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Univariate right fractional Ostrowski inequalities

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ABSTRACT

Very general univariate right Caputo fractional Ostrowski inequalities are presented. One of them is proved sharp and attained. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$.

RESUMEN

Se presenta de manera muy general desigualdades univariadas derechas de Caputo fraccionarias de Ostrowski. Se prueba que una de ellas es aguda Las estimaciones con respecto a $\|\cdot\|_p$, $1 \leq p \leq \infty$.

Keywords and Phrases: Ostrowski inequality, right Caputo fractional derivative.

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1 Introduction

In 1938, A. Ostrowski [7] proved the following important inequality:

Theorem 1. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b) *whose derivative* $f' : (a, b) \to \mathbb{R}$ *is bounded on* (a, b)*, i.e.,* $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < +\infty$*. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_\infty , \tag{1}$$

for any $x \in [a, b]$*. The constant* $\frac{1}{4}$ *is the best possible.*

Since then there has been a lot of activity around these inequalities with important applications to Numerical Analysis and Probability.

This paper is greatly motivated and inspired also by the following result.

Theorem 2. (see [1]) Let $f \in C^{n+1}([a, b])$, $n \in \mathbb{N}$ and $x \in [a, b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then it holds

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \le \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+2)!} \cdot \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a} \right). \tag{2}$$

Inequality (2) is sharp. In particular, when n is odd is attained by $f^*(y) := (y-x)^{n+1} \cdot (b-a)$, while when n is even the optimal function is

$$\bar{f}(y) := |y-x|^{n+\alpha} \cdot (b-a), \quad \alpha > 1.$$

Clearly inequality (2) generalizes inequality (1) for higher order derivatives of f.

Also in [2], see Chapters 24-26, we presented a complete theory of left fractional Ostrowski inequalities.

2 Main Results

We need

Definition 3. ([3], [4], [5], [6], [8]) Let $f \in L_1([a, b])$, $\alpha > 0$. The right Riemann-Liouville fractional operator of order α by

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} f(J)\, dJ, \tag{3}$$

$\forall\, x \in [a, b]$, where Γ is the gamma function. We set $I_{b-}^{0} := I$ (the identity operator).

Definition 4. ([3], [4], [5], [6], [8]) Let $f \in AC^m([a, b])$ ($f^{(m-1)}$ is in $AC([a, b])$), $m \in \mathbb{N}$, $m = \lceil \alpha \rceil$, $\alpha > 0$ ($\lceil \cdot \rceil$ the ceiling of the number). We define the right Caputo fractional derivative of order $\alpha > 0$, by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \quad \forall\, x \le b. \tag{4}$$

If $\alpha = m \in \mathbb{N}$, then

$$D_{b-}^{m} f(x) = (-1)^{m} f^{(m)}(x), \quad \forall\, x \in [a,b].$$

If $x > b$ we define $D_{b-}^{\alpha} f(x) = 0$.

We also need

Theorem 5. ([3]) Let $f \in AC^{m}([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Then

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^{k} + \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (J-x)^{\alpha-1} D_{b-}^{\alpha} f(J)\, dJ, \tag{5}$$

the right Caputo fractional Taylor formula with integral remainder.

We present

Theorem 6. Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^{m}([a,b])$. Assume $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, and $D_{b-}^{\alpha} f \in L_{\infty}([a,b])$. Then

$$\left| \frac{1}{b-a} \int_{a}^{b} f(x)\, dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^{\alpha}. \tag{6}$$

Proof. Let $x \in [a,b]$. We have

$$f(x) - f(b) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (J-x)^{\alpha-1} D_{b-}^{\alpha} f(J)\, dJ.$$

Then

$$\begin{aligned}
|f(x) - f(b)| &\leq \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (J-x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| dJ \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_{x}^{b} (J-x)^{\alpha-1} dJ \right) \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]} \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{(J-x)^{\alpha}}{\alpha} \Big|_{x}^{b} \right) \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]} \\
&= \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}.
\end{aligned}$$

Therefore

$$|f(x) - f(b)| \leq \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}, \quad \forall\, x \in [a,b].$$

Hence it holds

$$\left| \frac{1}{b-a} \int_{a}^{b} f(x)\, dx - f(b) \right| = \left| \frac{1}{b-a} \int_{a}^{b} (f(x) - f(b))\, dx \right|$$

$$\leq \frac{1}{b-a} \int_{a}^{b} |f(x) - f(b)|\, dx \leq \frac{1}{b-a} \int_{a}^{b} \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]} dx$$

$$
\begin{aligned}
&= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+1)} \int_a^b (b-x)^{\alpha}\, dx \\
&= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+1)} \left(- \left(\frac{(b-x)^{\alpha+1}}{\alpha+1} \Big|_a^b \right) \right) \\
&= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+1)} (-1) \left(0 - \frac{(b-a)^{\alpha+1}}{\alpha+1} \right) \\
&= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+2)} \cdot (b-a)^{\alpha+1} = \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]} \cdot (b-a)^{\alpha}}{\Gamma(\alpha+2)},
\end{aligned}
$$

proving the claim. ∎

We also give

Theorem 7. Let $\alpha \geq 1$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. Assume that $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, and $D_{b-}^{\alpha} f \in L_1([a,b])$. Then

$$
\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha+1)} (b-a)^{\alpha-1}. \tag{7}
$$

Proof. We have again

$$
\begin{aligned}
|f(x) - f(b)| &\leq \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| dJ \\
&\leq \frac{1}{\Gamma(\alpha)} (b-x)^{\alpha-1} \int_x^b \left| D_{b-}^{\alpha} f(J) \right| dJ \\
&\leq \frac{1}{\Gamma(\alpha)} (b-x)^{\alpha-1} \left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}.
\end{aligned}
$$

Hence

$$
|f(x) - f(b)| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha)} (b-x)^{\alpha-1}, \quad \forall\, x \in [a,b].
$$

Therefore

$$
\begin{aligned}
\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| &\leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\, dx \\
&\leq \frac{1}{b-a} \int_a^b \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha)} (b-x)^{\alpha-1}\, dx \\
&= \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{(b-a)\,\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1}\, dx \\
&= \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{(b-a)\,\Gamma(\alpha)} \frac{(b-x)^{\alpha}}{\alpha} = \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha+1)} (b-x)^{\alpha-1},
\end{aligned}
$$

proving the claim. ∎

We continue with

Theorem 8. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 1 - \frac{1}{p}$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. Assume that $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, and $D_{b-}^{\alpha} f \in L_q([a,b])$. Then

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} (b-a)^{\alpha - 1 + \frac{1}{p}}. \tag{8}$$

Proof. We have again

$$|f(x) - f(b)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| dJ$$

$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (J-x)^{p(\alpha-1)} dJ \right)^{\frac{1}{p}} \left(\int_x^b \left| D_{b-}^{\alpha} f(J) \right|^q dJ \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{(\alpha-1)+\frac{1}{p}}}{\left(p(\alpha-1)+1\right)^{\frac{1}{p}}} \left(\int_x^b \left| D_{b-}^{\alpha} f(J) \right|^q dJ \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{(\alpha-1)+\frac{1}{p}}}{\left(p(\alpha-1)+1\right)^{\frac{1}{p}}} \left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}.$$

Therefore

$$|f(x) - f(b)| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}} (b-x)^{\alpha-1+\frac{1}{p}}, \quad \forall\, x \in [a,b].$$

Hence

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| \leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\, dx$$

$$\leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{(b-a)\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}} \int_a^b (b-x)^{\alpha-1+\frac{1}{p}} dx$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}} \frac{(b-a)^{\alpha-1+\frac{1}{p}}}{\left(\alpha + \frac{1}{p}\right)}.$$

∎

Corollary 9. Let $\alpha > \frac{1}{2}$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. Assume $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, $D_{b-}^{\alpha} f \in L_2([a,b])$. Then

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_2([a,b])}}{\Gamma(\alpha)\left(\sqrt{2\alpha-1}\right)\left(\alpha + \frac{1}{2}\right)} (b-a)^{\alpha-\frac{1}{2}}. \tag{9}$$

We finish with

Proposition 10. Inequality (6) is sharp, namely it is attained by

$$f(x) = (b-x)^{\alpha}, \quad \alpha > 0,\ \alpha \notin \mathbb{N},\ x \in [a,b].$$

Proof. Notice that $(b-x)^{\alpha} \in AC^m([a,b])$. We see that

$$f'(x) = -\alpha (b-x)^{\alpha-1},$$

$$f''(x) = (-1)^2 \alpha(\alpha-1)(b-x)^{\alpha-2},$$

$$\ldots,$$

$$f^{(m-1)}(x) = (-1)^{m-1} \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(b-x)^{\alpha-m+1},$$

and

$$f^{(m)}(x) = (-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(\alpha-m+1)(b-x)^{\alpha-m}.$$

Thus

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^{2m}}{\Gamma(m-\alpha)} \alpha(\alpha-1)\ldots(\alpha-m+1) \int_x^b (J-x)^{m-\alpha-1} (b-J)^{\alpha-m}\, dJ$$

$$= \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{\Gamma(m-\alpha)} \int_x^b (b-J)^{(\alpha-m+1)-1} (J-x)^{(m-\alpha)-1}\, dJ$$

$$= \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{\Gamma(m-\alpha)} \frac{\Gamma(\alpha-m+1)\Gamma(m-\alpha)}{\Gamma(1)}$$

$$= \alpha(\alpha-1)\ldots(\alpha-m+1)\Gamma(\alpha-m+1) = \Gamma(\alpha+1).$$

That is

$$D_{b-}^{\alpha} f(x) = \Gamma(\alpha+1), \quad \forall\, x \in [a,b].$$

Also we see that $f^{(k)}(b) = 0$, $k = 0, 1, \ldots, m-1$, and $D_{b-}^{\alpha} f \in L_{\infty}([a,b])$. So f fulfills all assumptions. Next we see

$$\text{R.H.S.}(6) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)} (b-a)^{\alpha} = \frac{(b-a)^{\alpha}}{(\alpha+1)}.$$

$$\text{L.H.S.}(6) = \frac{1}{b-a} \int_a^b (b-x)^{\alpha}\, dx$$

$$= \frac{1}{b-a} \frac{(b-a)^{\alpha+1}}{(\alpha+1)} = \frac{(b-a)^{\alpha}}{\alpha+1},$$

proving attainability and sharpness of (6). ∎

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