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Uncertainty principle for the Riemann-Liouville operator

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ABSTRACT

A Beurling-Hörmander theorem's is proved for the Fourier transform connected with the Riemann-Liouville operator. Nextly, Gelfand-Shilov and Cowling-Price type theorems are established.

RESUMEN

Se demuestra el teorema de Beurling-Hörmander por la transformada de Fourier conectada con el operador de Riemann-Liouville. Además, se establecen teoremas tipo de Gelfand-Shilov y Cowling-Price.

Keywords: Beurling-Hörmander theorem, Gelfand-Shilov theorem, Cowling- Price theorem, Fourier transform, Riemann-Liouville operator.

Mathematics Subject Classification: 43A32; 42B10.

1 Introduction

The uncertainty principles play an important role in harmonic analysis and have been studied by many authors, and from many points of view [12, 15]. These principles state that a function f and its Fourier transform $\widehat{f}$ cannot be simultaneously sharply localized. Theorems of Hardy, Morgan, Gelfand-Shilov, or Cowlong-Price,... are established for several Fourier transforms [8, 14, 19, 20, 21], the most recent being the well known Beurling-Hörmander theorem's which has been proved by Hörmander [16], who took an idea of Beurling [4]. This theorem states that if f is an integrable function on $\mathbb{R}$ with respect to the Lebesgue measure, and if

$$\iint_{\mathbb{R}^2} |f(x)||\widehat{f}(y)|e^{|xy|}\,dx\,dy < +\infty,$$

then $f = 0$ almost everywhere.
Later, Bonami, Demange and Jaming [5] have generalized the above theorem and have established a strong multidimensional version of this uncertainty principle [15], by showing the following result if f is a square integrable function on $\mathbb{R}^n$ with respect to the Lebesgue measure, then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\widehat{f}(y)|}{(1+|x|+|y|)^d} e^{|\langle x/y\rangle|}\,dx\,dy < +\infty,$$

if and only if f may be written as

$$f(x) = P(x)e^{-\langle Ax/x\rangle},$$

where A is a real positive definite symmetric matrix and P is a polynomial with degree$(P) < \frac{d-n}{2}$.
In particular for $d \leqslant n$, f is identically zero.
The Beurling-Hörmander uncertainty principle in its weak and strong forms has been studied by many authors, and for various Fourier transforms. In particular, Bouattour and Trimèche [6] have showed this theorem for the hypergroup of Chébli-Trimèche, Kamoun and Trimèche [17] have proved an analogue of the Beurling-Hörmander theorem for some singular partial differential operators, Trimèche [22] has showed this uncertainty principle for the Dunkl transform, we cite also Yakubovich [26], who has established the same result for the Kontorovich-Lebedev transform.
The Beurling-Hörmander uncertainty principle implies many other known quantitative uncertainty principles as those of Gelfand-Shilov [13], Cowling-Price [8], Morgan [3, 19] or also the one of Hardy [14].

In [2], the third author with the others have considered the singular partial differential operators defined by

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}, \\ \\ \Delta_2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2} \ ; \ (r,x) \in]0,+\infty[\times\mathbb{R} \ ; \ \alpha \geq 0, \end{cases}$$

and they associated to Δ_1 and Δ_2 the following integral transform, called the Riemann-Liouville operator which is defined on $\mathscr{C}_*(\mathbb{R}^2)$ $\big($The space of continuous functions on $\mathbb{R}^2$, even with respect to he first variable$\big)$ by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\,ds, & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\frac{dt}{\sqrt{(1-t^2)}}; & \text{if } \alpha = 0. \end{cases}$$

The Fourier transform connected with the operator $\mathscr{R}_\alpha$ is defined by

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x),$$

where

$$\varphi_{\mu,\lambda}(r,x) = \mathscr{R}_\alpha\big(\cos(\mu.)e^{-i\lambda.}\big)(r,x).$$

$d\nu_\alpha$ is the measure defined on $[0,+\infty[\times\mathbb{R}$ by,

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}dr\otimes dx.$$

Many harmonic analysis results are established for the Fourier transform $\mathscr{F}_\alpha$ (Inversion formula, Plancherel's formula, Paley-Winer and Plancherel's theorems...).

The aim of this work is to establish the Beurling-Hörmander theorem for the fourier transform $\mathscr{F}_\alpha$ and to deduce the analogues of the Gelfand-Shilov and the Cowling-Price theorems for this transform.

More precisely, in the second section, we give some basic harmonic analysis results related to the Fourier transform $\mathscr{F}_\alpha$. The third section is devoted to establish the main result of this paper, that is the the Beurling-Hörmander theorem

• Let f be a square integrable function on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$. Let d be a real number, $d \geqslant 0$. If

$$\iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d} e^{|(r,x)||\theta(\mu,\lambda)|} d\nu_\alpha(r,x)\, d\widetilde{\gamma}_\alpha(\mu,\lambda) < +\infty.$$

Then
$i)$ For $d \leqslant 2$, $f = 0$.
$ii)$ For $d > 2$, there exist a positive constant a and a polynomial P on $\mathbb{R}^2$ even with respect to the first variable, such that

$$f(r,x) = P(r,x)e^{-a(r^2+x^2)},$$

with degree$(P) < \frac{d}{2} - 1$,
where
$\Gamma_+ = [0,+\infty[\times\mathbb{R} \cup \{(it,x) \mid (t,x) \in [0,+\infty[\times\mathbb{R}\ ,\ t \leqslant |x|\}$.

θ is the function defined on the set Γ_+ by

$$\theta(\mu,\lambda) = (\sqrt{\mu^2+\lambda^2},\lambda).$$

$d\widetilde{\gamma}_\alpha$ the measure defined on the set Γ_+ by

$$\iint_{\Gamma_+} g(\mu,\lambda)\ d\widetilde{\gamma}_\alpha(\mu,\lambda) = \frac{1}{\pi}\Big(\int_0^{+\infty}\int_{\mathbb{R}} g(\mu,\lambda)(\mu^2+\lambda^2)^{-\frac{1}{2}}\mu\, d\mu\, d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} g(i\mu,\lambda)(\lambda^2-\mu^2)^{-\frac{1}{2}}\mu d\mu d\lambda\Big).$$

The last section of this paper contains the following results that are respectively the Gelfand-Shilov and the Cowling-Price theorems for $\mathscr{F}_\alpha$

• Let p, q be two conjugate exponents, $p, q \in]1,+\infty[$. Let d, ξ, η be non negative real numbers such that $\xi\eta \geqslant 1$. Let f be a measurable function on $\mathbb{R}^2$, even with respect to the first variable, such that $f \in L^2(d\nu_\alpha)$. If

$$\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)|e^{\frac{\xi^p|(r,x)|^p}{p}}}{(1+|(r,x)|)^d}\, d\nu_\alpha(r,x) < +\infty,$$

and

$$\iint_{\Gamma_+} \frac{|\mathscr{F}_\alpha(f)(\mu,\lambda)|e^{\frac{\eta^q|\theta(\mu,\lambda)|^q}{q}}}{(1+|\theta(\mu,\lambda)|)^d}\ d\widetilde{\gamma}_\alpha(\mu,\lambda) < +\infty,$$

then
$i)$ For $d \leqslant 1$, $f = 0$.
$2i)$ For $d > 1$, we have

a) $f = 0$ for $\xi\eta > 1$.
b) $f = 0$ for $\xi\eta = 1$, and $p \neq 2$.
c) $f(r,x) = P(r,x)e^{-a(r^2+x^2)}$, for $\xi\eta = 1$, and $p = q = 2$,
where $a > 0$, and P is a polynomial on $\mathbb{R}^2$ even with respect to the first variable, with degree$(P) < d-1$.

- Let $\xi, \eta, \omega_1, \omega_2$ be non negative real numbers such that $\xi\eta \geqslant \frac{1}{4}$. Let p, q be two exponents, $p, q \in [1, +\infty]$, and let f be a measurable function on $\mathbb{R}^2$, even with respect to the first variable such that $f \in L^2(d\nu_\alpha)$. If

$$\Big\| \frac{e^{\xi|(.,.)|^2}}{(1+|(.,.)|)^{\omega_1}} f \Big\|_{p,\nu_\alpha} < +\infty,$$

and

$$\Big\| \frac{e^{\eta|\theta(.,.)|^2}}{(1+|\theta(.,.)|)^{\omega_2}} \mathscr{F}_\alpha(f) \Big\|_{q,\widetilde{\gamma}_\alpha} < +\infty,$$

then
i) For $\xi\eta > \frac{1}{4}$, $f = 0$.
ii) For $\xi\eta = \frac{1}{4}$, there exist a positive constant a and a polynomial P on $\mathbb{R}^2$, even with respect to the first variable, such that

$$f(r,x) = P(r,x)e^{-a(r^2+x^2)}.$$

2 The Fourier transform associated with the Riemann-Liouville operator

It's well known [2] that for all $(\mu, \lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} \Delta_1 u(r,x) = -i\lambda u(r,x), \\ \Delta_2 u(r,x) = -\mu^2 u(r,x), \\ u(0,0) = 1 \ , \ \dfrac{\partial u}{\partial r}(0,x) = 0 \ , \ \forall x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$, given by

$$\forall (r,x) \in \mathbb{R}^2; \quad \varphi_{\mu,\lambda}(r,x) = j_\alpha(r\sqrt{\mu^2+\lambda^2})e^{-i\lambda x},$$

where

$$j_\alpha(z) = \frac{2^\alpha \Gamma(\alpha+1)}{z^\alpha} J_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(\alpha+n+1)} (\frac{z}{2})^{2n}, \quad z \in \mathbb{C}, \tag{2.1}$$

and J_α is the Bessel function of the first kind and index α [9, 10, 18, 25].
The modified Bessel function j_α has the following integral representation [18, 25], for all $z \in \mathbb{C}$, we have

$$j_\alpha(z) = \begin{cases} \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\displaystyle\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}\cos(zt)dt, & \text{if } \alpha > -\frac{1}{2}; \\ \cos(z), & \text{if } \alpha = -\frac{1}{2}. \end{cases} \tag{2.2}$$

From the relation (2.2), we deduce that for all $z \in \mathbb{C}$, we have

$$\left|j_\alpha(z)\right| \leqslant e^{|\mathrm{Im}(z)|}. \tag{2.3}$$

From the properties of the modified Bessel function j_α, we deduce that the eigenfunction $\varphi_{\mu,\lambda}$ satisfies the following properties

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1, \tag{2.4}$$

if and only if (μ,λ) belongs to the set

$$\Gamma = \mathbb{R}^2 \cup \left\{(it,x) \mid (t,x) \in \mathbb{R}^2\ ,\ |t| \leq |x|\right\}.$$

The eigenfunction $\varphi_{\mu,\lambda}$ has the following Mehler integral representation

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^1\int_{-1}^1 \cos(\mu rs\sqrt{1-t^2})e^{-i\lambda(x+rt)}(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\,ds; & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^1 \cos(r\mu\sqrt{1-t^2})e^{-i\lambda(x+rt)}\dfrac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha = 0. \end{cases}$$

This integral representation allows to define the so-called Riemann-Liouville operator associated with Δ_1, Δ_2 by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^1\int_{-1}^1 f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\,ds; & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^1 f(r\sqrt{1-t^2}, x+rt)\dfrac{dt}{\sqrt{(1-t^2)}}; & \text{if } \alpha = 0. \end{cases}$$

where f is a continuous function on $\mathbb{R}^2$, even with respect to the first variable.
The transform $\mathscr{R}_\alpha$ generalizes the "mean operator" defined by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x + r\cos\theta)\,d\theta.$$

In the following, we denote by

dm_{n+1} the measure defined on $[0,+\infty[\times\mathbb{R}^n$ by,

$$dm_{n+1}(r,x) = \sqrt{\frac{2}{\pi}}\frac{1}{(2\pi)^{\frac{n}{2}}}dr\otimes dx.$$

$L^p(dm_{n+1})$ the space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$, such that

$$\begin{aligned} \|f\|_{p,m_{n+1}} &= \Big(\int_0^{+\infty}\int_{\mathbb{R}^n}|f(r,x)|^p\, dm_{n+1}(r,x)\Big)^{\frac{1}{p}} < +\infty, \quad \text{if } p\in[1,+\infty[,\\ \|f\|_{\infty,m_{n+1}} &= \operatorname{ess\,sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}^n}|f(r,x)| < +\infty, \qquad \text{if } p=+\infty. \end{aligned}$$

$d\nu_\alpha$ the measure defined on $[0,+\infty[\times\mathbb{R}$, by

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}dr\otimes dx.$$

$L^p(d\nu_\alpha)$ the space of measurable functions f on $[0,+\infty[\times\mathbb{R}$ such that $\|f\|_{p,\nu_\alpha} < +\infty$.

$\Gamma_+ = [0,+\infty[\times\mathbb{R}\cup\big\{(it,x) \mid (t,x)\in[0,+\infty[\times\mathbb{R}\ ,\ t\leqslant|x|\big\}$.

$\mathscr{B}_{\Gamma_+}$ the σ-algebra defined on Γ_+ by

$$\mathscr{B}_{\Gamma_+} = \{\theta^{-1}(B)\ ,\ B\in\mathscr{B}([0,+\infty[\times\mathbb{R})\},$$

where θ is the bijective function defined on the set Γ_+ by

$$\theta(\mu,\lambda) = (\sqrt{\mu^2+\lambda^2},\lambda).$$

$d\gamma_\alpha$ the measure defined on $\mathscr{B}_{\Gamma_+}$ by

$$\forall\ A\in\mathscr{B}_{\Gamma_+};\ \gamma_\alpha(A) = \nu_\alpha(\theta(A)).$$

$L^p(d\gamma_\alpha)$ the space of measurable functions f on Γ_+, such that $\|f\|_{p,\gamma_\alpha} < +\infty$.

$d\widetilde{\gamma}_\alpha$ the measure defined on $\mathscr{B}_{\Gamma_+}$ by

$$d\widetilde{\gamma}_\alpha(\mu,\lambda) = \frac{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}(\mu^2+\lambda^2)^{\alpha+\frac{1}{2}}}d\gamma_\alpha(\mu,\lambda).$$

$S_*(\mathbb{R}^2)$ the Shwartz's space formed by the infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, and even with respect to the first variable.

Then we have the following properties.

Proposition 2.1. i) *For all non negative measurable function* g *on* Γ_+*, we have*

$$\iint_{\Gamma_+} g(\mu,\lambda)\,d\gamma_\alpha(\mu,\lambda) = \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\Big(\int_0^{+\infty}\int_{\mathbb{R}} g(\mu,\lambda)(\mu^2+\lambda^2)^\alpha\mu\,d\mu\,d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} g(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha\mu\,d\mu\,d\lambda\Big).$$

ii) *For all measurable function* f *on* $[0,+\infty[\times\mathbb{R}$*, the function* $f o\theta$ *is measurable on* Γ_+*. Furthermore if* f *is non negative or integrable function on* $[0,+\infty[\times\mathbb{R}$ *with respect to the measure* $d\nu_\alpha$*, then we have*

$$\iint_{\Gamma_+}(f\circ\theta)(\mu,\lambda)\,d\gamma_\alpha(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\,d\nu_\alpha(r,x).$$

iii) *For all non negative measurable function* f*, respectively integrable on* $[0,+\infty[\times\mathbb{R}$ *with respect to the measure* dm_2*, we have*

$$\iint_{\Gamma_+}(f\circ\theta)(\mu,\lambda)\;d\widetilde{\gamma}_\alpha(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\;dm_2(r,x). \tag{2.5}$$

In the following we shall define the Fourier transform $\mathscr{F}_\alpha$ associated with the operator $\mathscr{R}_\alpha$, and we shall give some properties that we use in the sequel.

Definition 2.1. The Fourier transform $\mathscr{F}_\alpha$ associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$ is defined on $L^1(d\nu_\alpha)$ by

$$\forall(\mu,\lambda)\in\Gamma\;;\;\mathscr{F}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{\mu,\lambda}(r,x)\,d\nu_\alpha(r,x).$$

Then, for all $(\mu,\lambda)\in\Gamma$,

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathscr{F}_\alpha}(f)\circ\theta(\mu,\lambda), \tag{2.6}$$

where for all $(\mu,\lambda)\in[0,+\infty[\times\mathbb{R}$,

$$\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)j_\alpha(r\mu)e^{-i\lambda x}\,d\nu_\alpha(r,x). \tag{2.7}$$

Moreover, the relation (2.4) implies that the Fourier transform $\mathscr{F}_\alpha$ is a bounded linear operator from $L^1(d\nu_\alpha)$ into $L^\infty(d\gamma_\alpha)$, and that for all
$f\in L^1(d\nu_\alpha)$, we have

$$\|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha} \leqslant \|f\|_{1,\nu_\alpha}. \tag{2.8}$$

Theorem 2.1 (Inversion formula)**.** *Let* $f\in L^1(d\nu_\alpha)$ *such that*
$\mathscr{F}_\alpha(f)\in L^1(d\gamma_\alpha)$*, then for almost every* $(r,x)\in[0,+\infty[\times\mathbb{R}$*, we have*

$$\begin{aligned} f(r,x) &= \iint_{\Gamma_+}\mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}\,d\gamma_\alpha(\mu,\lambda) \\ &= \int_0^{+\infty}\int_{\mathbb{R}}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)j_\alpha(r\mu)e^{i\lambda x}\,d\nu_\alpha(\mu,\lambda). \end{aligned}$$

Lemma 2.2. *Let $\mathfrak{R}_\alpha$ be the mapping defined for all non negative measurable function* g *on* $[0,+\infty[\times\mathbb{R}$ *by*

$$\mathfrak{R}_\alpha(g)(r,x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-s^2)^{\alpha-\frac{1}{2}} g(rs,x)\,ds$$
$$= \frac{2\Gamma(\alpha+1)r^{-2\alpha}}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^r (r^2-s^2)^{\alpha-\frac{1}{2}} f(s,x)\,ds, \quad r>0. \tag{2.9}$$

Then for all non negative measurable functions f, g *on* $[0,+\infty[\times\mathbb{R}$*, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\mathfrak{R}_\alpha(g)(r,x)\,d\nu_\alpha(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} \mathscr{W}_\alpha(f)(r,x)g(r,x)\,dm_2(r,x), \tag{2.10}$$

where $\mathscr{W}_\alpha$ *is the classical Weyl transform defined for all non negative measurable function on* $[0,+\infty[\times\mathbb{R}$ *by*

$$\mathscr{W}_\alpha(f)(r,x) = \frac{1}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}\int_r^{+\infty}(t^2-r^2)^{\alpha-\frac{1}{2}} f(t,x)2t\,dt. \tag{2.11}$$

Proposition 2.2. *For all* $f\in L^1(d\nu_\alpha)$*, the function* $\mathscr{W}_\alpha(f)$ *belongs to* $L^1(dm_2)$*, and we have*

$$\|\mathscr{W}_\alpha(f)\|_{1,m_2} \leqslant \|f\|_{1,\nu_\alpha}. \tag{2.12}$$

Moreover, for all $(\mu,\lambda)\in[0,+\infty[\times\mathbb{R}$*, we have*

$$\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda) = (\Lambda_2\circ\mathscr{W}_\alpha)(f)(\mu,\lambda), \tag{2.13}$$

where Λ_2 *is the usual Fourier transform defined on* $L^1(dm_2)$ *by*

$$\Lambda_2(g)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} g(r,x)\cos(r\mu)e^{-i\lambda x}\,dm_2(r,x).$$

Remark 2.1. It's well known [23, 24] that the transforms $\widetilde{\mathscr{F}_\alpha}$ and Λ_2 are topological isomorphisms from $S_*(\mathbb{R}^2)$ onto itself. Then by the relation (2.13), we deduce that the classical Weyl transform $\mathscr{W}_\alpha$ is also a topological isomorphism from $S_*(\mathbb{R}^2)$ onto itself.

Proposition 2.3. *For all* $f\in S_*(\mathbb{R}^2)$*, we have*

$$\mathscr{W}_\alpha^{-1}(f) = (-1)^{1+[\alpha+\frac{1}{2}]}\mathscr{W}_{[\alpha+\frac{1}{2}]-\alpha+\frac{1}{2}}\Big(\big(\frac{\partial}{\partial t^2}\big)^{1+[\alpha+\frac{1}{2}]}(f)\Big), \tag{2.14}$$

where

$$\big(\frac{\partial}{\partial t^2}\big)(f)(t,x) = \frac{1}{t}\frac{\partial f}{\partial t}(t,x).$$

Proof. For $\sigma\in\mathbb{R}$, $\sigma>0$, let us define the so-called fractional transform $\mathscr{H}_\sigma$, defined on $S_*(\mathbb{R}^2)$ by

$$\mathscr{H}_\sigma(f)(r,x) = \frac{1}{2^\sigma\Gamma(\sigma)}\int_r^{+\infty}(t^2-r^2)^{\sigma-1}f(t,x)2t\,dt = \mathscr{W}_{\sigma-\frac{1}{2}}(f)(r,x).$$

From the remark 2.1, it follows that for all real number $\sigma > 0$, the mapping $\mathscr{H}_\sigma$ is a topological isomorphism from $S_*(\mathbb{R}^2)$ onto itself.
Moreover, we have the following properties
For all $\sigma, \delta \in \mathbb{R}$; $\sigma, \delta > 0$ and for every $f \in S_*(\mathbb{R}^2)$, we have

$$\big(\mathscr{H}_\sigma \circ \mathscr{H}_\delta\big)(f) = \mathscr{H}_{\sigma+\delta}(f).$$

For all $\sigma \in \mathbb{R}$, $\sigma > 0$, and for every integer k, we have

$$\mathscr{H}_\sigma(f) = (-1)^k \mathscr{H}_{\sigma+k}\Big(\big(\frac{\partial}{\partial t^2}\big)^k(f)\Big). \tag{2.15}$$

where $\dfrac{\partial}{\partial t^2}$ is the linear continuous operator defined on $S_*(\mathbb{R}^2)$ by

$$\frac{\partial}{\partial t^2}(f)(t,x) = \frac{1}{t}\frac{\partial f}{\partial t}(t,x).$$

The relation (2.15) allows us to extend the mapping $\mathscr{H}_\sigma$ on $\mathbb{R}$, by setting

$$\mathscr{H}_\sigma(f)(r,x) = (-1)^k \mathscr{H}_{\sigma+k}\Big(\big(\frac{\partial}{\partial t^2}\big)^k(f)\Big),$$

where k is any integer such that $\sigma + k > 0$, $\sigma \in \mathbb{R}$.
The extension $\mathscr{H}_\sigma$, $\sigma \in \mathbb{R}$ satisfies

$$\big(\mathscr{H}_\sigma \circ \mathscr{H}_\delta\big)(f) = \mathscr{H}_{\sigma+\delta}(f), \quad \sigma, \delta \in \mathbb{R}, \quad f \in S_*(\mathbb{R}^2),$$

and $\mathscr{H}_0(f) = f$, for all $f \in S_*(\mathbb{R}^2)$.
In particular, for all $\sigma \in \mathbb{R}$, the transform $\mathscr{H}_\sigma$ is a topological isomorphism from $S_*(\mathbb{R}^2)$ onto itself, and the isomorphism inverse is given by

$$\mathscr{H}_\sigma^{-1} = \mathscr{H}_{-\sigma}.$$

Thus, for all real number σ, we have

$$\mathscr{H}_\sigma^{-1}(f) = (-1)^{1+[\sigma]} \mathscr{H}_{1+[\sigma]-\sigma}\Big(\big(\frac{\partial}{\partial t^2}\big)^{1+[\sigma]}(f)\Big).$$

In particular

$$\mathscr{W}_\alpha^{-1}(f) = \mathscr{H}_{\alpha+\frac{1}{2}}^{-1}(f) = (-1)^{1+[\alpha+\frac{1}{2}]} \mathscr{H}_{[\alpha+\frac{1}{2}]-\alpha+\frac{1}{2}}\Big(\big(\frac{\partial}{\partial t^2}\big)^{1+[\alpha+\frac{1}{2}]}(f)\Big).$$

□

3 The Beurling-Hörmander theorem for the Riemann-Liouville operator

In this section, we shall establish the main result of this paper, that is the Beurling-Hörmander theorem for the Fourier transform $\mathscr{F}_\alpha$.
We recall firstly the following result that has been established by Bonami, Demange and Jaming [5].

Theorem 3.1. *Let* f *be a measurable function on* $\mathbb{R}\times\mathbb{R}^n$*, even with respect to the first variable such that* $f\in L^2(dm_{n+1})$*, and let* d *be a real number,* $d\geq 0$*. If*

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\Lambda_{n+1}(f)(s,y)|}{(1+|(r,x)|+|(s,y)|)^d}e^{|(r,x)||(s,y)|}\,dm_{n+1}(r,x)\,dm_{n+1}(s,y)<+\infty,$$

then there exist a positive constant a *and a polynomial* P *on* $\mathbb{R}\times\mathbb{R}^n$*, even with respect to the first variable, such that*

$$f(r,x)=P(r,x)e^{-a(r^2+|x|^2)},$$

with degree$(P)<\dfrac{d-(n+1)}{2}$.

In the following, we will establish some intermediary results that we use nextly.

Lemma 3.2. *Let* $f\in L^2(d\nu_\alpha)$ *such that*

$$\iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}}\frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}e^{|(r,x)||\theta(\mu,\lambda)|}\,d\nu_\alpha(r,x)\,d\widetilde{\gamma}_\alpha(\mu,\lambda)<+\infty, \tag{3.1}$$

then the function f *belongs to the space* $L^1(d\nu_\alpha)$.

Proof. From the hypothesis, and the relations (2.5) and (2.6), we have

$$\begin{aligned}&\iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}}\frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}e^{|(r,x)||\theta(\mu,\lambda)|}\,d\nu_\alpha(r,x)\,d\widetilde{\gamma}_\alpha(\mu,\lambda)\\&=\int_0^{+\infty}\int_{\mathbb{R}}\int_0^{+\infty}\int_{\mathbb{R}}\frac{|f(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d}e^{|(r,x)||(\mu,\lambda)|}\,d\nu_\alpha(r,x)dm_2(\mu,\lambda)<+\infty.\end{aligned}$$

We assume of course that $f\neq 0$. Then, there exists $(\mu_0,\lambda_0)\in[0,+\infty[\times\mathbb{R}$, such that $(\mu_0,\lambda_0)\neq(0,0)$, $\widetilde{\mathscr{F}}_\alpha(f)(\mu_0,\lambda_0)\neq 0$, and

$$|\widetilde{\mathscr{F}}_\alpha(f)(\mu_0,\lambda_0)|\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|\frac{e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}\,d\nu_\alpha(r,x)<+\infty,$$

hence

$$\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|\frac{e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}\,d\nu_\alpha(r,x)<+\infty.$$

Let h be the function defined on $[0,+\infty[$ by

$$h(s)=\frac{e^{s|(\mu_0,\lambda_0)|}}{(1+s+|(\mu_0,\lambda_0)|)^d},$$

then the function h admits a minimum attained at

$$s_0=\begin{cases}\dfrac{d}{|(\mu_0,\lambda_0)|}-1-|(\mu_0,\lambda_0)|, & \text{if } \dfrac{d}{|(\mu_0,\lambda_0)|}>1+|(\mu_0,\lambda_0)|;\\ 0, & \text{if } \dfrac{d}{|(\mu_0,\lambda_0)|}\leqslant 1+|(\mu_0,\lambda_0)|.\end{cases}$$

Consequently,

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|\, d\nu_\alpha(r,x) &\leqslant \frac{1}{h(s_0)}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)|e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}\, d\nu_\alpha(r,x)\\ &< +\infty.\end{aligned}$$

□

Lemma 3.3. *Let* $f \in L^2(d\nu_\alpha)$ *such that*

$$\iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d} e^{|(r,x)||\theta(\mu,\lambda)|}\, d\nu_\alpha(r,x)\, d\widetilde{\gamma}_\alpha(\mu,\lambda) < +\infty.$$

Then, there exists $a > 0$ *such that the function* $\widetilde{\mathscr{F}_\alpha}(f)$ *is analytic on the set*

$$B_a = \left\{(\mu,\lambda)\in\mathbb{C}^2 \mid \left|Im(\mu)\right| < a\ ,\ \left|Im(\lambda)\right| < a\right\}.$$

Proof. From the proof of the lemma 3.2, there exists $(\mu_0,\lambda_0) \neq (0,0)$, such that

$$\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)|e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}\, d\nu_\alpha(r,x) < +\infty.$$

Let $a > 0$, such that $0 < 2a < |(\mu_0,\lambda_0)|$. Then we have

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)|e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}\, d\nu_\alpha(r,x)\\ &=\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|e^{2a|(r,x)|}\frac{e^{|(r,x)|(|(\mu_0,\lambda_0)|-2a)}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}\, d\nu_\alpha(r,x) < +\infty.\end{aligned}$$

Let g be the function defined on $[0,+\infty[$ by

$$g(s) = \frac{e^{s(|(\mu_0,\lambda_0)|-2a)}}{(1+s+|(\mu_0,\lambda_0)|)^d},$$

then g admits a minimum attained at

$$s_0 = \begin{cases} \dfrac{d}{|(\mu_0,\lambda_0)|-2a} - 1 - |(\mu_0,\lambda_0)|, & \text{if } \dfrac{d}{|(\mu_0,\lambda_0)|-2a} > 1+|(\mu_0,\lambda_0)|;\\ 0, & \text{if } \dfrac{d}{|(\mu_0,\lambda_0)|-2a} \leqslant 1+|(\mu_0,\lambda_0)|.\end{cases}$$

Consequently,

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|e^{2a|(r,x)|}\, d\nu_\alpha(r,x)\\ &\qquad\leqslant \frac{1}{g(s_0)}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)|e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}\, d\nu_\alpha(r,x)\\ &\qquad< +\infty. \end{aligned}\tag{3.2}$$

On the other hand, from the relation (2.1) we deduce that for all $(r,x) \in [0,+\infty[\times\mathbb{R}$, the function

$$(\mu,\lambda) \longmapsto j_\alpha(r\mu)e^{-ix\lambda}$$

is analytic on $\mathbb{C}^2$ [7], even with respect to the first variable, and by the relation (2.3) we have

$$\left|j_\alpha(r\mu)e^{-i\lambda x}\right| \leqslant e^{|(r,x)|(|\mathrm{Im}(\mu)|+|\mathrm{Im}(\lambda)|)}. \tag{3.3}$$

From the relations (2.7), (3.2), and (3.3), it follows that the function $\widetilde{\mathscr{F}_\alpha}(f)$ is analytic on B_a, even with respect to the first variable. □

Corollary 3.1. Let $f \in L^2(d\nu_\alpha)$; $f \neq 0$; and let d be a real number, $d \geqslant 0$. If

$$\iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d} e^{|(r,x)||\theta(\mu,\lambda)|}\, d\nu_\alpha(r,x)\, d\widetilde{\gamma}_\alpha(\mu,\lambda) < +\infty.$$

then for all real number a; $a > 0$, we have

$$\nu_\alpha\Big(\Big\{(r,x) \in \mathbb{R}^2 \mid \widetilde{\mathscr{F}_\alpha}(f)(r,x) \neq 0 \quad \text{and} \quad |(r,x)| > a\Big\}\Big) > 0.$$

Proof. From lemma 3.2, the function f belongs to $L^1(d\nu_\alpha)$, and consequently the function $\widetilde{\mathscr{F}_\alpha}(f)$ is continuous on $\mathbb{R}^2$, even with respect to the first variable.
Then for all $a > 0$, the set

$$\Big\{(r,x) \in \mathbb{R}^2 \mid \widetilde{\mathscr{F}_\alpha}(f)(r,x) \neq 0 \quad \text{and} \quad |(r,x)| > a\Big\},$$

is on open subset of $\mathbb{R}^2$.
Assume that

$$\nu_\alpha\Big(\Big\{(r,x) \in \mathbb{R}^2 \mid \widetilde{\mathscr{F}_\alpha}(f)(r,x) \neq 0 \quad \text{and} \quad |(r,x)| > a\Big\}\Big) = 0,$$

then for all $(r,x) \in \mathbb{R}^2$; $|(r,x)| > a$, we have $\widetilde{\mathscr{F}_\alpha}(f)(r,x) = 0$.
Applying lemma 3.3 and analytic continuation, we deduce that $\widetilde{\mathscr{F}_\alpha}(f)$ vanishes on $\mathbb{R}^2$, and by theorem 2.1, it follows that $f = 0$. □

Lemma 3.4. *Let $f \in L^2(d\nu_\alpha)$ and let d be a real number $d \geqslant 0$. If*

$$\iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d} e^{|(r,x)||\theta(\mu,\lambda)|}\, d\nu_\alpha(r,x)\, d\widetilde{\gamma}_\alpha(\mu,\lambda) < +\infty,$$

then the function $\mathscr{W}_\alpha(f)$, belongs to $L^2(dm_2)$, where $\mathscr{W}_\alpha$ is the mapping defined by the relation (2.11).

Proof. From the hypothesis and the relations (2.5) and (2.6), we have

$$\begin{aligned}
&\iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d} e^{|(r,x)||\theta(\mu,\lambda)|}\, d\nu_\alpha(r,x)\, d\widetilde{\gamma}_\alpha(\mu,\lambda) \\
&= \int_0^{+\infty}\int_{\mathbb{R}}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|}\, d\nu_\alpha(r,x) dm_2(\mu,\lambda) < +\infty.
\end{aligned}$$

By the same way as inequality (3.2) of the lemma 3.3, there exists $b \in \mathbb{R}$, $b > 0$, such that

$$\int_0^{+\infty}\int_{\mathbb{R}} |\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)| e^{b|(\mu,\lambda)|} dm_2(\mu,\lambda) < +\infty. \tag{3.4}$$

Consequently, the function $\widetilde{\mathscr{F}}_\alpha(f)$ lies in $L^1(d\nu_\alpha)$ and by theorem 2.1, we get

$$f(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda) j_\alpha(r\mu) e^{i\lambda x} d\nu_\alpha(\mu,\lambda); \quad \text{a.e.}$$

In particular the function f is bounded and

$$\|f\|_{\infty,\nu_\alpha} \leqslant \|\widetilde{\mathscr{F}}_\alpha(f)\|_{1,\nu_\alpha}. \tag{3.5}$$

Now, we have

$$\begin{aligned} |\mathscr{W}_\alpha(f)(r,x)| &\leqslant \frac{1}{2^{\alpha+\frac12}\Gamma(\alpha+\frac12)} \int_r^{+\infty} (t^2-r^2)^{\alpha-\frac12} |f(t,x)| 2t dt \\ &= \frac{r^{2\alpha+1}}{2^{\alpha+\frac12}\Gamma(\alpha+\frac12)} \int_1^{+\infty} (u^2-1)^{\alpha-\frac12} |f(ru,x)| 2u du. \end{aligned}$$

Using Minkowski's inequality for integrals [11], we get

$$\begin{aligned} &\Big(\int_0^{+\infty}\int_{\mathbb{R}} |\mathscr{W}_\alpha(f)(r,x)|^2 \, dm_2(r,x)\Big)^{\frac12} \\ &\leqslant \frac{1}{2^{\alpha+\frac12}\Gamma(\alpha+\frac12)} \Big(\int_0^{+\infty}\int_{\mathbb{R}} \big(\int_1^{+\infty} r^{2\alpha+1}(u^2-1)^{\alpha-\frac12}|f(ru,x)| 2u du\big)^2 \, dm_2(r,x)\Big)^{\frac12} \\ &\leqslant \frac{1}{2^{\alpha+\frac12}\Gamma(\alpha+\frac12)} \int_1^{+\infty} (u^2-1)^{\alpha-\frac12} \Big(\int_0^{+\infty}\int_{\mathbb{R}} r^{4\alpha+2}|f(ru,x)|^2 \, dm_2(r,x)\Big)^{\frac12} 2u du \\ &= \frac{\Gamma(\alpha+1)^{\frac12}}{2^{\frac{\alpha}{2}-\frac34}\pi^{\frac14}\Gamma(\alpha+\frac12)} \Big(\int_1^{+\infty} (u^2-1)^{\alpha-\frac12} u^{-2\alpha-\frac12} du\Big) \Big(\int_0^{+\infty}\int_{\mathbb{R}} |f(t,x)|^2 t^{2\alpha+1} d\nu_\alpha(t,x)\Big)^{\frac12} \\ &= \frac{\Gamma(\alpha+1)^{\frac12}}{2^{\frac{\alpha}{2}-\frac74}\pi^{\frac14}\Gamma(\alpha+\frac12)} \Big(\int_0^{1} (1-s)^{\alpha-\frac12} s^{\frac94} ds\Big) \Big(\int_0^{+\infty}\int_{\mathbb{R}} |f(t,x)|^2 t^{2\alpha+1} d\nu_\alpha(t,x)\Big)^{\frac12} \\ &= C_\alpha \Big(\int_0^{+\infty}\int_{\mathbb{R}} |f(t,x)|^2 t^{2\alpha+1} d\nu_\alpha(t,x)\Big)^{\frac12} \end{aligned}$$

and by the relations (3.2) and (3.5), we get

$$\begin{aligned} \Big(\int_0^{+\infty}\int_{\mathbb{R}} |\mathscr{W}_\alpha(f)(r,x)|^2 \, dm_2(r,x)\Big)^{\frac12} &\leqslant M_\alpha \|f\|_{\infty,\nu_\alpha}^{\frac12} \Big(\int_0^{+\infty}\int_{\mathbb{R}} |f(t,x)| e^{2a|(t,x)|} d\nu_\alpha(t,x)\Big)^{\frac12} \\ &< +\infty. \end{aligned}$$

□

Remark 3.1. Let f be a function satisfying the hypothesis (3.1), then from the relations (3.2) and (3.4), we can prove that the function f belongs to the Schwartz's space $S_*(\mathbb{R}^2)$. Since the Weyl transform $\mathscr{W}_\alpha$ is an isomorphism from $S_*(\mathbb{R}^2)$ onto itself, then the function $\mathscr{W}_\alpha(f)$ belongs to $S_*(\mathbb{R}^2)$, in particular $\mathscr{W}_\alpha(f) \in L^2(dm_2)$.

Remark 3.2. Let σ be a positive real number such that $\sigma + \sigma^2 > d \geqslant 0$. Then, the function

$$t \longmapsto \frac{e^{\sigma t}}{(1+t+\sigma)^d},$$

is increasing on $[0, +\infty[$.

Theorem 3.5. *Let* $f \in L^2(d\nu_\alpha)$, *and let* d *be a real number,* $d \geqslant 0$. *If*

$$\iint_{\Gamma_+} \int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d} e^{|(r,x)||\theta(\mu,\lambda)|} \, d\nu_\alpha(r,x) \, d\widetilde{\gamma}_\alpha(\mu,\lambda) < +\infty,$$

then

$$\int_0^{+\infty}\int_{\mathbb{R}}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|\mathscr{W}_\alpha(f)(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|} \, dm_2(r,x)dm_2(\mu,\lambda) < +\infty.$$

Proof. From the hypothesis and the relations (2.5) and (2.6), we have

$$\int_0^{+\infty}\int_{\mathbb{R}}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|} \, d\nu_\alpha(r,x)dm_2(\mu,\lambda) < +\infty. \quad (3.6)$$

i) If $d = 0$, then by Fubini's theorem we have

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}}\int_0^{+\infty}\int_{\mathbb{R}} |\mathscr{W}_\alpha(f)(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|e^{|(r,x)||(\mu,\lambda)|} \, dm_2(r,x)dm_2(\mu,\lambda) \\
&\quad \leqslant \int_0^{+\infty}\int_{\mathbb{R}} |\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|\Big(\int_0^{+\infty}\int_{\mathbb{R}} |\mathscr{W}_\alpha(f)(r,x)|e^{|(r,x)||(\mu,\lambda)|} \, dm_2(r,x)\Big)dm_2(\mu,\lambda) \\
&\quad \leqslant \int_0^{+\infty}\int_{\mathbb{R}} |\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|\Big(\int_0^{+\infty}\int_{\mathbb{R}} \mathscr{W}_\alpha(|f|)(r,x)e^{|(r,x)||(\mu,\lambda)|} \, dm_2(r,x)\Big)dm_2(\mu,\lambda). \quad (3.7)
\end{aligned}$$

Using the relation (2.10), we deduce that

$$\int_0^{+\infty}\int_{\mathbb{R}} \mathscr{W}_\alpha(|f|)(r,x)e^{|(r,x)||(\mu,\lambda)|} \, dm_2(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|\mathfrak{R}_\alpha(e^{|(.,.)||(\mu,\lambda)|})(r,x) \, d\nu_\alpha(r,x), \quad (3.8)$$

but for all $(r,x) \in [0,+\infty[\times\mathbb{R}$

$$\mathfrak{R}_\alpha(e^{|(.,.)||(\mu,\lambda)|})(r,x) \leqslant e^{|(r,x)||(\mu,\lambda)|}. \quad (3.9)$$

Combining the relations (3.6), (3.7), (3.8), and (3.9), we get

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}}\int_0^{+\infty}\int_{\mathbb{R}} |\mathscr{W}_\alpha(f)(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|e^{|(r,x)||(\mu,\lambda)|} \, dm_2(r,x)dm_2(\mu,\lambda) \\
&\quad \leqslant \int_0^{+\infty}\int_{\mathbb{R}} |\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|\Big(\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|e^{|(r,x)||(\mu,\lambda)|} \, d\nu_\alpha(r,x)\Big)dm_2(\mu,\lambda) \\
&\quad < +\infty.
\end{aligned}$$

ii) If $d > 0$, let

$$B_d = \{(u,v) \in [0,+\infty[\times\mathbb{R} \mid |(u,v)| \leqslant d\}.$$

. By Fubini's theorem, we have

$$\begin{aligned}&\iint_{B_d^c}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)||\mathscr{W}_\alpha(f)(r,x)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|}\, dm_2(r,x)dm_2(\mu,\lambda)\\ &\leqslant \iint_{B_d^c} |\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|\Big(\int_0^{+\infty}\int_{\mathbb{R}} \mathscr{W}_\alpha(|f|)(r,x)\frac{e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}\\ &\times\ dm_2(r,x)\Big)dm_2(\mu,\lambda),\end{aligned}$$

and by the relation (2.10), we get

$$\begin{aligned}&\iint_{B_d^c}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)||\mathscr{W}_\alpha(f)(r,x)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|}\, dm_2(r,x)dm_2(\mu,\lambda)\\ &\leqslant \iint_{B_d^c} |\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|\Big(\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|\mathfrak{R}_\alpha\Big(\frac{e^{|(.,.)||(\mu,\lambda)|}}{(1+|(.,.)|+|(\mu,\lambda)|)^d}\Big)(r,x)\\ &\times\ d\nu_\alpha(r,x)\Big)dm_2(\mu,\lambda). \qquad (3.10)\end{aligned}$$

However, by the relation (2.9) and remark 3.2, we have
for all $(\mu,\lambda) \in B_d^c$

$$\mathfrak{R}_\alpha\Big(\frac{e^{|(.,.)||(\mu,\lambda)|}}{(1+|(.,.)|+|(\mu,\lambda)|)^d}\Big)(r,x) \leqslant \frac{e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}. \qquad (3.11)$$

Combining the relations (3.10) and (3.11), we obtain

$$\begin{aligned}&\iint_{B_d^c}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)||\mathscr{W}_\alpha(f)(r,x)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|}\, dm_2(r,x)dm_2(\mu,\lambda)\\ &\leqslant \iint_{B_d^c} |\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|\Big(\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|\frac{e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}\Big)\, d\nu_\alpha(r,x)\Big)dm_2(\mu,\lambda)\\ &\leqslant \int_0^{+\infty}\int_{\mathbb{R}}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)||f(r,x)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|}\, d\nu_\alpha(r,x)dm_2(\mu,\lambda) < +\infty.\end{aligned}$$

$$\begin{aligned}\bullet &\iint_{B_d}\iint_{B_d^c} \frac{|\mathscr{W}_\alpha(f)(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|}\, dm_2(r,x)dm_2(\mu,\lambda)\\ &\leqslant \iint_{B_d} |\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|\Big(\iint_{B_d^c} \mathscr{W}_\alpha(|f|)(r,x)\frac{e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}\, dm_2(r,x)\Big)dm_2(\mu,\lambda).\end{aligned}$$

But for $(\mu,\lambda)\in B_d$,

$$\begin{aligned}&\iint_{B_d^c}\mathscr{W}_\alpha(|f|)(r,x)\frac{e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}\,dm_2(r,x)\\&=\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|\mathfrak{R}_\alpha\Big(\frac{e^{|(.,.)||(\mu,\lambda)|}}{(1+|(.,.)|+|(\mu,\lambda)|)^d}\mathbf{1}_{B_d^c}\Big)(r,x)\,d\nu_\alpha(r,x)\\&\leqslant\iint_{B_d^c}|f(r,x)|\frac{e^{d|(r,x)|}}{(1+|(r,x)|+d)^d}\,d\nu_\alpha(r,x).\end{aligned}$$

Hence,

$$\begin{aligned}&\iint_{B_d}\iint_{B_d^c}\frac{|\mathscr{W}_\alpha(f)(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d}e^{|(r,x)||(\mu,\lambda)|}\,dm_2(r,x)dm_2(\mu,\lambda)\\&\leqslant\Big(\iint_{B_d}|\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|dm_2(\mu,\lambda)\Big)\Big(\iint_{B_d^c}|f(r,x)|\frac{e^{d|(r,x)|}}{(1+|(r,x)|+d)^d}\,d\nu_\alpha(r,x)\Big).\end{aligned}$$

In virtue of the relation (2.8), we have

$$\begin{aligned}&\iint_{B_d}\iint_{B_d^c}\frac{|\mathscr{W}_\alpha(f)(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d}e^{|(r,x)||(\mu,\lambda)|}\,dm_2(r,x)dm_2(\mu,\lambda)\\&\leqslant\ \|f\|_{1,\nu_\alpha}m_2(B_d)\Big(\iint_{B_d^c}|f(r,x)|\frac{e^{d|(r,x)|}}{(1+|(r,x)|+d)^d}\,d\nu_\alpha(r,x)\Big).\end{aligned}\tag{3.12}$$

On he other hand, from corollary 3.1 and the relation (3.6), there exists $(\mu_0,\lambda_0)\in[0,+\infty[\times\mathbb{R},\ |(\mu_0,\lambda_0)|>d,\ \widetilde{\mathscr{F}}_\alpha(f)(\mu_0,\lambda_0)\neq 0$, and

$$\iint_{B_d^c}|f(r,x)|\frac{e^{|(\mu_0,\lambda_0)||(r,x)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}\,d\nu_\alpha(r,x)<+\infty,\tag{3.13}$$

so, by remark 3.2,

$$\begin{aligned}&\iint_{B_d^c}|f(r,x)|\frac{e^{d|(r,x)|}}{(1+|(r,x)|+d)^d}\,d\nu_\alpha(r,x)\\&\qquad\leqslant\iint_{B_d^c}|f(r,x)|\frac{e^{|(\mu_0,\lambda_0)||(r,x)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}\,d\nu_\alpha(r,x)\\&\qquad<+\infty.\end{aligned}\tag{3.14}$$

The relations (3.12), (3.13), and (3.14) imply that

$$\iint_{B_d}\iint_{B_d^c}\frac{|\mathscr{W}_\alpha(f)(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d}e^{|(r,x)||(\mu,\lambda)|}\,dm_2(r,x)dm_2(\mu,\lambda)<+\infty.$$

Finally

$$\bullet \iint_{B_d}\iint_{B_d} \frac{|\mathscr{W}_\alpha(f)(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|} dm_2(r,x)dm_2(\mu,\lambda)$$
$$\leqslant e^{d^2}\Big(\iint_{B_d} |\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|dm_2(\mu,\lambda)\Big)\Big(\iint_{B_d} |\mathscr{W}_\alpha(f)(r,x)|\, dm_2(r,x)\Big)$$
$$\leqslant e^{d^2} m_2(B_d)\|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha}\|\mathscr{W}_\alpha(f)\|_{1,m_2},$$

and therefore by the relations (2.8) and (2.12), we deduce that

$$\iint_{B_d}\iint_{B_d} \frac{|\mathscr{W}_\alpha(f)(r,x)||\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d} e^{|(r,x)||(\mu,\lambda)|} dm_2(r,x)dm_2(\mu,\lambda)$$
$$\leqslant e^{d^2} m_2(B_d)\|f\|^2_{1,\nu_\alpha}$$
$$< +\infty,$$

and the proof of theorem 3.5 is complete. □

Theorem 3.6 (Beurling-Hörmander for $\mathscr{R}_\alpha$). *Let* $f \in L^2(d\nu_\alpha)$*, and let* d *be a real number,* $d \geqslant 0$*. If*

$$\iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d} e^{|(r,x)||\theta(\mu,\lambda)|}\, d\nu_\alpha(r,x)\, d\widetilde{\gamma}_\alpha(\mu,\lambda) < +\infty.$$

Then

i) *For* $d \leqslant 2$*,* $f = 0$*.*

ii) *For* $d > 2$*, there exist a positive constant* a *and a polynomial* P*, even with respect to the first variable, such that*

$$f(r,x) = P(r,x)e^{-a(r^2+x^2)},$$

with degree$(P) < \dfrac{d}{2} - 1$.

Proof. Let $f \in L^2(d\nu_\alpha)$, satisfying the hypothesis.
From proposition 2.2, lemma 3.2, and lemma 3.4, we deduce that the function $\mathscr{W}_\alpha(f)$ belongs to the space $L^1(dm_2) \cap L^2(dm_2)$ and that

$$\widetilde{\mathscr{F}}_\alpha(f) = \Lambda_2 \circ \mathscr{W}_\alpha(f).$$

Thus from theorem 3.5, we get

$$\int_0^{+\infty}\int_{\mathbb{R}}\int_0^{+\infty}\int_{\mathbb{R}} \frac{\big|\mathscr{W}_\alpha(f)(r,x)\big|\big|\Lambda_2\big(\mathscr{W}_\alpha(f)\big)(\mu,\lambda)\big|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}\, dm_2(r,x)dm_2(\mu,\lambda) < +\infty.$$

Applying theorem 3.1, when f is replaced by $\mathscr{W}_\alpha(f)$, we deduce that

If $d \leqslant 2$, $\mathscr{W}_\alpha(f) = 0$, and by remark 2.1, $f = 0$.

If $d > 2$, then there exist $a > 0$ and a polynomial Q even with respect to the first variable such that

$$\mathscr{W}_\alpha(f)(r,x) = Q(r,x)e^{-a(r^2+x^2)} = \sum_{2p+q\leqslant m} a_{p,q} r^{2p}x^q e^{-a(r^2+x^2)}.$$

In particular, the function $\mathscr{W}_\alpha(f)$ belongs to the space $S_*(\mathbb{R}^2)$. From remark 2.1, the function f belongs to $S_*(\mathbb{R}^2)$ and from the relation (2.14), we get

$$\begin{aligned} f(r,x) &= \mathscr{H}_{-\alpha-\frac{1}{2}}\Big(Q(t,y)e^{-a(t^2+y^2)}\Big)(r,x) \\ &= (-1)^{[\alpha+\frac{1}{2}]+1}\mathscr{H}_{[\alpha+\frac{1}{2}]-\alpha+\frac{1}{2}}\Big(\big(\frac{\partial}{\partial t^2}\big)^{[\alpha+\frac{1}{2}]+1}\big(P(t,y)e^{-a(t^2+y^2)}\big)\Big)(r,x) \\ &= \sum_{2p+q\leqslant m} a_{p,q}(-1)^{[\alpha+\frac{1}{2}]+1}\mathscr{H}_{[\alpha+\frac{1}{2}]-\alpha+\frac{1}{2}}\Big(\big(\frac{\partial}{\partial t^2}\big)^{[\alpha+\frac{1}{2}]+1}\big(t^{2p}y^q e^{-a(t^2+y^2)}\big)\Big)(r,x). \end{aligned} \tag{3.15}$$

However, for all $k \in \mathbb{N}$,

$$\big(\frac{\partial}{\partial t^2}\big)^k\Big(t^{2p}y^q e^{-a(t^2+y^2)}\Big) = \Big(\sum_{j=0}^{\min(p,k)} C_k^j \frac{2^j p!}{(p-j)!}(-2a)^{k-j}t^{2(p-j)}\Big)y^q e^{-a(t^2+y^2)}, \tag{3.16}$$

and for all $\sigma \in \mathbb{R}$, $\sigma > 0$,

$$\mathscr{H}_\sigma\big(t^{2p}y^q e^{-a(t^2+y^2)}\big)(r,x) = \frac{1}{2^\sigma\Gamma(\sigma)}\Big(\sum_{j=0}^{p} C_p^j \frac{\Gamma(\sigma+p-j)}{a^{\sigma+p-j}} r^{2j}\Big)x^q e^{-a(r^2+x^2)}. \tag{3.17}$$

Combining the relations (3.15), (3.16) and (3.17), we deduce that

$$f(r,x) = P(r,x)e^{-a(r^2+x^2)}.$$

Where P is a polynomial, even with respect to the first variable and degree(P) = degree(Q). $\square$

4 Applications of Beurling-Hörmander theorem

In this section, we shall deduce from the precedent Beurling-Hörmander theorem two most important uncertainty principles for the Fourier transform $\mathscr{F}_\alpha$, that are the Gelfand-Shilov and the Cowling-Price theorems.

Lemma 4.1. *Let P be a polynomial on $\mathbb{R}^2$, $P \neq 0$, with degree$(P) = m$. Then there exist two positive constants A and C such that*

$$\forall t \geqslant A, \quad p(t) = \int_0^{2\pi} \big|P(t\cos(\theta), t\sin(\theta)\big|d\theta \geqslant Ct^m.$$

Proof. Let P be a polynomial on $\mathbb{R}^2$, $P \neq 0$ and with degree$(P) = m$. We have

$$p(t) = \int_0^{2\pi} \Big| \sum_{j=0}^{m} a_j(\theta) t^j \Big| d\theta,$$

where the functions a_j, $0 \leqslant j \leqslant m$, are continuous on $[0, 2\pi]$. It's clear that the function p is continuous on $[0, +\infty[$, and by dominate convergence theorem's, we have

$$p(t) \sim C_m t^m \quad (t \longrightarrow +\infty), \tag{4.1}$$

where $C_m = \displaystyle\int_0^{2\pi} |a_m(\theta)| d\theta > 0$.
Now the relation (4.1) involves that there exists $A > 0$ such that

$$\forall t \geqslant A, \; p(t) \geqslant \frac{C_m}{2} t^m.$$

□

Theorem 4.2 (Gelfand-Shilov for $\mathscr{R}_\alpha$). *Let p, q be two conjugate exponents, $p, q \in]1, +\infty[$. Let ξ, η be non negative real numbers such that $\xi\eta \geqslant 1$. Let f be a measurable function on $\mathbb{R}^2$, even with respect to the first variable, such that $f \in L^2(d\nu_\alpha)$.*
If

$$\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)| e^{\frac{\xi^p |(r,x)|^p}{p}}}{(1 + |(r,x)|)^d} d\nu_\alpha(r,x) < +\infty,$$

and

$$\iint_{\Gamma_+} \frac{|\mathscr{F}_\alpha(f)(\mu,\lambda)| e^{\frac{\eta^q |\theta(\mu,\lambda)|^q}{q}}}{(1 + |\theta(\mu,\lambda)|)^d} d\widetilde{\gamma}_\alpha(\mu,\lambda) < +\infty \; ; \; d \geqslant 0.$$

Then
i) *For $d \leqslant 1$, $f = 0$.*
ii) *For $d > 1$, we have*
   a) $f = 0$ *for* $\xi\eta > 1$.
   b) $f = 0$ *for* $\xi\eta = 1$, *and* $p \neq 2$.
   c) $f(r,x) = P(r,x) e^{-a(r^2+x^2)}$ *for* $\xi\eta = 1$ *and* $p = q = 2$,
where $a > 0$ and P is a polynomial on $\mathbb{R}^2$ even with respect to the first variable, with degree$(P) < d - 1$.

Proof. Let f be a function satisfying the hypothesis. Since $\xi\eta \geqslant 1$, and by a convexity argument, we have

$$\begin{aligned}
&\iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^{2d}} e^{|(r,x)||\theta(\mu,\lambda)|}\, d\nu_\alpha(r,x)\, d\widetilde{\gamma}_\alpha(\mu,\lambda)\\
&\leqslant \iint_{\Gamma_+}\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|(r,x)|)^d(1+|\theta(\mu,\lambda)|)^d} e^{\xi\eta|(r,x)||\theta(\mu,\lambda)|}\, d\nu_\alpha(r,x)\, d\widetilde{\gamma}_\alpha(\mu,\lambda)\\
&\leqslant \Big(\iint_{\Gamma_+} \frac{|\mathscr{F}_\alpha(f)(\mu,\lambda)|}{(1+|\theta(\mu,\lambda)|)^d} e^{\frac{\eta^q|\theta(\mu,\lambda)|^q}{q}}\, d\widetilde{\gamma}_\alpha(\mu,\lambda)\Big)\\
&\times\Big(\int_0^{+\infty}\int_{\mathbb{R}} \frac{|f(r,x)|}{(1+|(r,x)|)^d} e^{\frac{\xi^p|(r,x)|^p}{p}}\, d\nu_\alpha(r,x)\Big)\\
&< +\infty.
\end{aligned} \tag{4.2}$$

Then from the Beurling-Hörmander theorem, we deduce that
i) For $d \leqslant 1$, $f = 0$.
ii) For $d > 1$, there exist a positive constant a, and a polynomial P on $\mathbb{R}^2$, even with respect to the first variable such that

$$f(r,x) = P(r,x)e^{-a(r^2+x^2)}, \tag{4.3}$$

with degree$(P) < d-1$, and by a standard calculus, we obtain

$$\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda) = Q(\mu,\lambda)e^{-\frac{1}{4a}(\mu^2+\lambda^2)}, \tag{4.4}$$

where Q is a polynomial on $\mathbb{R}^2$, even with respect to the first variable, with degree$(P) =$ degree(Q). On the other hand, from the relations (2.5), (2.6), (4.2), (4.3) and (4.4), we get

$$\begin{aligned}
\int_0^{+\infty}\int_{\mathbb{R}}\int_0^{+\infty}\int_{\mathbb{R}} &\frac{|P(r,x)||Q(\mu,\lambda)|}{(1+|(r,x)|)^d(1+|(\mu,\lambda)|)^d} e^{\xi\eta|(r,x)||(\mu,\lambda)|-a(r^2+x^2)}\\
&\times e^{-\frac{1}{4a}(\mu^2+\lambda^2)}\, d\nu_\alpha(r,x)d\mu d\lambda < +\infty,
\end{aligned}$$

so

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\varphi(t)}{(1+t)^d}\frac{\psi(\rho)}{(1+\rho)^d} e^{\xi\eta t\rho}e^{-at^2}e^{-\frac{1}{4a}\rho^2}t^{2\alpha+2}\rho dt d\rho < +\infty, \tag{4.5}$$

where

$$\varphi(t) = \int_0^{2\pi}\big|P(t\cos(\theta), t\sin(\theta))\big|\big|\cos(\theta)\big|^{2\alpha+1}d\theta,$$

and

$$\psi(\rho) = \int_0^{2\pi}\big|Q(\rho\cos(\theta), \rho\sin(\theta))\big|d\theta.$$

. Suppose that $\xi\eta > 1$. If $f \neq 0$, then each of the polynomials P and Q is not identically zero, let $m =$ degree$(P) =$ degree(Q).
From lemma 4.1, there exist two positive constants A and C such that

$$\forall t \geqslant A, \quad \varphi(t) \geqslant Ct^m,$$

and

$$\forall \rho \geqslant A, \quad \psi(\rho) \geqslant C\rho^m.$$

Then, the inequality (4.5) leads to

$$\int_A^{+\infty}\int_A^{+\infty} \frac{e^{\xi\eta t\rho}}{(1+t)^d(1+\rho)^d} e^{-at^2} e^{-\frac{1}{4a}\rho^2} dt d\rho < +\infty. \tag{4.6}$$

Let $\varepsilon > 0$, such that $\xi\eta - \varepsilon = \sigma > 1$. The relation (4.6) implies that

$$\int_A^{+\infty}\int_A^{+\infty} \frac{e^{\varepsilon t\rho}}{(1+t)^d(1+\rho)^d} e^{\sigma t\rho} e^{-at^2} e^{-\frac{1}{4a}\rho^2} dt d\rho < +\infty. \tag{4.7}$$

However, for all $t \geqslant A \geqslant \dfrac{d}{\varepsilon}$ and $\rho \geqslant A$, we have

$$\frac{e^{\varepsilon\rho t}}{(1+t)^d(1+\rho)^d} \geqslant \frac{e^{\varepsilon A^2}}{(1+A)^{2d}},$$

and by the relation (4.7) it follows that

$$\int_A^{+\infty}\int_A^{+\infty} e^{\sigma t\rho} e^{-at^2} e^{-\frac{1}{4a}\rho^2} dt d\rho < +\infty. \tag{4.8}$$

Let $F(t) = \displaystyle\int_A^{+\infty} e^{\sigma\rho t - \frac{1}{4a}\rho^2} d\rho$, then F can be written

$$F(t) = e^{a\sigma^2 t^2}\Big(\int_A^{+\infty} e^{-\frac{1}{4a}\rho^2} d\rho + 2a\sigma e^{-\frac{A^2}{4a}} \int_0^t e^{A\sigma s - a\sigma^2 s^2} ds\Big),$$

in particular

$$F(t) \geqslant e^{a\sigma^2 t^2} \int_A^{+\infty} e^{-\frac{1}{4a}\rho^2} d\rho.$$

Thus

$$\begin{aligned}\int_A^{+\infty}\int_A^{+\infty} e^{\sigma t\rho} e^{-at^2} e^{-\frac{1}{4a}\rho^2} dt d\rho &= \int_A^{+\infty} e^{-at^2} F(t) dt \\ &\geqslant \int_A^{+\infty} e^{-\frac{1}{4a}\rho^2} d\rho \int_A^{+\infty} e^{a(\sigma^2-1)t^2} dt = +\infty,\end{aligned}$$

because $\sigma > 1$. This contradics the relation (4.8) and shows that $f = 0$.

. Suppose that $\xi\eta = 1$ and $p \neq 2$. In this case we have $p > 2$ or $q > 2$. Suppose that $q > 2$, then from the second hypothesis and the relation (4.4), we have

$$\int_0^{+\infty} \frac{\psi(\rho) e^{-\frac{\rho^2}{4a}} e^{\frac{\eta^q\rho^q}{q}}}{(1+\rho)^d} \rho d\rho < +\infty. \tag{4.9}$$

If $f \neq 0$, then the polynomial Q is not identically zero, and by lemma 4.1 and the relation (4.9), it follows that

$$\int_0^{+\infty} \frac{e^{-\frac{\rho^2}{4a}} e^{\frac{\eta^q\rho^q}{q}}}{(1+\rho)^d} d\rho < +\infty,$$

which is impossible because $q > 2$.
The proof of theorem 4.2 is complete.

□

Theorem 4.3 (Cowling-Price for $\mathscr{R}_\alpha$). *Let $\xi, \eta, \omega_1, \omega_2$ be non negative real numbers such that $\xi\eta \geqslant \frac{1}{4}$. Let p, q be two exponents, $p, q \in [1, +\infty]$, and let f be a measurable function on $\mathbb{R}^2$, even with respect to the first variable such that $f \in L^2(d\nu_\alpha)$.*
If

$$\Big\| \frac{e^{\xi|(.,.)|^2}}{(1 + |(.,.)|)^{\omega_1}} f \Big\|_{p,\nu_\alpha} < +\infty, \tag{4.10}$$

and

$$\Big\| \frac{e^{\eta|\theta(.,.)|^2}}{(1 + |\theta(.,.)|)^{\omega_2}} \mathscr{F}_\alpha(f) \Big\|_{q,\widetilde{\gamma}_\alpha} < +\infty, \tag{4.11}$$

then
i) *For $\xi\eta > \frac{1}{4}$, $f = 0$.*
ii) *For $\xi\eta = \frac{1}{4}$, there exist a positive constant a and a polynomial P on $\mathbb{R}^2$, even with respect to the first variable, such that*

$$f(r, x) = P(r, x)e^{-a(r^2+x^2)}.$$

Proof. Let p' and q' be the conjugate exponents of p respectively q. Let us pick $d_1, d_2 \in \mathbb{R}$, such that $d_1 > 2\alpha + 3$ and $d_2 > 2$. Finally, let d be a positive real number such that $d > \max(\omega_1 + \frac{d_1}{p'}, \omega_2 + \frac{d_2}{q'}, 1)$.
From Hölder's inequality and the relations (4.10) and (4.11), we deduce that

$$\int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r, x)|e^{\xi|(r,x)|^2}}{(1 + |(r, x)|)^{\omega_1 + \frac{d_1}{p'}}} d\nu_\alpha(r, x) < +\infty,$$

and

$$\iint_{\Gamma_+} \frac{|\mathscr{F}_\alpha(f)(\mu, \lambda)|e^{\eta|\theta(\mu,\lambda)|^2}}{(1 + |\theta(\mu, \lambda)|)^{\omega_2 + \frac{d_2}{q'}}} d\widetilde{\gamma}_\alpha(\mu, \lambda) < +\infty.$$

Consequently we have

$$\int_0^{+\infty} \int_{\mathbb{R}} \frac{|f(r, x)|e^{\xi|(r,x)|^2}}{(1 + |(r, x)|)^{d}} d\nu_\alpha(r, x) < +\infty,$$

and

$$\iint_{\Gamma_+} \frac{|\mathscr{F}_\alpha(f)(\mu, \lambda)|e^{\eta|\theta(\mu,\lambda)|^2}}{(1 + |\theta(\mu, \lambda)|)^{d}} d\widetilde{\gamma}_\alpha(\mu, \lambda) < +\infty.$$

Then, the desired result follows from theorem 4.2. □

Remark 4.1. The Hardy's theorem is a special case of theorem 4.3 when $p = q = +\infty$.

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