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Some Uniqueness Results On Meromorphic Functions Sharing Three Sets II

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ABSTRACT

With the help of the notion of weighted sharing we investigate the uniqueness of meromorphic functions concerning three set sharing and significantly improve two results of Zhang [16] and as a corollary of the main result we improve a result of the present author [2] as well.

RESUMEN

Con la ayuda del concepto de peso repartido, investigamos la unicidad de funciones meromorfas sobre un conjunto compartido y mejoramos significativamente dos resultados de Zhang [16] y como corolario del resultado principal que mejoramos también el resultado de la autora [2].

Mathematics Subject Classification: 30D35.

Keywords. Meromorphic functions, uniqueness, weighted sharing, shared set.

[1]The author dedicates the paper to the memory of his respected teacher Late Prof. B. K. Lahiri who first germinated the inquisition for research work in the author's mind.

1 Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We shall use the standard notations of value distribution theory :

$$T(r,f),\ \ m(r,f),\ \ N(r,\infty;f),\ \ \overline{N}(r,\infty;f),\ldots$$

(see [5]). For any constant a, we define

$$\Theta(a;f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

In [4] Gross posed the following question:

Can one find two finite sets S_j $(j = 1,2)$ *such that any two non-constant entire functions* f *and* g *satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1,2$ *must be identical ?*

In the last couple of years or so several attempts have been made in many papers to answer the above question under weaker hypothesis (see [1], [2], [3], [9], [10], [13], [15], [16])).

A recent increment to uniqueness theory has been to considering weighted sharing instead of sharing IM/CM which implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2001 in [7, 8] and since then this notion played a vital role on the uniqueness of meromorphic or entire functions sharing sets concerning the question of Gross. Below we are giving the definition.

Definition 1.1. *[7, 8] Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C}\cup\{\infty\}$ *we denote by* $E_k(a;f)$ *the set of all* a*-points of* f*, where an* a*-point of multiplicity* m *is counted* m *times if* $m \le k$ *and* $k+1$ *times if* $m > k$*. If* $E_k(a;f) = E_k(a;g)$*, we say that* f,g *share the value* a *with weight* k*.*

We write f, g share (a,k) to mean that f,g share the value a with weight k. Clearly if f,g share (a,k) then f,g share (a,p) for any integer p, $0 \le p < k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively.

Definition 1.2. *[7] Let* S *be a set of distinct elements of* $\mathbb{C}\cup\{\infty\}$ *and* k *be a nonnegative integer or* ∞*. Let* $E_f(S,k) = \bigcup_{a\in S} E_k(a;f)$*.*

Clearly $E_f(S) = E_f(S,\infty)$ *and* $\overline{E}_f(S) = E_f(S,0)$.

Improving the result of Lahiri-Banerjee [10] and Yi-Lin [15] the present author have recently proved the following result.

Theorem A. *[1] Let* $S_1 = \{z : z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ *and* $S_3 = \{\infty\}$, *where* a, b *are nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no repeated root and* $n\ (\geq 4)$ *is an integer. If for two non-constant meromorphic functions* f *and* g $E_f(S_1,4) = E_g(S_1,4)$, $E_f(S_2,0) = E_g(S_2,0)$ *and* $E_f(S_3,\infty) = E_g(S_3,\infty)$ *and* $\Theta(\infty;f) + \Theta(\infty;g) > 0$ *then* $f \equiv g$.

In [2] the present author further improved *Theorem A* as follows.

Theorem B. *[2] Let* $S_i, i = 1,2,3$ *be defined as in Theorem A. If for two non-constant meromorphic functions* f *and* g $E_f(S_1,4) = E_g(S_1,4)$, $E_f(S_2,0) = E_g(S_2,0)$ *and* $E_f(S_3,6) = E_g(S_3,6)$ *and* $\Theta(\infty;f) + \Theta(\infty;g) > 0$ *then* $f \equiv g$.

Now it is quite natural to ask the following question.

i) *What happens in Theorem B if no conditions over the ramification indexes of* f *and* g *are imposed ?*

In the direction of the above question some investigations have already been carried out by Zhang [16] in the following theorems.

Theorem C. *Let* $S_1 = \{z : z^n(z+a) - b = 0\}$, $S_2 = \{0\}$ *and* $S_3 = \{\infty\}$, *where* a, b *are nonzero constants such that* $z^n(z+a) - b = 0$ *has no repeated root and* $n\ (\geq 3)$ *is an integer. If for two nonconstant meromorphic functions* f *and* g $E_f(S_1,\infty) = E_g(S_1,\infty)$, $E_f(S_2,0) = E_g(S_2,0)$ *and* $E_f(S_3,\infty) = E_g(S_3,\infty)$ *then* $f \equiv g$ *or* $f = \frac{-ae^{\gamma}(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1}$, $g = \frac{-a(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1}$, *where* γ *is a non-constant entire function.*

Theorem D. *Let* $S_i, i = 1,2,3$ *be defined as in Theorem C and* $n\ (\geq 4)$ *is an integer. If for two non-constant meromorphic functions* f *and* g $E_f(S_1,\infty) = E_g(S_1,\infty)$, $E_f(S_2,0) = E_g(S_2,0)$ *and* $E_f(S_3,0) = E_g(S_3,0)$ *then* $f \equiv g$ *or* $f = \frac{-ae^{\gamma}(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1}$, $g = \frac{-a(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1}$, *where* γ *is a non-constant entire function.*

The following example shows that in *Theorems A-C* $a \neq 0$ is necessary.

Example 1.1. *Let* $f(z) = e^z$ *and* $g(z) = e^{-z}$ *and* $S_1 = \{z : z^4 - 1 = 0\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$. *Since* $f - \omega^l = g - \omega^{4-l}$, *where* $\omega = \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}$, $0 \leq l \leq 3$, *clearly* $E_f(S_i,\infty) = E_g(S_i,\infty)$ *for* $i = 1,2,3$ *but* f *and* g *do not satisfy the conclusions of Theorems A-B.*

Regarding *Theorems A-C* following example establishes the fact that the set S_1 can not be replaced by any arbitrary set containing three distinct elements. However it still remains open for investigations whether the degree of the equation defining S_1 in *Theorem A-C* can be reduced to three or less.

Example 1.2. *Let* $f(z) = \sqrt{ab}\, e^{\sqrt{ab}z}$ *and* $g(z) = \sqrt{ab}\, e^{-\sqrt{ab}z}$ *and* $S_1 = \{a, b, \sqrt{ab}\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$, *where* a *and* b *are nonzero complex numbers. Clearly* $E_f(S_i, \infty) = E_g(S_i, \infty)$ *for* $i = 1, 2, 3$ *but* f *and* g *do not satisfy the conclusions of Theorems A-C.*

In the paper we also concentrate our attention to the above problem as investigated by Zhang [16] and provide a better solution in this direction. We now state the following two theorems which are the main results of the paper.

Theorem 1.1. *Let* $S_1 = \{z : z^n(z + a) - b = 0\}$, $S_2 = \{0\}$ *and* $S_3 = \{\infty\}$, *where* a, b *are nonzero constants such that* $z^n(z + a) - b = 0$ *has no repeated root and* n (≥ 3) *is an integer. If for two non-constant meromorphic functions* f *and* g $E_f(S_1, 3) = E_g(S_1, 3)$, $E_f(S_2, 0) = E_g(S_2, 0)$ *and* $E_f(S_3, 6) = E_g(S_3, 6)$ *then* $f \equiv g$ *or* $f = \frac{-ae^{\gamma}(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1}$, $\quad g = \frac{-a(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1}$, *where* γ *is a non-constant entire function.*

Corollary 1.1. *Let* S_1, S_2 *and* S_3 *be defined as in Theorem 1.1 and* $n(\geq 3)$ *be an integer. If for two non-constant meromorphic functions* f *and* g $E_f(S_1, 3) = E_g(S_1, 3)$, $E_f(S_2, 0) = E_g(S_2, 0)$ *and* $E_f(S_3, 6) = E_g(S_3, 6)$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 0$ *then* $f \equiv g$

Theorem 1.2. *Let* S_1, S_2 *and* S_3 *be defined as in Theorem 1.1 and* $n(\geq 4)$ *be an integer. If for two non-constant meromorphic functions* f *and* g $E_f(S_1, 3) = E_g(S_1, 3)$, $E_f(S_2, 0) = E_g(S_2, 0)$ *and* $E_f(S_3, 0) = E_g(S_3, 0)$ *then the conclusion of Theorem 1.1 holds .*

Remark 1. *Theorem 1.1*, *Corollary 1.1* and *Theorem 1.2* are respectively the improvements of *Theorems C, B and D* respectively.

We now explain some notations which are used in the paper.

Definition 1.3. *[6] After* $a \in \mathbb{C} \cup \{\infty\}$, *we denote by* $N(r, a; f \mid= 1)$ *the counting function of simple* a *points of* f. *For a positive integer* m *we denote by* $N(r, a; f \mid\leq m)$ $(N(r, a; f \mid\geq m))$ *the counting function of those* a *points of* f *whose multiplicities are not greater(less) than* m *where each* a *point is counted according to its multiplicity.*

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ *are defined similarly, where in counting the* a*-points of* f *we ignore the multiplicities.*

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ *are defined analogously.*

Definition 1.4. *[2] We denote by* $\overline{N}(r, a; f \mid= k)$ *the reduced counting function of those* a*-points of* f *whose multiplicities is exactly* k, *where* $k \geq 2$ *is an integer.*

Definition 1.5. *[2] Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share* (a, k) *where* $a \in \mathbb{C} \cup \{\infty\}$. *Let* z_0 *be a* a*-point of* f *with multiplicity* p, *a* a*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p > q$, *by* $\overline{N}_E^{(k+1}(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p = q \geq k+1$; *each point in these counting functions is counted only once. In the same way we can define* $\overline{N}_L(r, a; g)$ *and* $\overline{N}_E^{(k+1}(r, a; g)$.

Definition 1.6. *[8] We denote by* $N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2)$

Definition 1.7. *[7, 8] Let* f, g *share a value* a *IM. We denote by* $\overline{N}_*(r,a;f,g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ *and* $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

Definition 1.8. *[11] Let* $a, b \in \mathbb{C}\cup\{\infty\}$*. We denote by* $N(r,a;f \mid\; g=b)$ *the counting function of those* a*-points of* f*, counted according to multiplicity, which are* b*-points of* g.

Definition 1.9. *[11] Let* $a, b_1, b_2, \ldots, b_q \in \mathbb{C}\cup\{\infty\}$*. We denote by* $N(r,a;f \mid\; g \neq b_1, b_2, \ldots, b_q)$ *the counting function of those* a*-points of* f*, counted according to multiplicity, which are not the* b_i*-points of* g *for* $i = 1, 2, \ldots, q$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined as follows.

$$F = \frac{f^n(f+a)}{b}, \quad G = \frac{g^n(g+a)}{b}. \tag{2.1}$$

Henceforth we shall denote by H, Φ and V the following three functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1. *Let* F, G *share* $(1,1)$ *and* $H \not\equiv 0$*. Then*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Proof. The lemma can be proved in the line of proof of Lemma 1 [8]. □

Lemma 2.2. *Let* S_1, S_2 *and* S_3 *be defined as in Theorem 1.1 and* F, G *be given by (2.1). If for two non-constant meromorphic functions* f *and* g $E_f(S_1,0) = E_g(S_1,0)$, $E_f(S_2,0) = E_g(S_2,0)$, $E_f(S_3,0) = E_g(S_3,0)$ *and* $H \not\equiv 0$ *then*

$$\begin{aligned} N(r,H) \leq\;& \overline{N}_*(r,0,f,g) + \overline{N}(r,0;f+a \mid\geq 2) + \overline{N}(r,0;g+a \mid\geq 2) + \overline{N}_*(r,1;F,G) \\ &+ \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned}$$

where $\overline{N}_0(r, 0; F^{'})$ *is the reduced counting function of those zeros of* $F^{'}$ *which are not the zeros of* $F(F-1)$ *and* $\overline{N}_0(r, 0; G^{'})$ *is similarly defined.*

Proof. The lemma can be proved in the line of proof of Lemma 2.2 [2]. □

Lemma 2.3. *[12] Let* f *be a nonconstant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f *with constant coefficients* $\{a_k\}$ *and* $\{b_j\}$ *where* $a_n \neq 0$ *and* $b_m \neq 0$. *Then*

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.4. *Let* F *and* G *be given by (2.1),* $n \geq 3$ *an integer and* $F \not\equiv G$. *If* F, G *share* $(1, m)$, f, g *share* $(0, p)$, (∞, k), *where* $0 \leq p < \infty$ *then*

$$[np + n - 1]\, \overline{N}(r, 0; f \mid\geq p + 1) \quad \leq \quad \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + S(r, f) + S(r, g).$$

Proof. Suppose 0 is an e.v.P. (Picard exceptional value) of f and g then the lemma follows immediately.

Next suppose 0 is not an e.v.P. of f and g. If $\Phi \equiv 0$, then by integration we obtain

$$F - 1 \equiv C(G - 1).$$

It is clear that if z_0 is a zero of f then it is a zero of g. So it follows that $F(z_0) = G(z_0) = 0$. So $C = 1$ which contradicts $F \not\equiv G$. So $\Phi \not\equiv 0$. Since f, g share $(0, p)$ it follows that a common zero of f and g of order $r \leq p$ is a zero of Φ of order exactly $nr - 1$ where as a common zero of f and g of order $r > p$ is a zero of Φ of order at least $np + n - 1$. Let z_0 is a zero of f with multiplicity q and a zero of g with multiplicity t. From (2.1) we know that z_0 is a zero of F with multiplicity nq and a zero of G with multiplicity nt. So from the definition of Φ it is clear that

$$\begin{aligned} & [np + n - 1]\overline{N}(r, 0; f \mid\geq p + 1) \\ = \;& [np + n - 1]\overline{N}(r, 0; g \mid\geq p + 1) \\ = \;& [np + n - 1]\overline{N}\,(r, 0; F \mid\geq n(p + 1)) \\ \leq \;& N(r, 0; \Phi) \\ \leq \;& N(r, \infty; \Phi) + S(r, f) + S(r, g) \\ \leq \;& \overline{N}_*(r, \infty; F, G) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \end{aligned}$$

The lemma follows from above. □

Lemma 2.5. *Let* F, G *be given by (2.1),* F, G *share* $(1,m)$, $0 \le m < \infty$ *and* $\omega_1, \omega_2, \ldots, \omega_n$ *are the distinct roots of the equation* $z^n + az^{n-1} + b = 0$ *and* $n \ge 3$. *Then*

$$\overline{N}_*(r,1;F,G) \le \frac{1}{m}\left[\overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\otimes}(r,0;f^{'})\right] + S(r,f),$$

where $N_{\otimes}(r,0;f^{'}) = N(r,0;f^{'} \mid f \ne 0, \omega_1, \omega_2, \ldots, \omega_n)$

Proof. We omit the proof since it can be proved in the line of proof of Lemma 2.15 [2]. □

Lemma 2.6. *Let* F *and* G *be given by (2.1),* $n \ge 3$ *an integer and* $F \not\equiv G$. *If* F, G *share* $(1,m)$, f, g *share* $(0,0)$, (∞,k) *then*

$$\begin{aligned}\overline{N}(r,0;f) \quad &\le \quad \frac{m}{mn-m-1}\overline{N}(r,\infty;f \mid \ge k+1) + \frac{1}{mn-m-1}\overline{N}(r,\infty;f) + S(r,f) \\ &\quad +S(r,g).\end{aligned}$$

Proof. Since using *Lemma 2.5* in *Lemma 2.4* we get for $p = 0$ that

$$\begin{aligned}(n-1)\overline{N}(r,0;f) \quad &\le \quad \overline{N}(r,\infty;f \mid \ge k+1) + \frac{1}{m}[\overline{N}(r,0;f) + \overline{N}(r,\infty;f)] \\ &\quad +S(r,f) + S(r,g),\end{aligned}$$

the lemma follows. □

Lemma 2.7. *Let* F, G *be given by (2.1),* $n \ge 3$ *an integer and* $F \not\equiv G$. *If* f, g *share* $(0,0)$, (∞,k), *where* $0 \le k < \infty$, *and* F, G *share* $(1,m)$ *then the poles of* F *and* G *are the zeros of* V *and*

$$\begin{aligned}\text{(i)} \qquad & n\overline{N}(r,\infty;f \mid = 1) + (2n+1)\overline{N}(r,\infty;f \mid = 2) + \ldots + [(n+1)k-1]\overline{N}(r,\infty;f \mid = k) \\ & +[(n+1)k+n]\overline{N}(r,\infty;f \mid \ge k+1) \le \frac{1}{n-1}\overline{N}(r,\infty;f \mid \ge k+1) + \overline{N}(r,0;f+a) \\ & +\overline{N}(r,0;g+a) + \frac{n}{n-1}\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).\end{aligned}$$

$$\begin{aligned}\text{(ii)} \qquad \overline{N}(r,\infty;f \mid \ge k+1) \quad &\le \quad \frac{n-1}{(n-1)[(n+1)k+n]-1}[\overline{N}(r,0;f+a) + \overline{N}(r,0;g+a)] \\ &\quad + \frac{n}{(n-1)[(n+1)k+n]-1}\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).\end{aligned}$$

Proof. Suppose ∞ is an e.v.P. of f and g then the lemma follows immediately.

Next suppose ∞ is not an e.v.P. of f and g. If $V \equiv 0$, then by integration we obtain $1 - \frac{1}{F} \equiv A\left(1 - \frac{1}{G}\right)$. If z_0 is a pole of f then it is a pole of g. Hence from the definition of F and G we have $\frac{1}{F(z_0)} = 0$ and $\frac{1}{G(z_0)} = 0$. So $A = 1$ which contradicts $F \not\equiv G$. So $V \not\equiv 0$. Since f, g share (∞,k), we note that F and G have no pole of multiplicity q where $(n+1)k < q < (n+1)(k+1)$ and so it follows that F, G share $(\infty, (n+1)k+n)$. So using *Lemma 2.3* and *Lemma 2.4* for $p=0$ we get from the definition of V

$$\begin{aligned}
& n\overline{N}(r,\infty;f\mid=1)+(2n+1)\overline{N}(r,\infty;f\mid=2)+\ldots+[(n+1)k-1]\overline{N}(r,\infty;f\mid=k) \quad (2.2)\\
& +[(n+1)k+n]\overline{N}(r,\infty;f\mid\geq k+1)\\
\leq\ & N(r,0;V)\\
\leq\ & N(r,\infty;V)+S(r,f)+S(r,g)\\
\leq\ & \overline{N}_*(r,0;f,g)+\overline{N}(r,0;f+a)+\overline{N}(r,0;g+a)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\leq\ & \frac{1}{n-1}\overline{N}(r,\infty;f\mid\geq k+1)+\overline{N}(r,0;f+a)+\overline{N}(r,0;g+a)+\frac{n}{n-1}\overline{N}_*(r,1;F,G)\\
& +S(r,f)+S(r,g),
\end{aligned}$$

from which (i) follows. Again from (2.2) we note that

$$\begin{aligned}
& \frac{(n-1)[(n+1)k+n]-1}{n-1}\ \overline{N}(r,\infty;f\mid\geq k+1)\\
\leq\ & \overline{N}(r,0;f+a)+\overline{N}(r,0;g+a)+\frac{n}{n-1}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g),
\end{aligned}$$

from which (ii) follows. □

Lemma 2.8. *([2], Lemma 2.9) Let* F, G *be given by (2.1) and they share* $(1, m)$. *If* f, g *share* $(0, p)$, (∞, k) *where* $2 \leq m < \infty$ *and* $H \not\equiv 0$. *Then*

$$\begin{aligned}
T(r,F)\ \leq\ & \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,0;f,g)+N_2(r,0;f+a)+N_2(r,0;g+a)\\
& +\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)-m(r,1;G)-\overline{N}(r,1;F\mid=3)\\
& -\ldots-(m-2)\overline{N}(r,1;F\mid=m)-(m-2)\ \overline{N}_L(r,1;F)-(m-1)\overline{N}_L(r,1;G)\\
& -(m-1)\overline{N}_E^{(m+1}(r,1;F)+S(r,F)+S(r,G)
\end{aligned}$$

Lemma 2.9. *([14], Lemma 6) If* $H \equiv 0$, *then* F, G *share* $(1, \infty)$. *If further* F, G *share* $(\infty, 0)$ *then* F, G *share* (∞, ∞).

3 Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then F and G share $(1, 3)$, $(\infty, 7n+6)$. We consider the following cases.

Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. Noting that f, g share $(0, 0)$ and $(\infty, 6)$ implies $\overline{N}_*(r,0;f,g) \leq \overline{N}(r,0;f) = \overline{N}(r,0;g)$ and $\overline{N}_*(r,\infty;f,g) \leq \overline{N}(r,\infty;f\mid\geq 7) = \overline{N}(r,\infty;g\mid\geq 7)$, using *Lemmas 2.3* and *2.6* for $m = 3$ in *Lemma 2.8* we obtain

$$\begin{aligned}
& (n+1)\{T(r,f)+T(r,g)\} \qquad (3.1)\\
\leq\; & 6\overline{N}(r,0;f)+2T(r,f)+2T(r,g)+4\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;f\mid\geq 7)\\
& -3\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\leq\; & 2T(r,f)+2T(r,g)+\left(\frac{6n+10}{3n-4}\right)\overline{N}(r,\infty;f\mid\geq 7)+\left(\frac{12n-10}{3n-4}\right)\\
& \overline{N}(r,\infty;f)-3\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)
\end{aligned}$$

So using *Lemma 2.7 (i)* for $k = 6$ in (3.1) we get

$$\begin{aligned}
& (n-1)\{T(r,f)+T(r,g)\} \qquad (3.2)\\
\leq\; & \left(\frac{6n+10}{(3n-4)(7n+6)}\right)\left[T(r,f)+T(r,g)+\frac{1}{n-1}\left\{\overline{N}(r,\infty;f\mid\geq 7)+n\overline{N}_*(r,1;F,G)\right\}\right]\\
& +\left(\frac{12n-10}{n(3n-4)}\right)\left[T(r,f)+T(r,g)+\frac{1}{n-1}\left\{\overline{N}(r,\infty;f\mid\geq 7)+n\overline{N}_*(r,1;F,G)\right\}\right]\\
& -3\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\leq\; & \left[\frac{6n+10}{(3n-4)(7n+6)}+\frac{12n-10}{n(3n-4)}\right]\{T(r,f)+T(r,g)\}+\frac{1}{n-1}\left[\frac{n(6n+10)}{(3n-4)(7n+6)}\right.\\
& \left.+\frac{12n-10}{3n-4}\right]\overline{N}_*(r,1;F,G)+\frac{1}{(n-1)}\left[\frac{6n+10}{(3n-4)(7n+6)}+\frac{12n-10}{n(3n-4)}\right]\overline{N}(r,\infty;f\mid\geq 7)\\
& -3\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

Now using *Lemma 2.7 (ii)* for $k = 6$ in (3.2) we get

$$\begin{aligned}
& \left[n-1-\frac{6n+10}{(3n-4)(7n+6)}-\frac{12n-10}{n(3n-4)}\right]\{T(r,f)+T(r,g)\}\\
\leq\; & \left[\frac{n(6n+10)}{(n-1)(3n-4)(7n+6)}+\frac{12n-10}{(n-1)(3n-4)}\right]\overline{N}_*(r,1;F,G)\\
& +\left[\frac{6n+10}{(3n-4)(7n+6)}+\frac{12n-10}{n(3n-4)}\right]\left[\frac{1}{7n^2-n-7}\{T(r,f)+T(r,g)\}\right.\\
& \left.+\frac{n}{(n-1)(7n^2-n-7)}\overline{N}_*(r,1;F,G)\right]-3\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g),
\end{aligned}$$

from which we get a contradiction for $n \geq 3$.

Case 2. Let $H \equiv 0$. Now from *Lemma 2.9* we have F and G share $(1,\infty)$ and (∞,∞). This implies $E_f(S_1,\infty) = E_g(S_1,\infty)$, $E_f(S_2,0) = E_g(S_2,0)$ and $E_f(S_3,\infty) = E_g(S_3,\infty)$. Now the theorem follows from *Theorem C.* □

Proof of Corollary 1.1. Let F, G be given by (2.1). Then F and G share $(1,3)$, $(\infty,7n+6)$. By *Theorem 1.1* we get either $f \equiv g$ or $f = \frac{-ae^{\gamma}(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1}$, $g = \frac{-a(e^{n\gamma}-1)}{e^{(n+1)\gamma}-1}$, where γ is a non-constant entire function. If $f \not\equiv g$ then using *Lemma 2.3* clearly $\Theta(\infty;f) = \Theta(\infty;g) = 1 -$

$\limsup_{r\longrightarrow\infty} \frac{\sum_{k=1}^{n} \overline{N}(r,u_k;e^{\gamma})}{nT(r,e^{\gamma})} = 0$, where $u_k = exp\ (\frac{2k\pi i}{n+1})$ for $k = 1,2,\ldots,n$ and hence we deduce a contradiction. This proves the corollary. □

Proof of Theorem 1.2. Let F, G be given by (2.1). Then F and G share $(1,3)$, (∞,n). We consider the following cases.

Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. Noting that f, g share $(0,0)$ and $(\infty,0)$ implies $\overline{N}_*(r,0;f,g) \le \overline{N}(r,0;f) = \overline{N}(r,0;g)$ and $\overline{N}_*(r,\infty;f,g) \le \overline{N}(r,\infty;f \mid\ge 7) = \overline{N}(r,\infty;g \mid\ge 7)$, using *Lemmas 2.3* and *2.6* for $m=3$ and $k=0$ in *Lemma 2.8* we obtain

$$\begin{aligned} & (n+1)\{T(r,f)+T(r,g)\} \qquad (3.3)\\ \le\ & 6\overline{N}(r,0;f)+2T(r,f)+2T(r,g)+6\overline{N}(r,\infty;f)-3\overline{N}_*(r,1;F,G)\\ & +S(r,f)+S(r,g)\\ \le\ & 2T(r,f)+2T(r,g)+\left(\frac{18n}{3n-4}\right)\overline{N}(r,\infty;f)-3\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \end{aligned}$$

So using *Lemma 2.7 (ii)* for $k=0$ and *Lemma 2.5* in (3.3) we get

$$\begin{aligned} & (n-1)\{T(r,f)+T(r,g)\} \qquad (3.4)\\ \le\ & \left(\frac{18n(n-1)}{(3n-4)(n^2-n-1)}\right)[T(r,f)+T(r,g)]+\left(\frac{18n^2}{(3n-4)(n^2-n-1)}\right)\overline{N}_*(r,1;F,G)\\ & -3\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ \le\ & \left(\frac{18n(n-1)}{(3n-4)(n^2-n-1)}\right)[T(r,f)+T(r,g)]+\left(\frac{18n^2}{6(3n-4)(n^2-n-1)}-\frac{1}{2}\right)\\ & \left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right\}+S(r,f)+S(r,g)\\ \le\ & \left(\frac{18n(n-1)}{(3n-4)(n^2-n-1)}+\frac{18n^2}{3(3n-4)(n^2-n-1)}-1\right)[T(r,f)+T(r,g)]\\ & +S(r,f)+S(r,g). \end{aligned}$$

Clearly (3.4) implies a contradiction for $n \ge 4$.

Case 2. Let $H \equiv 0$. Now from *Lemma 2.9* we have F and G share $(1,\infty)$ and (∞,∞). This implies $E_f(S_1,\infty) = E_g(S_1,\infty)$, $E_f(S_2,0) = E_g(S_2,0)$ and $E_f(S_3,\infty) = E_g(S_3,\infty)$. Now the theorem follows from *Theorem C.* □

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