xi)\Big[\frac{(2n-1)\{(\alpha^2-\beta^2)\eta(U)-(U\beta)\}-((\phi U)\alpha)-\frac{n-2}{n(n-1)}r\eta(U)}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{n-2}{n(n-1)}r}\Big] \\ &- \frac{2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-(\xi(\xi\beta))\}-\frac{dr(\xi)}{n}}{[2n\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{r}{n}][(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{n-2}{n(n-1)}r]} \\ &\quad \Big[\{2n(\alpha^2-\beta^2)-(\xi\beta)-\frac{r}{n}\}\eta(U)-(2n-1)(U\beta)-((\phi U)\alpha)\Big].\end{aligned} \tag{3.5}$$

Next, setting $X=U=\xi$ in (3.1) and proceeding in a similar manner as above, we get

$$\begin{aligned}B(Z) &= \frac{[2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-(\xi(\xi\beta))-\frac{dr(\xi)}{n}]\eta(Z)}{(2n-1)[\alpha^2-(\xi\beta)-\beta^2]-\frac{n-2}{n(n-1)}r} \\ &- \frac{(2n-1)(Z(\xi\beta))+(\phi Z(\xi\alpha))}{(2n-1)[\alpha^2-(\xi\beta)-\beta^2]-\frac{n-2}{n(n-1)}r} \\ &+ B(\xi)\Big[\frac{(2n-1)\{(\alpha^2-\beta^2)\eta(Z)-(Z\beta)\}-((\phi Z)\alpha)-\frac{n-2}{n(n-1)}r\eta(Z)}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{n-2}{n(n-1)}r}\Big] \\ &- \frac{2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-(\xi(\xi\beta))\}-\frac{dr(\xi)}{n}}{[2n\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{r}{n}][(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{n-2}{n(n-1)}r]} \\ &\quad \Big[\{2n(\alpha^2-\beta^2)-(\xi\beta)-\frac{r}{n}\}\eta(Z)-(2n-1)(Z\beta)-((\phi Z)\alpha)\Big].\end{aligned} \tag{3.6}$$

Again, setting $Z = U = \xi$ in (3.1), we get

$$
\begin{aligned}
(\nabla_X S)(\xi,\xi) - \frac{dr(X)}{n} &= A(X)\big[S(\xi,\xi) - \frac{r}{n}\big] + [B(\xi) + D(\xi)][S(X,\xi) \\
&- \frac{n-2}{n(n-1)} r\eta(X)] + B(R(X,\xi)\xi) \\
&+ D(R(X,\xi)\xi) - \frac{r}{n(n-1)}[B(X) + D(X)] \\
&= [2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}]A(X) + [B(\xi) \\
&+ D(\xi)]\big[S(X,\xi) - \{\frac{n-2}{n(n-1)} r + \alpha^2 - (\xi\beta) - \beta^2\}\eta(X)\big] \\
&+ [B(X) + D(X)]\big[\alpha^2 - (\xi\beta) - \beta^2 - \frac{r}{n(n-1)}\big].
\end{aligned} \tag{3.7}
$$

Now we have

$$
(\nabla_X S)(\xi,\xi) = \nabla_X S(\xi,\xi) - 2S(\nabla_X \xi, \xi),
$$

which yields by using (2.5) and (2.9) that

$$
\begin{aligned}
(\nabla_X S)(\xi,\xi) &= 2n[2\alpha(X\alpha) - 2\beta(X\beta) - (X(\xi\beta))] \\
&+ 2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n-1)((\phi X)\beta)] \\
&+ 2\beta[((\phi X)\alpha) + (2n-1)\{(X\beta) - (\xi\beta)\eta(X)\}].
\end{aligned} \tag{3.8}
$$

In view of (3.8), (3.7) yields

$$
\begin{aligned}
&\big[2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}\big]A(X) + \big[\alpha^2 - (\xi\beta) - \beta^2 - \frac{r}{n(n-1)}\big]\big[B(X) + D(X)\big] \\
&= 2n\big[2\alpha(X\alpha) - 2\beta(X\beta) - (X(\xi\beta))\big] + 2\alpha\big[(X\alpha) - \eta(X)(\xi\alpha) \\
&-(2n-1)((\phi X)\beta)\big] + 2\beta\big[((\phi X)\alpha) + (2n-1)\{(X\beta) - (\xi\beta)\eta(X)\}\big] \\
&-\frac{dr(X)}{n} - \{B(\xi) + D(\xi)\}\big[\{(2n-1)(\alpha^2 - \beta^2) \\
&-\frac{n-2}{n(n-1)} r\}\eta(X) - ((\phi X)\alpha) - (2n-1)(X\beta)\big].
\end{aligned} \tag{3.9}
$$

Using (3.5) and (3.6) in (3.9), we obtain

$$
\begin{aligned}
&[2n\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{r}{n}]A(X) \\
&= 2n[2\alpha(X\alpha)-2\beta(X\beta)-(X(\xi\beta))]+2\alpha[(X\alpha)-\eta(X)(\xi\alpha) \\
&-(2n-1)((\phi X)\beta)]+2\beta[((\phi X)\alpha)+(2n-1)\{(X\beta)-(\xi\beta)\eta(X)\}]-\frac{dr(X)}{n} \\
&-\frac{2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-(\xi(\xi\beta))\}-\frac{dr(\xi)}{n}}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{n-2}{n(n-1)}r}\Big[\{(2n-1)(\alpha^2 \\
&-\beta^2)-\frac{n-2}{n(n-1)}r\}\eta(X)-((\phi X)\alpha)-(2n-1)(X\beta)\Big] \\
&+A(\xi)\frac{2n\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{r}{n}}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{n-2}{n(n-1)}r}\Big[\{(2n-1)(\alpha^2 \\
&-\beta^2)-\frac{n-2}{n(n-1)}r\}\eta(X)-((\phi X)\alpha)-(2n-1)(X\beta)\Big] \\
&-\frac{2\{\alpha^2-(\xi\beta)-\beta^2-\frac{r}{n(n-1)}\}}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{n-2}{n(n-1)}r}\Big[2n\{2\alpha(\xi\alpha) \\
&-2\beta(\xi\beta)\}-(\xi(\xi\beta))-\frac{dr(\xi)}{n}\Big]\eta(X) \\
&+\frac{2\{\alpha^2-(\xi\beta)-\beta^2-\frac{r}{n(n-1)}\}}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{n-2}{n(n-1)}r}\Big[(2n-1)(X(\xi\beta))+(\phi X(\xi\alpha))\Big] \\
&+\frac{2\{\alpha^2-(\xi\beta)-\beta^2-\frac{r}{n(n-1)}\}[2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)-(\xi(\xi\beta))\}-\frac{dr(\xi)}{n}]}{[2n\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{r}{n}][(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}-\frac{n-2}{n(n-1)}r]} \\
&\Big[\{2n(\alpha^2-\beta^2)-(\xi\beta)-\frac{r}{n}\}\eta(X)-(2n-1)(X\beta)-((\phi X)\alpha)\Big].
\end{aligned}
\tag{3.10}
$$

This leads to the following:

Theorem 3.2. *In a weakly concircular symmetric trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$, *the associated 1-forms* D, B *and* A *are given by (3.5), (3.6) and (3.10) respectively.*

4 Weakly concircular Ricci symmetric trans-Sasakian manifolds

Definition 4.1. *A trans-Sasakian manifold* $(M^{2n+1}, g)(n > 1)$ *is said to be weakly concircular Ricci symmetric if its concircular Ricci tensor* P *of type (0, 2) satisfies (1.9).*

In view of (1.8), (1.9) yields

$$\begin{aligned}(\nabla_X S)(Y,Z) - \frac{dr(X)}{n} g(Y,Z) &= A(X)\left[S(Y,Z) - \frac{r}{n} g(Y,Z)\right] \\ &+ B(Y)\left[S(X,Z) - \frac{r}{n} g(X,Z)\right] \\ &+ D(Z)\left[S(X,Y) - \frac{r}{n} g(X,Y)\right].\end{aligned} \tag{4.1}$$

Setting $X = Y = Z = \xi$ in (4.1), we get the relation (3.2) and hence we can state the following:

Theorem 4.1. *In a weakly concircular Ricci symmetric trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$, *the relation (3.2) holds.*

Next, substituting X and Y by ξ in (4.1), we obtain

$$\begin{aligned}(\nabla_\xi S)(\xi,Z) - \frac{dr(\xi)}{n}\eta(Z) &= [A(\xi) + B(\xi)]\big[S(\xi,Z) \\ &- \frac{r}{n}\eta(Z)\big] + D(Z)\big[S(\xi,\xi) - \frac{r}{n}\big].\end{aligned} \tag{4.2}$$

Using (3.2) and (3.4) in (4.2), we get

$$\begin{aligned} D(Z) &= \frac{\left[2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - (\xi(\xi\beta)) - \frac{dr(\xi)}{n}\right]\eta(Z)}{2n[\alpha^2 - (\xi\beta) - \beta^2] - \frac{r}{n}} \\ &- \frac{(2n-1)(Z(\xi\beta)) + (\phi Z(\xi\alpha))}{2n[\alpha^2 - (\xi\beta) - \beta^2] - \frac{r}{n}} \\ &+ D(\xi)\left[\frac{2n\{(\alpha^2 - \beta^2) - (\xi\beta) - \frac{r}{n}\}\eta(Z) - ((\phi Z)\alpha) - (2n-1)(Z\beta)}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}}\right] \\ &- \frac{2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - (\xi(\xi\beta))\} - \frac{dr(\xi)}{n}}{[2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}]^2}\Big[\{2n(\alpha^2 - \beta^2) \\ &- (\xi\beta) - \frac{r}{n}\}\eta(Z) - (2n-1)(Z\beta) - ((\phi Z)\alpha)\Big] \quad \text{for all } Z.\end{aligned} \tag{4.3}$$

Again putting $X = Z = \xi$ in (4.1) and proceeding in a similar manner as above we get

$$\begin{aligned} B(Y) &= \frac{\left[2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - (\xi(\xi\beta)) - \frac{dr(\xi)}{n}\right]\eta(Y)}{2n[\alpha^2 - (\xi\beta) - \beta^2] - \frac{r}{n}} \\ &- \frac{(2n-1)(Y(\xi\beta)) + (\phi Y(\xi\alpha))}{2n[\alpha^2 - (\xi\beta) - \beta^2] - \frac{r}{n}} \\ &+ B(\xi)\left[\frac{2n\{(\alpha^2 - \beta^2) - (\xi\beta) - \frac{r}{n}\}\eta(Y) - ((\phi Y)\alpha) - (2n-1)(Y\beta)}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}}\right] \\ &- \frac{2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - (\xi(\xi\beta))\} - \frac{dr(\xi)}{n}}{[2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}]^2}\Big[\{2n(\alpha^2 - \beta^2) \\ &- (\xi\beta) - \frac{r}{n}\}\eta(Y) - (2n-1)(Y\beta) - ((\phi Y)\alpha)\Big] \quad \text{for all } Y.\end{aligned} \tag{4.4}$$

Again, setting $Y = Z = \xi$ in (4.1) and using (2.9) and (2.11), we get

$$\begin{aligned}(\nabla_X S)(\xi,\xi) - \frac{dr(X)}{n} &= \Big[2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}\Big]A(X) \\ &+ [B(\xi) + D(\xi)]\big[\{2n(\alpha^2 - \beta^2) - (\xi\beta)\}\eta(X) \\ &- ((\phi X)\alpha) - (2n-1)(X\beta)\big].\end{aligned} \tag{4.5}$$

Using (3.2) and (3.8) in (4.5), we get

$$\begin{aligned}A(X) &= \frac{2n[2\alpha(X\alpha) - 2\beta(X\beta) - (X(\xi\beta))]}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}} \\ &+ \frac{2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n-1)((\phi X)\beta)]}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}} \\ &+ \frac{2\beta[((\phi X)\alpha) + (2n-1)\{(X\beta) - (\xi\beta)\eta(X)\}]}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}} \\ &+ A(\xi)\Big[\frac{\{2n(\alpha^2 - \beta^2) - (\xi\beta) - \frac{r}{n}\}\eta(X) - ((\phi X)\alpha) - (2n-1)(X\beta)}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}}\Big] \\ &- \frac{2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - (\xi(\xi\beta))\} - \frac{dr(\xi)}{n}}{[2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}]^2}\Big[\{2n(\alpha^2 - \beta^2) \\ &- (\xi\beta) - \frac{r}{n}\}\eta(X) - (2n-1)(X\beta) - ((\phi X)\alpha)\Big] \quad \text{for all } X.\end{aligned} \tag{4.6}$$

This leads to the following:

Theorem 4.2. *In a weakly concircular Ricci symmetric trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$*, the associated 1-forms* D*,* B *and* A *are given by (4.3), (4.4) and (4.6) respectively.*

Adding (4.3), (4.4) and (4.6) and using (3.2), we get

$$\begin{aligned}&A(X) + B(X) + D(X) \\ &= \frac{2n[2\alpha(X\alpha) - 2\beta(X\beta) - (X(\xi\beta))]}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}} \\ &+ \frac{2\alpha[(X\alpha) - \eta(X)(\xi\alpha) - (2n-1)((\phi X)\beta)]}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}} \\ &+ \frac{2\beta[((\phi X)\alpha) + (2n-1)\{(X\beta) - (\xi\beta)\eta(X)\}]}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}} \\ &\frac{2\Big[2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta)\} - (\xi(\xi\beta)) - \frac{dr(\xi)}{n}\Big]\eta(X)}{2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}} \\ &- \frac{(2n-1)(X(\xi\beta)) + (\phi X(\xi\alpha))}{n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{2n}} \\ &- \frac{2\Big[2n\{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - (\xi(\xi\beta))\} - \frac{dr(\xi)}{n}\Big]}{[2n\{\alpha^2 - (\xi\beta) - \beta^2\} - \frac{r}{n}]^2}\Big[\{2n(\alpha^2 - \beta^2) \\ &-(\xi\beta) - \frac{r}{n}\}\eta(X) - (2n-1)(X\beta) - ((\phi X)\alpha)\Big]\end{aligned} \tag{4.7}$$

for any vector field X. This leads to the following:

Theorem 4.3. *In a weakly concircular Ricci symmetric trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$*, the sum of the associated 1-forms is given by (4.7).*

In particular, if $\phi(\mathrm{grad}\,\alpha) = \mathrm{grad}\,\beta$, then $(\xi\beta) = 0$ and hence the relation (4.7) reduces to the following form

$$\begin{aligned} & A(X) + B(X) + D(X) \\ &= \frac{2n[2\alpha(X\alpha) - 2\beta(X\beta)]}{2n(\alpha^2 - \beta^2) - \frac{r}{n}} + \frac{2\alpha\{(X\alpha) - \eta(X)(\xi\alpha) - (2n-1)((\phi X)\beta)\}}{2n(\alpha^2 - \beta^2) - \frac{r}{n}} \\ &+ \frac{2\beta\{((\phi X)\alpha) + (2n-1)(X\beta)\} + 2\{4n\alpha(\xi\alpha) - \frac{dr(\xi)}{n}\}\eta(X) - 2(\phi X(\xi\alpha))}{2n(\alpha^2 - \beta^2) - \frac{r}{n}} \\ &- \frac{2[4n\alpha(\xi\alpha) - \frac{dr(\xi)}{n}]}{[2n(\alpha^2 - \beta^2) - \frac{r}{n}]^2}\Big[\{2n(\alpha^2 - \beta^2) - \frac{r}{n}\}\eta(X) - ((\phi X)\alpha) - (2n-1)(X\beta)\Big]. \end{aligned} \tag{4.8}$$

for any vector field X. This leads to the following:

Corollary 4.1. *If a weakly concircular Ricci symmetric trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$ *satisfies the condition* $\phi(\mathrm{grad}\,\alpha) = \mathrm{grad}\,\beta$*, then the sum of the associated 1-forms is given by (4.8).*

If $\beta = 0$ and $\alpha = 1$, then (4.7) yields $A(X)+B(X)+D(X) = 0$ for all X and hence we can state the following:

Corollary 4.2. *There is no weakly concircular Ricci symmetric Sasakian manifold* $M^{2n+1}(n > 1)$*, unless the sum of the 1-forms is everywhere zero.*

Corollary 4.3. *If an* α*-Sasakian manifold is weakly concircular Ricci symmetric, then the sum of the 1-forms, i.e.,* $A + B + D$ *is given by*

$$\begin{aligned} A(X) + B(X) + D(X) &= \frac{2\alpha[(2n+1)(X\alpha) - \eta(X)(\xi\alpha)] - 2(\phi X(\xi\alpha))}{2n\alpha^2 - \frac{r}{n}} \\ &+ \frac{2[4n\alpha(\xi\alpha) - \frac{dr(\xi)}{n}]((\phi X)\alpha)}{(2n\alpha^2 - \frac{r}{n})^2}. \end{aligned}$$

Again, if $\alpha = 0$ and $\beta = 1$, then (4.7) yields $A(X) + B(X) + D(X) = 0$ for all X. This leads to the following:

Corollary 4.4. *There is no weakly concircular Ricci symmetric Kenmotsu manifold* $M^{2n+1}(n > 1)$*, unless the sum of the 1-forms is everywhere zero.*

Corollary 4.5. *If a* β*-Kenmotsu manifold is weakly concircular Ricci symmetric, then the sum of* *the 1-forms, i.e.,* $A+B+D$ *is given by*

$$\begin{aligned}
&A(X)+B(X)+D(X)\\
&=\frac{2n\{2\beta(X\beta)+(X(\xi\beta))\}-2(2n-1)\beta\{(X\beta)-(\xi\beta)\eta(X)\}}{2n\{(\xi\beta)+\beta^2\}+\frac{r}{n}}\\
&+\frac{2[\{4n\beta(\xi\beta)+(\xi(\xi\beta)+\frac{dr(\xi)}{n}\}\eta(X)+(2n-1)(X(\xi\beta))]}{2n\{(\xi\beta)+\beta^2\}+\frac{r}{n}}\\
&-\frac{2[2n\{2\beta(\xi\beta)+(\xi(\xi\beta))\}+\frac{dr(\xi)}{n}][\{2n\beta^2+(\xi\beta)+\frac{r}{n}\}\eta(X)+(2n-1)(X\beta)]}{[2n\{(\xi\beta)+\beta^2\}+\frac{r}{n}]^2}.
\end{aligned}$$

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