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On the global behavior of $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n + |y_n|$

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ABSTRACT

In this paper we consider the system of piecewise linear difference equations in the title, where the initial conditions x_0 and y_0 are real numbers. We show that there exists a unique equilibrium solution and exactly two prime period-3 solutions, and that except for the unique equilibrium solution, every solution of the system is eventually one of the two prime period-3 solutions.

RESUMEN

En este artículo consideramos el sistema de ecuaciones en diferencia lineales por partes indicado en el título, donde las condiciones iniciales x_0 e y_0 son números reales. Demostramos que existe una única solución de equilibrio y exactamente dos soluciones de período 3-primo, y que exceptuando la solución única de equilibrio, toda solución del sistema es eventualmente una de las dos soluciones de periodo 3-primo.

Keywords and Phrases: Periodic solution; systems of piecewise linear difference equations

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1 Introduction

In this paper we consider the system of piecewise linear difference equations

$$\left\{\begin{array}{lll} x_{n+1} & = & |x_n| - y_n - 1 \\ & & \\ y_{n+1} & = & x_n + |y_n| \end{array}\right. , \quad n = 0, 1, \ldots \tag{1.1}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers. We show that every solution of System(1.1) is either (from the beginning) the unique equilibrium point

$$(\bar{x}, \bar{y}) = \left(-\frac{2}{5}, -\frac{1}{5}\right)$$

or else is eventually one of the following period-3 cycles:

$$\mathbf{P}_3^1 = \left(\begin{array}{cccccc} x_0 & = & 0 & , & y_0 & = & -1 \\ & & & & & & \\ x_1 & = & 0 & , & y_1 & = & 1 \\ & & & & & & \\ x_2 & = & -2 & , & y_2 & = & 1 \end{array}\right) \quad \text{or} \quad \mathbf{P}_3^2 = \left(\begin{array}{cccccc} x_0 & = & 0 & , & y_0 & = & -\frac{1}{3} \\ & & & & & & \\ x_1 & = & -\frac{2}{3} & , & y_1 & = & \frac{1}{3} \\ & & & & & & \\ x_2 & = & -\frac{2}{3} & , & y_2 & = & -\frac{1}{3} \end{array}\right).$$

This study of System(1.1) was motivated by Devaney's celebrated Gingerbreadman map

$$\left\{\begin{array}{lll} x_{n+1} & = & |x_n| - y_n + 1 \\ & & \\ y_{n+1} & = & x_n \end{array}\right. , \quad n = 0, 1, \ldots .$$

See Ref. [1, 2, 3, 4].

We believe that the methods and techniques used in this paper will be useful in discovering the global behavior of similar piecewise linear systems of the form

$$\left\{\begin{array}{l} x_{n+1} = |x_n| + a y_n + b \\ \\ y_{n+1} = x_n + c\,|y_n| + d \end{array}\right. , \qquad n = 0, 1, 2 \ldots$$

For another system of this form see [5].

2 The Global Behavior Of The Solutions Of System(1.1)

Set

$$
\begin{aligned}
l_1 &= \{(x, y) : x \geq 0, y = 0\} \\
l_2 &= \{(x, y) : x = 0, y \geq 0\} \\
l_3 &= \{(x, y) : x \leq 0, y = 0\} \\
l_4 &= \{(x, y) : x = 0, y \leq 0\} \\
Q_1 &= \{(x, y) : x > 0, y > 0\} \\
Q_2 &= \{(x, y) : x < 0, y > 0\} \\
Q_3 &= \{(x, y) : x < 0, y < 0\} \\
Q_4 &= \{(x, y) : x > 0, y < 0\}.
\end{aligned}
$$

Theorem 1. Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(1.1) with $(x_0, y_0) \in \mathbf{R}^2$. Then either $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium $(\bar{x}, \bar{y})$, or else there exists a non-negative integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System(1.1) is either the prime period-3 cycle $\mathbf{P}_3^1$ or the prime period-3 cycle $\mathbf{P}_3^2$.

The proof of Theorem 1 is a direct consequence of the following lemmas.

Lemma 2. Suppose there exists a non-negative integer $N \geq 0$ such that

$$y_N = -x_N - 1 \qquad \text{and} \qquad y_N \geq 0.$$

Then $(x_{N+1}, y_{N+1}) = (0, -1)$, and so $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Proof. Note that $x_N = -y_N - 1 \leq -1$, and so

$$
\begin{aligned}
x_{N+1} &= |x_N| - y_N - 1 &= -x_N - (-x_N - 1) - 1 &= 0 \\
y_{N+1} &= x_N + |y_N| &= x_N + (-x_N - 1) &= -1.
\end{aligned}
$$

The proof is complete. □

Lemma 3. Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in l_2$. Then $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Proof. We have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 &= 0 - y_N - 1 &= -y_N - 1 &< 0 \\ y_{N+1} &= x_N + |y_N| &= 0 + y_N &= y_N &\geq 0 \end{aligned}$$

and so it follows by Lemma 2 that $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. ☐

Lemma 4. Suppose there exists a non-negative integer $N \geq 0$ such that $x_N = 0$ and $y_N < -1$. Then

(1) $x_{N+3} = 2y_N + 2 < 0$.

(2) If $-\dfrac{3}{2} \leq y_N < -1$, then $y_{N+3} = -2y_N - 3 \leq 0$.

(3) If $y_N < -\dfrac{3}{2}$, then $\{(x_n, y_n)\}_{n=N+4}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Proof. We have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 &= -y_N - 1 &> 0 \\ y_{N+1} &= x_N + |y_N| &= -y_N &> 0 \\ x_{N+2} &= |x_{N+1}| - y_{N+1} - 1 &= -2 & \\ y_{N+2} &= x_{N+1} + |y_{N+1}| &= -2y_N - 1 &> 0 \\ x_{N+3} &= |x_{N+2}| - y_{N+2} - 1 &= 2y_N + 2 &< 0 \\ y_{N+3} &= x_{N+2} + |y_{N+2}| &= -2y_N - 3. & \end{aligned}$$

If $-\dfrac{3}{2} \leq y_N < -1$, then $y_{N+3} = -2y_N - 3 \leq 0$. If $y_N < -\dfrac{3}{2}$, then $y_{N+3} = -2y_N - 3 > 0$ and so by Lemma 2 $\{(x_n, y_n)\}_{n=N+4}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. The proof is complete. ☐

Lemma 5. Suppose there exists a non-negative integer $N \geq 0$ such that $x_N = 0$ and $-1 < y_N \leq 0$. Then

(1) If $-\dfrac{1}{4} < y_N \leq 0$, then $\{(x_n, y_n)\}_{n=N+5}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

(2) If $-\dfrac{1}{2} < y_N \leq -\dfrac{1}{4}$, then $x_{N+5} = 8y_N + 2$, $y_{N+5} = -8y_N - 3$, and $x_{N+6} = 0$.

(3) If $-1 < y_N \le -\frac{1}{2}$, then $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Proof. We have

$$\begin{array}{lllllll} x_{N+1} & = & |x_N| - y_N - 1 & = & -y_N - 1 & < & 0 \\ y_{N+1} & = & x_N + |y_N| & = & -y_N & \ge & 0 \\ x_{N+2} & = & |x_{N+1}| - y_{N+1} - 1 & = & 2y_N & \le & 0 \\ y_{N+2} & = & x_{N+1} + |y_{N+1}| & = & -2y_N - 1 & & \\ x_{N+3} & = & |x_{N+2}| - y_{N+2} - 1 & = & 0. & & \end{array}$$

If $-\frac{1}{4} < y_N \le 0$, then $y_{N+2} < 0$ and $y_{N+3} = x_{N+2} + |y_{N+2}| = 4y_N + 1 > 0$. It follows by Lemma 3 that $\{(x_n, y_n)\}_{n=N+5}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$, and so Statement 1 is true.

If $-\frac{1}{2} < y_N \le -\frac{1}{4}$, then $y_{N+2} < 0$ and

$$\begin{array}{lllllll} y_{N+3} & = & x_{N+2} + |y_{N+2}| & = & 4y_N + 1 & \le & 0 \\ x_{N+4} & = & |x_{N+3}| - y_{N+3} - 1 & = & -4y_N - 2 & < & 0 \\ y_{N+4} & = & x_{N+3} + |y_{N+3}| & = & -4y_N - 1 & \ge & 0 \\ x_{N+5} & = & |x_{N+4}| - y_{N+4} - 1 & = & 8y_N + 2 & \le & 0 \\ y_{N+5} & = & x_{N+4} + |y_{N+4}| & = & -8y_N - 3 & & \\ x_{N+6} & = & |x_{N+5}| - y_{N+5} - 1 & = & 0 & & \end{array}$$

and so Statement 2 is true.

If $-1 < y_N \le -\frac{1}{2}$, then $y_{N+6} = x_{N+5} + |y_{N+5}| = -1$ and so $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. The proof is complete. □

Lemma 6. Suppose there exists a non-negative integer $N \ge 0$ such that $(x_N, y_N) \in l_4$. Then the following five statements are true:

(1) Suppose $-\frac{1}{3} < y_N \leq 0$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$.

(2) Suppose $y_N = -\frac{1}{3}$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^2$.

(3) Suppose $-\frac{4}{3} < y_N < -\frac{1}{3}$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$.

(4) Suppose $y_N = -\frac{4}{3}$. Then $\{(x_n, y_n)\}_{n=N+3}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^2$.

(5) Suppose $y_N < -\frac{4}{3}$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$.

Proof. We have $x_N = 0$ and $y_N \leq 0$.

(1) Suppose $-\frac{1}{3} < y_N \leq 0$. Note that by Statement 1 of Lemma 5, that if $-\frac{1}{4} < y_N \leq 0$, then $\{(x_n, y_n)\}_{n=N+5}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

So suppose $-\frac{1}{3} < y_N \leq -\frac{1}{4}$. For each integer $n \geq 1$, let

$$a_n = \frac{-2^{2n} + 1}{3 \cdot 2^{2n}}.$$

Observe that

$$-\frac{1}{4} = a_1 > a_2 > a_3 > \ldots > -\frac{1}{3} \quad \text{and} \quad \lim_{n\to\infty} a_n = -\frac{1}{3}.$$

Thus there exists a unique integer $K \geq 1$ such that $y_N \in (a_{K+1}, a_K]$. We first consider the case $K = 1$; that is, $y_N \in \left(-\frac{5}{16}, -\frac{1}{4}\right]$. It follows from Statement 2 of Lemma 5 that $x_{N+5} = 8y_N + 2 \leq 0$, $y_{N+5} = -8y_N - 3 < 0$, and $x_{N+6} = 0$. Thus $y_{N+6} = x_{N+5} + |y_{N+5}| = 16y_N + 5 > 0$, and so by Lemma 3 we have $\{(x_n, y_n)\}_{n=N+8}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Hence without loss of generality, we may assume $K \geq 2$. For each integer $m \geq 1$, let $\mathcal{P}(m)$ be the following statement:

$$\begin{aligned} x_{N+3m+3} &= 0 \\ y_{N+3m+3} &= 2^{2m+2}y_N + \frac{2^{2m+2}-1}{3} \leq 0. \end{aligned}$$

Claim: $\mathcal{P}(m)$ is true for $1 \leq m \leq K-1$.
The proof of the Claim will be by induction on m. We shall first show that $\mathcal{P}(1)$ is true.
Recall that $x_N = 0$ and $y_N \in (a_{K+1}, a_K] \subset \left(-\frac{1}{3}, -\frac{5}{16}\right]$, and so by Statement 2 of Lemma 5

we have $x_{N+5} = 8y_N + 2 < 0$ and $y_{N+5} = -8y_N - 3 < 0$. Then

$$\begin{aligned} x_{N+3(1)+3} &= 0 \\ y_{N+3(1)+3} &= 16y_N + 5 = 2^{2(1)+2}y_N + \frac{2^{2(1)+2}-1}{3} \leq 0 \end{aligned}$$

and so $\mathcal{P}(1)$ is true. Thus if $K = 2$, then we have shown that for $1 \leq m \leq K-1$, $\mathcal{P}(m)$ is true. It remains to consider the case $K \geq 3$. So assume that $K \geq 3$. Let m be an integer such that $1 \leq m \leq K-2$, and suppose $\mathcal{P}(m)$ is true. We shall show that $\mathcal{P}(m+1)$ is true.

Since $\mathcal{P}(m)$ is true we know

$$\begin{aligned} x_{N+3m+3} &= 0 \\ y_{N+3m+3} &= 2^{2m+2}y_N + \frac{2^{2m+2}-1}{3} \leq 0. \end{aligned}$$

Recall that $y_N \in (a_{K+1}, a_K] = \left(\frac{-2^{2(K+1)}+1}{3 \cdot 2^{2(K+1)}}, \frac{-2^{2K}+1}{3 \cdot 2^{2K}}\right]$.

Then

$$x_{N+3m+4} = |x_{N+3m+3}| - y_{N+3m+3} - 1 = -2^{2m+2}y_N - \left(\frac{2^{2m+2}-1}{3}\right) - 1.$$

Note that $x_{N+3m+4} = -y_{N+3m+3} - 1$.

In particular,

$$\begin{aligned} x_{N+3m+4} &= -2^{2m+2}y_N - \left(\frac{2^{2m+2}-1}{3}\right) - 1 \\ &< -2^{2m+2}\left(\frac{-2^{2(K+1)}+1}{3 \cdot 2^{2(K+1)}}\right) - \left(\frac{2^{2m+2}-1}{3}\right) - 1 \\ &= \frac{2^{2m+2K+4}}{3 \cdot 2^{2K+2}} - \frac{2^{2m+2}}{3 \cdot 2^{2K+2}} - \frac{2^{2m+2}}{3} + \frac{1}{3} - 1 \\ &= -\frac{2^{2m-2K}}{3} - \frac{2}{3} \\ &< 0 \end{aligned}$$

and

$$y_{N+3m+4} = x_{N+3m+3} + |y_{N+3m+3}| = 0 + |y_{N+3m+3}| = -y_{N+3m+3} \geq 0.$$

Thus

$$\begin{aligned} x_{N+3m+5} &= |x_{N+3m+4}| - y_{N+3m+4} - 1 = y_{N+3m+3} + 1 - (-y_{N+3m+3}) - 1 \\ &= 2y_{N+3m+3} \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned} y_{N+3m+5} &= x_{N+3m+4} + |y_{N+3m+4}| = -y_{N+3m+3} - 1 + (-y_{N+3m+3}) \\ &= -2y_{N+3m+3} - 1. \end{aligned}$$

In particular,

$$\begin{aligned} y_{N+3m+5} &= -2\left(2^{2m+2}y_N + \frac{2^{2m+2}-1}{3}\right) - 1 \\ &< -2\left[2^{2m+2}\left(\frac{-2^{2(K+1)}+1}{3\cdot 2^{2(K+1)}}\right) + \frac{2^{2m+2}-1}{3}\right] - 1 \\ &= \frac{2^{2m+2K+5}}{3\cdot 2^{2K+2}} - \frac{2^{2m+3}}{3\cdot 2^{2K+2}} - \frac{2^{2m+3}}{3} + \frac{2}{3} - 1 \\ &= -\frac{2^{2m-2K+1}}{3} - \frac{1}{3} \\ &< 0. \end{aligned}$$

Finally,

$$\begin{aligned} x_{N+3(m+1)+3} &= x_{N+3m+6} \\ &= |x_{N+3m+5}| - y_{N+3m+5} - 1 \\ &= -2y_{N+3m+3} - (-2y_{N+3m+3} - 1) - 1 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
y_{N+3(m+1)+3} &= y_{N+3m+6} \\
&= x_{N+3m+5} + |y_{N+3m+5}| \\
&= 2y_{N+3m+3} + 2y_{N+3m+3} + 1 \\
&= 4y_{N+3m+3} + 1 \\
&= 2^2\left(2^{2m+2}y_N + \frac{2^{2m+2}-1}{3}\right) + 1 \\
&= 2^{2m+4}y_N + \frac{2^{2m+4}-4}{3} + 1 \\
&= 2^{2(m+1)+2}y_N + \frac{2^{2(m+1)+2}-1}{3}.
\end{aligned}$$

In particular,

$$\begin{aligned}
y_{N+3(m+1)+3} &\leq 2^{2(m+1)+2}\left(\frac{-2^{2K}+1}{3\cdot 2^{2K}}\right) + \frac{2^{2(m+1)+2}-1}{3} \\
&= -\frac{2^{2m+2K+4}}{3\cdot 2^{2K}} + \frac{2^{2m+4}}{3\cdot 2^{2K}} + \frac{2^{2m+4}}{3} - \frac{1}{3} \\
&= -\frac{1}{3}\left(1 - 2^{2m-2K+4}\right) \\
&\leq 0
\end{aligned}$$

and so $\mathcal{P}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{P}(m)$ is true for $1 \leq m \leq K-1$. Specifically, $\mathcal{P}(K-1)$ is true, and so

$$\begin{aligned}
x_{N+3(K-1)+3} &= x_{N+3K} = 0 \\
y_{N+3(K-1)+3} &= y_{N+3K} = 2^{2K}y_N + \frac{2^{2K}-1}{3} < 0.
\end{aligned}$$

Note that

$$2^{2K}\left(\frac{-2^{2K+2}+1}{3\cdot 2^{2K+2}}\right) + \frac{2^{2K}-1}{3} < y_{N+3K} \leq 2^{2K}\left(\frac{-2^{2K}+1}{3\cdot 2^{2K}}\right) + \frac{2^{2K}-1}{3}.$$

So as

$$2^{2K}\left(\frac{-2^{2K+2}+1}{3\cdot 2^{2K+2}}\right)+\frac{2^{2K}-1}{3}=\frac{-2^{4K+2}}{3\cdot 2^{2K+2}}+\frac{2^{2K}}{3\cdot 2^{2K+2}}+\frac{2^{2K}}{3}-\frac{1}{3}=\frac{1}{3}\left(\frac{1}{2^2}-1\right)=-\frac{1}{4}$$

and

$$2^{2K}\left(\frac{-2^{2K}+1}{3\cdot 2^{2K}}\right)+\frac{2^{2K}-1}{3}=\frac{-2^{2K}+1}{3}+\frac{2^{2K}-1}{3}=0$$

we have

$$-\frac{1}{4} < y_{N+3K} \le 0$$

and so it follows from Statement 1 of Lemma 5 that $\{(x_n, y_n)\}_{n=N+3K+5}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

(2) Suppose $y_n = -\frac{1}{3}$. Note that $(0, -\frac{1}{3}) \in \mathbf{P}_3^1$ and so $\{(x_n, y_n)\}_{n=N}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

(3) Suppose $-\frac{4}{3} < y_N \le -\frac{1}{3}$.
We shall first consider the case where $-\frac{4}{3} < y_N \le -1$.

So suppose $-\frac{4}{3} < y_N \le -1$. For each integer $n \ge 0$, let

$$b_n = \frac{-2^{2n+2}+1}{3\cdot 2^{2n}}.$$

Observe that

$$-1 = b_0 > b_1 > b_2 > \ldots > -\frac{4}{3} \quad \text{and} \quad \lim_{n\to\infty} b_n = -\frac{4}{3}.$$

Thus there exists a unique integer $K \ge 1$ such that $y_N \in (b_K, b_{K-1}]$. We first consider the case $K = 1$; that is, $y_N \in \left(-\frac{5}{4}, -1\right]$. Note that if $y_N = -1$ then $(x_N, y_N) = (0, -1)$ and $\{(x_n, y_n)\}_{n=N}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. So assume $y_N \in \left(-\frac{5}{4}, -1\right)$. By Statements 1 and 2 of Lemma 4, we have $x_{N+3} = 2y_N + 2 < 0$ and $y_{N+3} = -2y_N - 3 \le 0$. Then

$$x_{N+4} = |x_{N+3}| - y_{N+3} - 1 = 0$$

$$y_{N+4} = x_{N+3} + |y_{N+3}| = 4y_N + 5 > 0$$

and so it follows by Lemma 3 that $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Hence without loss of generality, we may assume $K \geq 2$. For each integer $m \geq 1$, let $\mathcal{Q}(m)$ be the following statement:

$$\begin{aligned} x_{N+3m+1} &= 0 \\ y_{N+3m+1} &= 2^{2m} y_N + \frac{2^{2m+2} - 1}{3} \leq 0. \end{aligned}$$

Claim: $\mathcal{Q}(m)$ is true for $1 \leq m \leq K - 1$.

The proof of the Claim will be by induction on m. We shall first show that $\mathcal{Q}(1)$ is true. Recall that $x_N = 0$ and $y_N \in (b_K, b_{K-1}] \subset \left(-\frac{4}{3}, -\frac{5}{4}\right]$, and so by Statements 1 and 2 of Lemma 4 we have

$$\begin{aligned} x_{N+3} &= 2y_N + 2 < 0 \\ y_{N+3} &= -2y_N - 3 < 0 \end{aligned}$$

$$\begin{aligned} x_{N+3(1)+1} &= |x_{N+3}| - y_{N+3} - 1 = 0 \\ y_{N+3(1)+1} &= x_{N+3} + |y_{N+3}| \\ &= 4y_N + 5 \leq 0 \\ &= 2^{2(1)} y_N + \frac{2^{2(1)+2} - 1}{3} \leq 0 \end{aligned}$$

and so $\mathcal{Q}(1)$ is true. Thus if $K = 2$, then we have shown that for $1 \leq m \leq K - 1$, $\mathcal{Q}(m)$ is true. It remains to consider the case $K \geq 3$. So assume that $K \geq 3$. Let m be an integer such that $1 \leq m \leq K-2$, and suppose $\mathcal{Q}(m)$ is true. We shall show that $\mathcal{Q}(m+1)$ is true.

Since $\mathcal{Q}(m)$ is true we know

$$\begin{aligned} x_{N+3m+1} &= 0 \\ y_{N+3m+1} &= 2^{2m} y_N + \frac{2^{2m+2} - 1}{3} \leq 0 \end{aligned}$$

and so

$$x_{N+3m+2} = |x_{N+3m+1}| - y_{N+3m+1} - 1 = 0 - y_{N+3m+1} - 1.$$

Recall that $y_N \in (b_K, b_{K-1}] = \left(\frac{-2^{2K+2} + 1}{3 \cdot 2^{2K}}, \frac{-2^{2K} + 1}{3 \cdot 2^{2K-2}} \right]$.

In particular,

$$\begin{aligned}
x_{N+3m+2} &= -2^{2m}y_N - \left(\frac{2^{2m+2}-1}{3}\right) - 1 \\
&< -2^{2m}\left(\frac{-2^{2K+2}+1}{3\cdot 2^{2K}}\right) - \left(\frac{2^{2m+2}-1}{3}\right) - 1 \\
&= \frac{2^{2K+2m+2}}{3\cdot 2^{2K}} - \frac{2^{2m}}{3\cdot 2^{2K}} - \frac{2^{2m+2}}{3} + \frac{1}{3} - 1 \\
&= -\frac{1}{3}\left(2^{2m-2K+2}+2\right) \\
&< 0
\end{aligned}$$

and

$$y_{N+3m+2} = x_{N+3m+1} + |y_{N+3m+1}| = 0 - y_{N+3m+1} \geq 0.$$

Hence

$$\begin{aligned}
x_{N+3m+3} &= |x_{N+3m+2}| - y_{N+3m+2} - 1 = y_{N+3m+1} + 1 - (-y_{N+3m+1}) - 1 \\
&= 2y_{N+3m+1} \\
&\leq 0
\end{aligned}$$

and

$$\begin{aligned}
y_{N+3m+3} &= x_{N+3m+2} + |y_{N+3m+2}| = -y_{N+3m+1} - 1 + (-y_{N+3m+1}) \\
&= -2y_{N+3m+1} - 1.
\end{aligned}$$

In particular,

$$\begin{aligned}
y_{N+3m+3} &= -2\left[2^{2m}y_N + \frac{2^{2m+2}-1}{3}\right] - 1 \\
&< -2\left[2^{2m}\left(\frac{-2^{2K+2}+1}{3\cdot 2^{2K}}\right) + \frac{2^{2m+2}-1}{3}\right] - 1 \\
&= \frac{2^{2K+2m+3}}{3\cdot 2^{2K}} - \frac{2^{2m+1}}{3\cdot 2^{2K}} - \frac{2^{2m+3}}{3} + \frac{2}{3} - 1 \\
&= -\frac{1}{3}\left(2^{2m-2K+1}+1\right) \\
&< 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
x_{N+3(m+1)+1} &= x_{N+3m+4} \\
&= |x_{N+3m+3}| - y_{N+3m+1} - 1 \\
&= -2y_{N+3m+1} - (-2y_{N+3m+1} - 1) - 1 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
y_{N+3(m+1)+1} &= y_{N+3m+4} \\
&= x_{N+3m+3} + |y_{N+3m+3}| \\
&= 2y_{N+3m+1} + 2y_{N+3m+1} + 1 \\
&= 4y_{N+3m+1} + 1 \\
&= 2^{2(m+1)}y_N + \frac{2^{2(m+1)+2}-1}{3}.
\end{aligned}$$

In particular,

$$y_{N+3(m+1)+1} \leq 2^{2m+2}\left(\frac{-2^{2K}+1}{3\cdot 2^{2K-2}}\right) + \frac{2^{2m+4}-1}{3}$$

$$
\begin{aligned}
&= -\frac{2^{2K+2m+2}}{3\cdot 2^{2K-2}} + \frac{2^{2m+2}}{3\cdot 2^{2K-2}} + \frac{2^{2m+4}}{3} - \frac{1}{3} \\
&= \frac{1}{3}\left(2^{2m-2K+4} - 1\right) \\
&\leq 0
\end{aligned}
$$

and so $\mathcal{Q}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{Q}(m)$ is true for $1 \leq m \leq K-1$. Specifically, $\mathcal{Q}(K-1)$ is true, and so

$$
\begin{aligned}
x_{N+3(K-1)+1} &= 0 \\
y_{N+3(K-1)+1} &= 2^{2(K-1)}y_N + \frac{2^{2(K-1)+2}-1}{3} \quad \leq \quad 0.
\end{aligned}
$$

Note that

$$
\begin{aligned}
0 \geq y_{N+3(K-1)+1} &> 2^{2(K-1)}\left(\frac{-2^{2K+2}+1}{3\cdot 2^{2K}}\right) + \frac{2^{2K}-1}{3} \\
&= -\frac{2^{4K}}{3\cdot 2^{2K}} + \frac{2^{2K-2}}{3\cdot 2^{2K}} + \frac{2^{2K}}{3} - \frac{1}{3} \\
&= \frac{1}{3}\left(\frac{1}{4} - 1\right) \\
&= -\frac{1}{4}
\end{aligned}
$$

and so it follows by Statement 1 of Lemma 5 that $\{(x_n, y_n)\}_{n=N+3K+3}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Suppose $-1 < y_N < -\frac{1}{2}$. By Statement 3 of Lemma 5 we have $\{(x_n, y_n)\}_{n=N+3}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

To complete the proof of Statement 3 we shall now suppose that $-\frac{1}{2} \leq y_N < -\frac{1}{3}$. For each integer $n \geq 1$, let

$$\alpha_n = \frac{-2^{2n-1}-1}{3\cdot 2^{2n-1}}.$$

Observe that

$$-\frac{1}{2} = \alpha_1 < \alpha_2 < \alpha_3 < \ldots < -\frac{1}{3} \quad \text{and} \quad \lim_{n\to\infty} \alpha_n = -\frac{1}{3}.$$

Thus there exists a unique integer $K \geq 1$ such that $y_N \in [\alpha_K, \alpha_{K+1})$. We first consider the case $K = 1$; that is, $y_N \in \left[-\frac{1}{2}, -\frac{3}{8}\right)$. By Statement 2 of Lemma 5 we have $x_{N+5} = 8y_N + 2 \leq 0$, $y_{N+5} = -8y_N - 3 > 0$, and so it follows by Lemma 2 that $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. Without loss of generality we may assume $K \geq 2$. For each integer $m \geq 1$, let $\mathcal{R}(m)$ be the following statement:

$$\begin{aligned} x_{N+3m+2} &= 2^{2m+1}y_N + \frac{2^{2m+1}-2}{3} && < 0 \\ y_{N+3m+2} &= -2^{2m+1}y_N - \left(\frac{2^{2m+1}+1}{3}\right) && \leq 0. \end{aligned}$$

Claim: $\mathcal{R}(m)$ is true for $1 \leq m \leq K-1$.

The proof of the Claim will be by induction on m. We shall first show that $\mathcal{R}(1)$ is true. Recall that $x_N = 0$ and $y_N \in [\alpha_K, \alpha_{K+1}) \subset \left[-\frac{3}{8}, -\frac{1}{3}\right)$, and so it follows from Statement 2 of Lemma 5 that

$$\begin{aligned} x_{N+3(1)+2} &= 8y_N + 2 &&= 2^{2(1)+1}y_N + \frac{2^{2(1)+1}-2}{3} && < 0 \\ y_{N+3(1)+2} &= -8y_N - 3 &&= -2^{2(1)+1}y_N - \left(\frac{2^{2(1)+1}+1}{3}\right) && \leq 0 \end{aligned}$$

and so $\mathcal{R}(1)$ is true. Thus if $K = 2$, then we have shown that for $1 \leq m \leq K-1$, $\mathcal{R}(m)$ is true. It remains to consider the case $K \geq 3$. So assume that $K \geq 3$. Let m be an integer such that $1 \leq m \leq K-2$, and suppose $\mathcal{R}(m)$ is true. We shall show that $\mathcal{R}(m+1)$ is true.

Since $\mathcal{R}(m)$ is true we know

$$\begin{aligned} x_{N+3m+2} &= 2^{2m+1}y_N + \frac{2^{2m+1}-2}{3} && < 0 \\ y_{N+3m+2} &= -2^{2m+1}y_N - \left(\frac{2^{2m+1}+1}{3}\right) && \leq 0. \end{aligned}$$

Then

$$\begin{aligned} x_{N+3m+3} &= |x_{N+3m+2}| - y_{N+3m+2} - 1 \\ &= -2^{2m+1}y_N - \frac{2^{2m+1}-2}{3} - \left(-2^{2m+1}y_N - \frac{2^{2m+1}+1}{3}\right) - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} y_{N+3m+3} &= x_{N+3m+2} + |y_{N+3m+2}| \\ &= 2^{2m+1} y_N + \frac{2^{2m+1}-2}{3} + 2^{2m+1} y_N + \frac{2^{2m+1}+1}{3} \\ &= 2^{2m+2} y_N + \frac{2^{2m+2}-1}{3}. \end{aligned}$$

Recall that $y_N \in [\alpha_K, \alpha_{K+1}) = \left[\frac{-2^{2K-1}-1}{3 \cdot 2^{2K-1}}, \frac{-2^{2(K+1)-1}-1}{3 \cdot 2^{2(K+1)-1}}\right)$.

In particular,

$$\begin{aligned} y_{N+3m+3} &< 2^{2m+2}\left(\frac{-2^{2(K+1)-1}-1}{3 \cdot 2^{2(K+1)-1}}\right) + \frac{2^{2m+2}-1}{3} \\ &= -\frac{2^{2K+2m+3}}{3 \cdot 2^{2K+1}} - \frac{2^{2m+2}}{3 \cdot 2^{2K+1}} + \frac{2^{2m+2}}{3} - \frac{1}{3} \\ &= -\frac{1}{3}\left(1 + 2^{2m-2K+1}\right) \\ &< 0. \end{aligned}$$

Then

$$x_{N+3m+4} = |x_{N+3m+3}| - y_{N+3m+3} - 1 = 0 - y_{N+3m+3} - 1 = -y_{N+3m+3} - 1.$$

In particular,

$$\begin{aligned} x_{N+3m+4} &= -2^{2m+2} y_N - \frac{2^{2m+2}-1}{3} - 1 \\ &\leq -2^{2m+2}\left(\frac{-2^{2K-1}-1}{3 \cdot 2^{2K-1}}\right) - \left(\frac{2^{2m+2}-1}{3}\right) - 1 \\ &= \frac{2^{2m+2K+1}}{3 \cdot 2^{2K-1}} + \frac{2^{2m+2}}{3 \cdot 2^{2K-1}} - \frac{2^{2m+2}}{3} + \frac{1}{3} - 1 \\ &= -\frac{2}{3}\left(1 - 2^{2m-2K+2}\right) \\ &< 0. \end{aligned}$$

Hence

$$y_{N+3m+4} = x_{N+3m+3} + |y_{N+3m+3}| = 0 + (-y_{N+3m+3}) = -y_{N+3m+3} > 0.$$

Finally,

$$
\begin{aligned}
x_{N+3(m+1)+2} &= x_{N+3m+5} \\
&= |x_{N+3m+4}| - y_{N+3m+4} - 1 \\
&= y_{N+3m+3} + 1 - (-y_{N+3m+3}) - 1 \\
&= 2y_{N+3m+3} & < \quad 0 \\
&= 2^{2(m+1)+1} y_N + \frac{2^{2(m+1)+1} - 2}{3} & < \quad 0
\end{aligned}
$$

and

$$
\begin{aligned}
y_{N+3(m+1)+2} &= y_{N+3m+5} \\
&= x_{N+3m+4} + |y_{N+3m+4}| \\
&= -y_{N+3m+3} - 1 + (-y_{N+3m+3}) \\
&= -2y_{N+3m+3} - 1 \\
&= -2^{2(m+1)+1} y_N - \left(\frac{2^{2(m+1)+1} + 1}{3} \right).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
y_{N+3(m+1)+2} &\leq -2^{2m+3} \left(\frac{-2^{2K-1} - 1}{3 \cdot 2^{2K-1}} \right) - \left(\frac{2^{2m+3} + 1}{3} \right) \\
&= \frac{2^{2m+2K+2}}{3 \cdot 2^{2K-1}} + \frac{2^{2m+3}}{3 \cdot 2^{2K-1}} - \frac{2^{2m+3}}{3} - \frac{1}{3} \\
&= \frac{1}{3} \left(2^{2m-2K+4} - 1 \right) \\
&\leq 0
\end{aligned}
$$

and so $\mathcal{R}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{R}(m)$ is true for $1 \leq m \leq K-1$. Specifically, $\mathcal{R}(K-1)$ is true, and so

$$
\begin{aligned}
x_{N+3(K-1)+2} &= 2^{2(K-1)+1} y_N + \frac{2^{2(K-1)+1} - 2}{3} & < \quad 0 \\
y_{N+3(K-1)+2} &= -2^{2(K-1)+1} y_N - \left(\frac{2^{2(K-1)+1} + 1}{3} \right) & \leq \quad 0.
\end{aligned}
$$

Then

$$\begin{aligned} x_{N+3K} &= x_{N+3(K-1)+3} \\ &= |x_{N+3(K-1)+2}| - y_{N+3(K-1)+2} - 1 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} y_{N+3K} &= y_{N+3(K-1)+3} \\ &= x_{N+3(K-1)+2} + |y_{N+3(K-1)+2}| \\ &= 2^{2K} y_N + \frac{2^{2K}-1}{3}. \end{aligned}$$

Note that

$$2^{2K}\left(\frac{-2^{2K-1}-1}{3\cdot 2^{2K-1}}\right) + \frac{2^{2K}-1}{3} \quad \leq \quad y_{N+3K} \quad < \quad 2^{2K}\left(\frac{-2^{2K+1}-1}{3\cdot 2^{2K+1}}\right) + \frac{2^{2K}-1}{3}.$$

So as

$$2^{2K}\left(\frac{-2^{2K-1}-1}{3\cdot 2^{2K-1}}\right) + \frac{2^{2K}-1}{3} = \frac{-2^{4K-1}}{3\cdot 2^{2K-1}} + \frac{2^{2K}}{3\cdot 2^{2K-1}} + \frac{2^{2K}}{3} - \frac{1}{3} = -\frac{2}{3} - \frac{1}{3} = -1$$

and

$$2^{2K}\left(\frac{-2^{2K+1}-1}{3\cdot 2^{2K+1}}\right) + \frac{2^{2K}-1}{3} = \frac{-2^{4K+1}}{3\cdot 2^{2K+1}} + \frac{2^{2K}}{3} - \frac{1}{3} = -\frac{1}{6} - \frac{1}{3} = -\frac{1}{2}$$

we have

$$-1 \leq y_{N+3K} < -\frac{1}{2}$$

and so it follows by Statement 3 of Lemma 5 and the fact $(0,-1) \in \mathbf{P}_3^1$ that the solution $\{(x_n, y_n)\}_{n=N+3K+3}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

(4) Suppose $y_N = -\frac{4}{3}$. By direct computations we have $(x_{N+3}, y_{N+3}) = (-\frac{2}{3}, -\frac{1}{3}) \in \mathbf{P}_3^2$, and so $\{(x_n, y_n)\}_{n=N+3}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^2$.

(5) Suppose $y_N < -\frac{4}{3}$.

First consider the case $-\frac{3}{2} \leq y_N < -\frac{4}{3}$. For each integer $n \geq 0$, let

$$\beta_n = \frac{-2^{2n+3} - 1}{3 \cdot 2^{2n+1}}.$$

Observe that

$$-\frac{3}{2} = \beta_0 < \beta_1 < \beta_2 < \ldots < -\frac{4}{3} \quad \text{and} \quad \lim_{n\to\infty} \beta_n = -\frac{4}{3}.$$

Thus there exists a unique integer $K \geq 1$ such that $y_N \in [\beta_{K-1}, \beta_K)$. We first consider the case $K = 1$; that is, $y_N \in \left[-\frac{3}{2}, -\frac{11}{8}\right)$. By Statements 1 and 2 of Lemma 4 we have

$$\begin{aligned} x_{N+3} &= 2y_N + 2 < 0 \\ y_{N+3} &= -2y_N - 3 \leq 0 \end{aligned}$$

and so

$$\begin{aligned} x_{N+4} &= |x_{N+3}| - y_{N+3} - 1 = 0 \\ y_{N+4} &= x_{N+3} + |y_{N+3}| = 4y_N + 5 < 0. \end{aligned}$$

In particular, $-1 \leq y_{N+4} < -\frac{1}{2}$. It follows by Statement 3 of Lemma 5 that the solution $\{(x_n, y_n)\}_{n=N+7}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Thus without loss of generality, we may assume that $K \geq 2$. For each integer $m \geq 1$, let $\mathcal{S}(m)$ be the following statement:

$$\begin{aligned} x_{N+3m+3} &= 2^{2m+1} y_N + \frac{2^{2m+3} - 2}{3} < 0 \\ y_{N+3m+3} &= -2^{2m+1} y_N - \left(\frac{2^{2m+3} - 2}{3}\right) - 1 \leq 0. \end{aligned}$$

Claim: $\mathcal{S}(m)$ is true for $1 \leq m \leq K - 1$.
The proof of the Claim will be by induction on m. We shall first show that $\mathcal{S}(1)$ is true. Recall that $x_N = 0$ and $y_N \in [\beta_{K-1}, \beta_K) \subset \left[-\frac{11}{8}, -\frac{4}{3}\right)$, and so by Statements 1 and 2 of Lemma 4 we have

$$x_{N+3} = 2y_N + 2 < 0$$

$$\begin{aligned}
y_{N+3} &= -2y_N - 3 < 0 \\
x_{N+4} &= |x_{N+3}| - y_{N+3} - 1 = 0 \\
y_{N+4} &= x_{N+3} + |y_{N+3}| = 4y_N + 5 < 0 \\
x_{N+5} &= |x_{N+4}| - y_{N+4} - 1 = -4y_N - 6 < 0 \\
y_{N+5} &= x_{N+4} + |y_{N+4}| = -4y_N - 5 > 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
x_{N+3(1)+3} &= x_{N+6} = |x_{N+5}| - y_{N+5} - 1 = 8y_N + 10 < 0 \\
y_{N+3(1)+3} &= y_{N+6} = x_{N+5} + |y_{N+5}| = -8y_N - 11 \leq 0.
\end{aligned}$$

It follows that $\mathcal{S}(1)$ is true. Thus if $K = 2$, then we have shown that for $1 \leq m \leq K-1$, $\mathcal{S}(m)$ is true. It remains to consider the case $K \geq 3$. So assume that $K \geq 3$. Let m be an integer such that $1 \leq m \leq K-2$, and suppose $\mathcal{S}(m)$ is true. We shall show that $\mathcal{S}(m+1)$ is true.

Since $\mathcal{S}(m)$ is true, we know

$$\begin{aligned}
x_{N+3m+3} &= 2^{2m+1}y_N + \frac{2^{2m+3}-2}{3} < 0 \\
y_{N+3m+3} &= -2^{2m+1}y_N - \left(\frac{2^{2m+3}-2}{3}\right) - 1 \leq 0.
\end{aligned}$$

Note that $y_{N+3m+3} = -x_{N+3m+3} - 1$, and so $-1 \leq x_{N+3m+3} < 0$.

Thus

$$\begin{aligned}
x_{N+3m+4} &= |x_{N+3m+3}| - y_{N+3m+3} - 1 \\
&= -x_{N+3m+3} - (-x_{N+3m+3} - 1) - 1 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned} y_{N+3m+4} &= x_{N+3m+3} + |y_{N+3m+3}| \\ &= x_{N+3m+3} + x_{N+3m+3} + 1 \\ &= 2x_{N+3m+3} + 1. \end{aligned}$$

Recall that $y_N \in [\beta_{K-1}, \beta_K) = \left[\frac{-2^{2(K-1)+3} - 1}{3 \cdot 2^{2(K-1)+1}}, \frac{-2^{2K+3} - 1}{3 \cdot 2^{2K+1}}\right)$.

In particular,

$$\begin{aligned} y_{N+3m+4} &= 2\left[2^{2m+1} y_N + \frac{2^{2m+3} - 2}{3}\right] + 1 \\ &< 2\left[2^{2m+1}\left(\frac{-2^{2K+3} - 1}{3 \cdot 2^{2K+1}}\right) + \frac{2^{2m+3} - 2}{3}\right] + 1 \\ &= -\frac{2^{2K+2m+5}}{3 \cdot 2^{2K+1}} - \frac{2^{2m+2}}{3 \cdot 2^{2K+1}} + \frac{2^{2m+4}}{3} - \frac{1}{3} \\ &= -\frac{1}{3}\left(2^{2m-2K+1} + 1\right) \\ &< 0. \end{aligned}$$

Also note that $-1 < x_{N+3m+3} < -\frac{1}{2}$.

Thus

$$\begin{aligned} x_{N+3m+5} &= |x_{N+3m+4}| - y_{N+3m+4} - 1 \\ &= 0 - (2x_{N+3m+3} + 1) - 1 \\ &= -2x_{N+3m+3} - 2 \\ &< 0 \end{aligned}$$

and

$$\begin{aligned}
y_{N+3m+5} &= x_{N+3m+4} + |y_{N+3m+4}| \\
&= 0 + (-2x_{N+3m+3} - 1) \\
&= -2x_{N+3m+3} - 1 \\
&> 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
x_{N+3(m+1)+3} &= x_{N+3m+6} \\
&= |x_{N+3m+5}| - y_{N+3m+5} - 1 \\
&= 2x_{N+3m+3} + 2 - (-2x_{N+3m+3} - 1) - 1 \\
&= 4x_{N+3m+3} + 2 \qquad < \quad 0 \\
&= 4\left[2^{2m+1}y_N + \left(\frac{2^{2m+3}-2}{3}\right)\right] + 2 \qquad < \quad 0 \\
&= 2^{2(m+1)+1}y_N + \left(\frac{2^{2(m+1)+3}-2}{3}\right) + 2 \quad < \quad 0
\end{aligned}$$

and

$$\begin{aligned}
y_{N+3(m+1)+3} &= y_{N+3m+6} \\
&= x_{N+3m+5} + |y_{N+3m+5}| \\
&= -2x_{N+3m+3} - 2 + (-2x_{N+3m+3} - 1) \\
&= -4x_{N+3m+3} - 3 \\
&= -4\left[2^{2m+1}y_N + \left(\frac{2^{2m+3}-2}{3}\right)\right] - 3 \\
&= -2^{2(m+1)+1}y_N - \left(\frac{2^{2(m+1)+3}-2}{3}\right) - 1.
\end{aligned}$$

In particular,

$$\begin{aligned} y_{N+3m+6} &\le -4\left[2^{2m+1}\left(\frac{-2^{2(K-1)+3}-1}{3\cdot 2^{2(K-1)+1}}\right)+\frac{2^{2m+3}-2}{3}\right]-3 \\ &= \frac{2^{2K+2m+4}}{3\cdot 2^{2K-1}}+\frac{2^{2m+3}}{3\cdot 2^{2K-1}}-\frac{2^{2m+5}}{3}-\frac{1}{3} \\ &= \frac{1}{3}\left(2^{2m-2K+4}-1\right) \\ &< 0 \end{aligned}$$

and so $\mathcal{S}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{S}(m)$ is true for $1 \le m \le K-1$. Specifically, $\mathcal{S}(K-1)$ is true, and so

$$x_{N+3(K-1)+3} = x_{N+3K} = 2^{2K-1}y_N + \frac{2^{2K+1}-2}{3} < 0$$

$$y_{N+3(K-1)+3} = y_{N+3K} = -2^{2K-1}y_N - \left(\frac{2^{2K+1}-2}{3}\right) - 1 < 0.$$

Note that $y_{N+3K} = -x_{N+3K} - 1$.

Thus

$$\begin{aligned} x_{N+3K+1} &= |x_{N+3K}| - y_{N+3K} - 1 \\ &= -x_{N+3K} - (-x_{N+3K} - 1) - 1 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} y_{N+3K+1} &= x_{N+3K} + |y_{N+3K}| \\ &= x_{N+3K} + x_{N+3K} + 1 \\ &= 2x_{N+3K} + 1 \\ &= 2\left(2^{2K-1}y_N + \frac{2^{2K+1}-2}{3}\right) + 1. \end{aligned}$$

Note that

$$2\left[2^{2K-1}\left(\frac{-2^{2(K-1)+3}-1}{3\cdot 2^{2(K-1)+1}}\right)+\frac{2^{2K+1}-2}{3}\right]+1 \leq y_{N+3K+1}$$

$$< 2\left[2^{2K-1}\left(\frac{-2^{2K+3}-1}{3\cdot 2^{2K+1}}\right)+\frac{2^{2K+1}-2}{3}\right]+1.$$

So as

$$2\left[2^{2K-1}\left(\frac{-2^{2(K-1)+3}-1}{3\cdot 2^{2(K-1)+1}}\right)+\frac{2^{2K+1}-2}{3}\right]+1 = \frac{-2^{4K+1}}{3\cdot 2^{2K-1}}-\frac{2^{2K}}{3\cdot 2^{2K-1}}+\frac{2^{2K+2}}{3}-\frac{1}{3}$$

$$= -\frac{1}{3}(2+1) = -1$$

and

$$2\left[2^{2K-1}\left(\frac{-2^{2K+3}-1}{3\cdot 2^{2K+1}}\right)+\frac{2^{2K+1}-2}{3}\right]+1 = \frac{-2^{2K+3}}{3\cdot 2}-\frac{1}{6}+\frac{2^{2K+2}}{3}-\frac{1}{3}$$

$$= -\frac{1}{6}-\frac{1}{3} = -\frac{1}{2}$$

we have

$$-1 \leq y_{N+3K+1} < -\frac{1}{2}$$

and hence it follows from case 3 of this Lemma and the fact that $(0,-1)\in \mathbf{P}_3^1$ that the solution $\{(x_n,y_n)\}_{n=N+3K+5}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$.

Finally, suppose $y_N < -\frac{3}{2}$. Then by Statement 3 of Lemma 4 the solution $\{(x_n,y_n)\}_{n=N+4}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

□

Lemma 7. Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N,y_N)\in Q_1$. Then $\{(x_n,y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Proof. We have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 &= x_N - y_N - 1 \\ y_{N+1} &= x_N + |y_N| &= x_N + y_N &> 0. \end{aligned}$$

If $x_{N+1} \geq 0$ then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} - 1 &= -2y_N - 2 &< 0 \\ y_{N+2} &= x_{N+1} + |y_{N+1}| &= 2x_N - 1 &> 0 \\ x_{N+3} &= |x_{N+2}| - y_{N+2} - 1 &= -2x_N + 2y_N + 2 &\leq 0 \\ y_{N+3} &= x_{N+2} + |y_{N+2}| &= 2x_N - 2y_N - 3 \\ x_{N+4} &= |x_{N+3}| - y_{N+3} - 1 &= 0 \end{aligned}$$

and so $(x_{N+4}, y_{N+4}) \in l_2 \cup l_4$. By Lemmas 3 and 6, the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

If $x_{N+1} < 0$ then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} - 1 &= -2x_N &< 0 \\ y_{N+2} &= x_{N+1} + |y_{N+1}| &= 2x_N - 1 \\ x_{N+3} &= |x_{N+2}| - y_{N+2} - 1 &= 0 \end{aligned}$$

and so $(x_{N+3}, y_{N+3}) \in l_2 \cup l_4$. By Lemmas 3 and 6, the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. □

Lemma 8. Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in Q_2$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Proof. We have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 &= -x_N - y_N - 1 \\ y_{N+1} &= x_N + |y_N| &= x_N + y_N. \end{aligned}$$

Case 1: Suppose $y_{N+1} \geq 0$. Then by Lemma 2, the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Case 2: Suppose $y_{N+1} < 0$ and $x_{N+1} \leq 0$. Then $x_{N+2} = |x_{N+1}| - y_{N+1} - 1 = 0$ and so $(x_{N+2}, y_{N+2}) \in l_2 \cup l_4$. By Lemmas 3 and 6, the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Case 3: Suppose $y_{N+1} < 0$ and $x_{N+1} > 0$. Then

$$\begin{aligned}
x_{N+2} &= |x_{N+1}| - y_{N+1} - 1 &=& -2x_N - 2y_N - 2 &>& 0 \\
y_{N+2} &= x_{N+1} + |y_{N+1}| &=& -2x_N - 2y_N - 1 &>& 0 \\
x_{N+3} &= |x_{N+2}| - y_{N+2} - 1 &=& -2 && \\
y_{N+3} &= x_{N+2} + |y_{N+2}| &=& -4x_N - 4y_N - 3 &>& 0 \\
x_{N+4} &= |x_{N+3}| - y_{N+3} - 1 &=& 4x_N + 4y_N + 4 &<& 0 \\
y_{N+4} &= x_{N+3} + |y_{N+3}| &=& -4x_N - 4y_N - 5 && \\
x_{N+5} &= |x_{N+4}| - y_{N+4} - 1 &=& 0 &&
\end{aligned}$$

and so $(x_{N+5}, y_{N+5}) \in l_2 \cup l_4$. By Lemmas 3 and 6, the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. □

Lemma 9. Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in Q_4$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Proof. We have

$$\begin{aligned}
x_{N+1} &= |x_N| - y_N - 1 &=& x_N - y_N - 1 && \\
y_{N+1} &= x_N + |y_N| &=& x_N - y_N &>& 0
\end{aligned}$$

Case 1: Suppose $x_{N+1} > 0$. Then $(x_{N+1}, y_{N+1}) \in Q_1$ and so by Lemma 7, the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Case 2: Suppose $x_{N+1} = 0$. Then $(x_{N+1}, y_{N+1}) \in l_2$ and so by Lemma 3, the solution $\{(x_n, y_n)\}_{n=N+4}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Case 3: Suppose $x_{N+1} < 0$. Then $(x_{N+1}, y_{N+1}) \in Q_2$ and so by Lemma 8, the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

□

Lemma 10. Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in l_1$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or $\mathbf{P}_3^2$.

Proof. We have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 = x_N - 1 \\ y_{N+1} &= x_N + |y_N| = x_N \end{aligned}$$

Case 1: Suppose $x_N = 0$. Then $(x_{N+1}, y_{N+1}) = (-1, 0)$, and so $(x_{N+2}, y_{N+2}) = (0, -1)$. Hence the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Case 2: Suppose $0 < x_N \leq 1$. Then $x_{N+1} \leq 0$ and $y_{N+1} > 0$. Thus $(x_{N+1}, y_{N+1}) \in Q_2 \cup l_2$, and hence by Lemmas 3 and 8, the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Case 3: Suppose $x_N > 1$. Then $x_{N+1} > 0$ and $y_{N+1} > 0$. Thus $(x_{N+1}, y_{N+1}) \in Q_1$ and by Lemma 7, the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. □

Lemma 11. Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in l_3$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Proof. We have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 = -x_N - 1 \\ y_{N+1} &= x_N + |y_N| = x_N < 0. \end{aligned}$$

Case 1: Suppose $-1 < x_N \leq 0$. Then $x_{N+2} = |x_{N+1}| - y_{N+1} - 1 = 0$, and so $(x_{N+2}, y_{N+2}) \in l_2 \cup l_4$. It follows by Lemmas 3 and 6, that the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Case 2: Suppose $x_N = -1$. Then $(x_{N+1}, y_{N+1}) = (0, -1) \in \mathbf{P}_3^1$, and so the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Case 3: Suppose $x_N < -1$. Then $(x_{N+1}, y_{N+1}) \in Q_4 \cup l_1$. It follows by Lemmas 9 and 10, the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. □

To complete the proof of Theorem 2.1 it remains to consider the case where the initial condition $(x_0, y_0) \in Q_3$.

Lemma 12. Suppose $(x_0, y_0) \in Q_3$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium solution $(\bar{x}, \bar{y}) = \left(-\frac{2}{5}, -\frac{1}{5}\right)$, or else is eventually either the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Proof. If $(x_0, y_0) = \left(-\frac{2}{5}, -\frac{1}{5}\right)$, then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the equilibrium. So suppose $(x_0, y_0) \in Q_3 \setminus \left\{\left(-\frac{2}{5}, -\frac{1}{5}\right)\right\}$. It suffices to show that there exists an integer $N \geq 0$ such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is either the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

For the sake of contradiction, assume that it is false that there exists an integer $N \geq 0$ such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is either the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. It follows from the previous lemmas that $x_n < 0$ and $y_n < 0$ for every integer $n \geq 0$.

Case 1: Suppose $x_0 \leq -2$ and $y_0 < 0$. Then

$$x_1 = |x_0| - y_0 - 1 = -x_0 - y_0 - 1 > 0$$

which is a contradiction, and the proof is complete.

Case 2: Suppose $-2 < x_0 < 0$ and $y_0 \leq -1$. Then

$$x_1 = |x_0| - y_0 - 1 = -x_0 - y_0 - 1 > 0$$

which is a contradiction, and the proof is complete.

Case 3: It remains to consider the case $(x_0, y_0) \in (-2, 0) \times (-1, 0)$. For each integer $n \geq 0$, let

$$a_n = \frac{-2^{4n-2} - 1}{5 \cdot 2^{4n-3}}, \quad b_n = \frac{-2^{4n} + 1}{5 \cdot 2^{4n-1}}, \quad c_n = \frac{-2^{4n-2} - 1}{5 \cdot 2^{4n-2}}, \quad d_n = \frac{-2^{4n} + 1}{5 \cdot 2^{4n}} \quad \text{and} \quad D_n = \frac{2^{4n} - 1}{5}.$$

Observe that

$$-2 = a_0 < a_1 < a_2 < \ldots < -\frac{2}{5} \qquad \text{and} \qquad \lim_{n \to \infty} a_n = -\frac{2}{5}$$

$$0 = b_0 > b_1 > b_2 > \ldots > -\frac{2}{5} \qquad \text{and} \qquad \lim_{n \to \infty} b_n = -\frac{2}{5}$$

$$-1 = c_0 < c_1 < c_2 < \ldots < -\frac{1}{5} \qquad \text{and} \qquad \lim_{n \to \infty} c_n = -\frac{1}{5}$$

$$0 = d_0 > d_1 > d_2 > \ldots > -\frac{1}{5} \qquad \text{and} \qquad \lim_{n \to \infty} d_n = -\frac{1}{5}.$$

There exists a unique integer $K \geq 0$ such that

$$(x_0, y_0) \in [a_K, b_K] \times [c_K, d_K] \setminus [a_{K+1}, b_{K+1}] \times [c_{K+1}, d_{K+1}].$$

We first consider the case $K = 0$; that is, $(x_0, y_0) \in [-2, 0] \times [-1, 0] \setminus [-\frac{1}{2}, -\frac{3}{8}] \times [-\frac{1}{4}, -\frac{3}{16}]$. Note that by Lemmas 6 and 11, and by Case 1 and Case 2 of this lemma, we know that the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually either the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$ when (x_0, y_0) is an element of the outer boundaries of $[-2, 0] \times [-1, 0]$.

Recall by assumption that $x_n < 0$ and $y_n < 0$ for every integer $n \geq 0$.

So suppose $(x_0, y_0) \in (-2, 0) \times (-1, 0) \setminus \left[-\frac{1}{2}, -\frac{3}{8}\right] \times \left[-\frac{1}{4}, -\frac{3}{16}\right]$. Then

$$x_1 = |x_0| - y_0 - 1 = -x_0 - y_0 - 1$$

$$y_1 = x_0 + |y_0| = x_0 - y_0$$

$$x_2 = |x_1| - y_1 - 1 = (x_0 + y_0 + 1) - (x_0 - y_0) - 1 = 2y_0$$

$$y_2 = x_1 + |y_1| = (-x_0 - y_0 - 1) + (-x_0 + y_0) = -2x_0 - 1.$$

If $-2 < x_0 < -\frac{1}{2}$, then $y_2 > 0$ which is a contradiction.

Thus $-\frac{1}{2} \leq x_0 < 0$. Then

$$x_3 = |x_2| - y_2 - 1 = (-2y_0) - (-2x_0 - 1) - 1 = 2x_0 - 2y_0$$

$$y_3 = x_2 + |y_2| = (2y_0) + (2x_0 + 1) = 2x_0 + 2y_0 + 1$$

$$x_4 = |x_3| - y_3 - 1 = (-2x_0 + 2y_0) - (2x_0 + 2y_0 + 1) - 1 = -4x_0 - 2$$

$$y_4 = x_3 + |y_3| = (2x_0 - 2y_0) + (-2x_0 - 2y_0 - 1) = -4y_0 - 1.$$

If $-1 < y_0 < -\frac{1}{4}$, then $y_4 > 0$ which is a contradiction.

Hence $-\frac{1}{4} \leq y_0 < 0$. Then

$$x_5 = |x_4| - y_4 - 1 = (4x_0 + 2) - (-4y_0 - 1) - 1 = 4x_0 + 4y_0 + 2$$

$$y_5 = x_4 + |y_4| = (-4x_0 - 2) + (4y_0 + 1) = -4x_0 + 4y_0 - 1$$

$$x_6 = |x_5| - y_5 - 1 = (-4x_0 - 4y_0 - 2) - (-4x_0 + 4y_0 - 1) - 1 = -8y_0 - 2$$

$$y_6 = x_5 + |y_5| = (4x_0 + 4y_0 + 2) + (4x_0 - 4y_0 + 1) = 8x_0 + 3.$$

If $-\frac{3}{8} < x_0 < 0$, then $y_6 > 0$ which is a contradiction.

Hence $-\frac{1}{2} < x_0 \leq -\frac{3}{8}$. Thus

$$x_7 = |x_6| - y_6 - 1 = (8y_0 + 2) - (8x_0 + 3) - 1 = -8x_0 + 8y_0 - 2$$

$$y_7 = x_6 + |y_6| = (-8y_0 - 2) + (-8x_0 - 3) = -8x_0 - 8y_0 - 5$$

$$x_8 = |x_7| - y_7 - 1 = (8x_0 - 8y_0 + 2) - (-8x_0 - 8y_0 - 5) - 1 = 16x_0 + 6$$

$$y_8 = x_7 + |y_7| = (-8x_0 + 8y_0 - 2) + (8x_0 + 8y_0 + 5) = 16y_0 + 3 > 0,$$

which is a contradiction. Thus the case $K = 0$ is complete.

Next consider the case $K \geq 1$. Recall that $x_n < 0$ and $y_n < 0$ for all $n \geq 0$.

For each integer m such that $0 \leq m \leq K - 1$, let $\mathcal{P}(m)$ be the following proposition:

$$\begin{aligned}
x_{8m+1} &= -2^{4m}x_0 - 2^{4m}y_0 - 3D_m - 1 \\
y_{8m+1} &= 2^{4m}x_0 - 2^{4m}y_0 + D_m \\
\\
x_{8m+2} &= 2^{4m+1}y_0 + 2D_m \\
y_{8m+2} &= -2^{4m+1}x_0 - 4D_m - 1 \\
\\
x_{8m+3} &= 2^{4m+1}x_0 - 2^{4m+1}y_0 + 2D_m \\
y_{8m+3} &= 2^{4m+1}x_0 + 2^{4m+1}y_0 + 6D_m + 1 \\
\\
x_{8m+4} &= -2^{4m+2}x_0 - 8D_m - 2 \\
y_{8m+4} &= -2^{4m+2}y_0 - 4D_m - 1 \\
\\
x_{8m+5} &= 2^{4m+2}x_0 + 2^{4m+2}y_0 + 12D_m + 2 \\
y_{8m+5} &= -2^{4m+2}x_0 + 2^{4m+2}y_0 - 4D_m - 1 \\
\\
x_{8m+6} &= -2^{4m+3}y_0 - 8D_m - 2 \\
y_{8m+6} &= 2^{4m+3}x_0 + 16D_m + 3
\end{aligned}$$

$$
\begin{aligned}
x_{8m+7} &= -2^{4m+3}x_0 + 2^{4m+3}y_0 - 8D_m - 2 \\
y_{8m+7} &= -2^{4m+3}x_0 - 2^{4m+3}y_0 - 24D_m - 5 \\
\\
x_{8m+8} &= 2^{4m+4}x_0 + 32D_m + 6 \\
y_{8m+8} &= 2^{4m+4}x_0 + 16D_m + 3.
\end{aligned}
$$

Claim: $\mathcal{P}(m)$ is true for $0 \leq m \leq K-1$. The proof of the Claim will be by induction on m. We shall first show that $\mathcal{P}(0)$ is true.

$$
\begin{aligned}
x_{8(0)+1} &= -x_0 - y_0 - 1 &&= -2^{4(0)}x_0 - 2^{4(0)}y_0 - 3D_0 - 1 \\
y_{8(0)+1} &= x_0 - y_0 &&= 2^{4(0)}x_0 - 2^{4(0)}y_0 - D_0 \\
\\
x_{8(0)+2} &= 2y_0 &&= 2^{4(0)+1}y_0 + 2D_0 \\
y_{8(0)+2} &= -2x_0 - 1 &&= -2^{4(0)+1}x_0 - 4D_0 - 1 \\
\\
x_{8(0)+3} &= 2x_0 - 2y_0 &&= 2^{4(0)+1}x_0 - 2^{4(0)+1}y_0 + 2D_0 \\
y_{8(0)+3} &= 2x_0 + 2y_0 + 1 &&= 2^{4(0)+1}x_0 + 2^{4(0)+1}y_0 + 6D_0 + 1 \\
\\
x_{8(0)+4} &= -4x_0 - 2 &&= -2^{4(0)+2}x_0 - 8D_0 - 2 \\
y_{8(0)+4} &= -4y_0 - 1 &&= -2^{4(0)+2}x_0 - 4D_0 - 1 \\
\\
x_{8(0)+5} &= 4x_0 + 4y_0 + 2 &&= 2^{4(0)+1}x_0 - 2^{4(0)+2}y_0 + 12D_0 + 2 \\
y_{8(0)+5} &= -4x_0 + 4y_0 - 1 &&= -2^{4(0)+2}x_0 + 2^{4(0)+2}y_0 - 4D_0 - 1 \\
\\
x_{8(0)+6} &= -8x_0 - 2 &&= -2^{4(0)+3}x_0 - 8D_0 - 2 \\
y_{8(0)+6} &= 8y_0 + 3 &&= 2^{4(0)+3}x_0 + 16D_0 + 3 \\
\\
x_{8(0)+7} &= -8x_0 + 8y_0 - 2 &&= -2^{4(0)+3}x_0 + 2^{4(0)+3}y_0 - 8D_0 - 2 \\
y_{8(0)+7} &= -8x_0 - 8y_0 - 5 &&= -2^{4(0)+3}x_0 - 2^{4(0)+3}y_0 - 24D_0 + 5 \\
\\
x_{8(0)+8} &= 16x_0 + 6 &&= 2^{4(0)+4}x_0 + 32D_0 + 6 \\
y_{8(0)+8} &= 16y_0 + 3 &&= 2^{4(0)+4}x_0 + 16D_0 + 3
\end{aligned}
$$

and so $\mathcal{P}(0)$ is true. Thus if $K = 1$, then we have shown that for $0 \leq m \leq K-1$, $\mathcal{P}(m)$ is true. It remains to consider the case $K \geq 2$. So assume that $K \geq 2$. Suppose that m is an integer such that $0 \leq m \leq K-2$, and that $\mathcal{P}(m)$ is true. We shall show that $\mathcal{P}(m+1)$ is true.

Since $\mathcal{P}(m)$ is true, we know

$$x_{8m+8} = 2^{4m+4}x_0 + 32D_m + 6$$

$$y_{8m+8} = 2^{4m+4}x_0 + 16D_m + 3.$$

Hence

$$\begin{aligned}
x_{8(m+1)+1} &= x_{8m+9} \\
&= |x_{8m+8}| - y_{8m+8} - 1 \\
&= -(2^{4m+4}x_0 + 32D_m + 6) - (2^{4m+4}y_0 + 16D_m + 3) - 1 \\
&= -2^{4m+4}x_0 - 2^{4m+4}y_0 - 48D_m - 10 \\
&= -2^{4m+4}x_0 - 2^{4m+4}y_0 - 48\left(\frac{2^{4m}-1}{5}\right) - 10 \\
&= -2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 - 3\left(\frac{2^{4(m+1)}-1}{5}\right) - 1 \\
&= -2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 - 3D_{m+1} - 1
\end{aligned}$$

and

$$\begin{aligned}
y_{8(m+1)+1} &= y_{8m+9} \\
&= x_{8m+8} + |y_{8m+8}| \\
&= 2^{4m+4}x_0 + 32D_m + 6 + (-2^{4m+4}y_0 - 16D_m - 3) \\
&= 2^{4m+4}x_0 - 2^{4m+4}y_0 + 16D_m + 3 \\
&= 2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 + 16\left(\frac{2^{4m}-1}{5}\right) + 3 \\
&= 2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 + D_{m+1}.
\end{aligned}$$

Thus

$$\begin{aligned}
x_{8(m+1)+2} &= x_{8m+10} \\
&= |x_{8m+9}| - y_{8m+9} - 1 \\
&= -(-2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 - 3D_{m+1} - 1) \\
&\quad -(2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 + D_{m+1}) - 1 \\
&= 2^{4(m+1)+1}y_0 + 2D_{m+1}
\end{aligned}$$

and

$$\begin{aligned}
y_{8(m+1)+2} &= y_{8m+10} \\
&= x_{8m+9} + |y_{8m+9}| \\
&= -2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 - 3D_{m+1} - 1 + (-2^{4(m+1)}x_0 + 2^{4(m+1)}y_0 - D_{m+1}) \\
&= -2^{4(m+1)+1}x_0 - 4D_{m+1} - 1.
\end{aligned}$$

Then

$$\begin{aligned}
x_{8(m+1)+3} &= x_{8m+11} \\
&= |x_{8m+10}| - y_{8m+10} - 1 \\
&= -2^{4(m+1)+1}y_0 - 2D_{m+1} + 2^{4(m+1)+1}x_0 + 4D_{m+1} + 1 - 1 \\
&= 2^{4(m+1)+1}x_0 - 2^{4(m+1)+1}y_0 + 2D_{m+1}
\end{aligned}$$

and

$$\begin{aligned}
y_{8(m+1)+3} &= y_{8m+11} \\
&= x_{8m+10} + |y_{8m+10}| \\
&= 2^{4(m+1)+1}y_0 + 2D_{m+1} + 2^{4(m+1)+1}x_0 + 4D_{m+1} + 1 \\
&= 2^{4(m+1)+1}x_0 + 2^{4(m+1)+1}y_0 + 6D_{m+1} + 1.
\end{aligned}$$

Hence

$$\begin{aligned} x_{8(m+1)+4} &= x_{8m+12} \\ &= |x_{8m+11}| - y_{8m+11} - 1 \\ &= -2^{4(m+1)+1}x_0 + 2^{4(m+1)+1}y_0 - 2D_{m+1} - 2^{4(m+1)+1}x_0 \\ &\quad -2^{4(m+1)+1}y_0 - 6D_{m+1} - 2 \\ &= -2^{4(m+1)+2}x_0 - 8D_{m+1} - 2 \end{aligned}$$

and

$$\begin{aligned} y_{8(m+1)+4} &= y_{8m+12} \\ &= x_{8m+11} + |y_{8m+11}| \\ &= 2^{4(m+1)+1}x_0 - 2^{4(m+1)+1}y_0 + 2D_{m+1} - 2^{4(m+1)+1}x_0 \\ &\quad -2^{4(m+1)+1}y_0 - 6D_{m+1} - 1 \\ &= -2^{4(m+1)+2}y_0 - 4D_{m+1} - 1. \end{aligned}$$

Thus

$$\begin{aligned} x_{8(m+1)+5} &= x_{8m+13} \\ &= |x_{8m+12}| - y_{8m+12} - 1 \\ &= 2^{4(m+1)+2}x_0 + 8D_{m+1} + 2 + 2^{4(m+1)+2}y_0 + 4D_{m+1} + 1 - 1 \\ &= 2^{4(m+1)+2}x_0 + 2^{4(m+1)+2}y_0 + 12D_{m+1} + 2 \end{aligned}$$

and

$$\begin{aligned} y_{8(m+1)+5} &= y_{8m+13} \\ &= x_{8m+12} + |y_{8m+12}| \\ &= -2^{4(m+1)+2}x_0 - 8D_{m+1} - 2 + 2^{4(m+1)+2}y_0 + 4D_{m+1} + 1 \\ &= -2^{4(m+1)+2}x_0 + 2^{4(m+1)+2}y_0 - 4D_{m+1} - 1. \end{aligned}$$

Hence

$$\begin{aligned} x_{8(m+1)+6} &= x_{8m+14} \\ &= |x_{8m+13}| - y_{8m+13} - 1 \\ &= -2^{4(m+1)+2}x_0 - 2^{4(m+1)+2}y_0 - 12D_{m+1} - 2 \\ &\quad +2^{4(m+1)+2}x_0 - 2^{4(m+1)+2}y_0 + 4D_{m+1} + 1 - 1 \\ &= -2^{4(m+1)+3}y_0 - 8D_{m+1} - 2 \end{aligned}$$

and

$$\begin{aligned} y_{8(m+1)+6} &= y_{8m+14} \\ &= x_{8m+13} + |y_{8m+13}| \\ &= 2^{4(m+1)+2}x_0 + 2^{4(m+1)+2}y_0 + 12D_{m+1} + 2 + 2^{4(m+1)+2}x_0 \\ &\quad -2^{4(m+1)+2}y_0 + 4D_{m+1} + 1 \\ &= 2^{4(m+1)+3}x_0 + 16D_{m+1} + 3. \end{aligned}$$

Then

$$\begin{aligned} x_{8(m+1)+7} &= x_{8m+15} \\ &= |x_{8m+14}| - y_{8m+14} - 1 \\ &= 2^{4(m+1)+3}y_0 + 8D_{m+1} + 2 - 2^{4(m+1)+3}x_0 - 16D_{m+1} - 3 - 1 \\ &= -2^{4(m+1)+3}x_0 + 2^{4(m+1)+3}y_0 - 8D_{m+1} - 2 \end{aligned}$$

and

$$\begin{aligned} y_{8(m+1)+7} &= y_{8m+15} \\ &= x_{8m+14} + |y_{8m+14}| \\ &= -2^{4(m+1)+3}y_0 - 8D_{m+1} - 2 - 2^{4(m+1)+3}x_0 - 16D_{m+1} - 3 \\ &= -2^{4(m+1)+3}x_0 - 2^{4(m+1)+3}y_0 - 24D_{m+1} - 5. \end{aligned}$$

Thus

$$\begin{aligned} x_{8(m+1)+8} &= x_{8m+16} \\ &= |x_{8m+15}| - y_{8m+15} - 1 \\ &= 2^{4(m+1)+3}x_0 - 2^{4(m+1)+3}y_0 + 8D_{m+1} + 2 \\ &\quad +2^{4(m+1)+3}x_0 + 2^{4(m+1)+3}y_0 + 24D_{m+1} + 5 - 1 \\ &= 2^{4(m+1)+4}x_0 + 32D_{m+1} + 6 \end{aligned}$$

and

$$\begin{aligned} y_{8(m+1)+8} &= y_{8m+16} \\ &= x_{8m+15} + |y_{8m+15}| \\ &= -2^{4(m+1)+3}x_0 + 2^{4(m+1)+3}y_0 - 8D_{m+1} - 2 \\ &\quad +2^{4(m+1)+3}x_0 + 2^{4(m+1)+3}y_0 + 24D_{m+1} + 5 \\ &= 2^{4(m+1)+4}y_0 + 16D_{m+1} + 3 \end{aligned}$$

and so $\mathcal{P}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{P}(m)$ is true for $0 \leq m \leq K-1$. In particular, $\mathcal{P}(K-1)$ is true. Thus

$$x_{8K} = x_{8(K-1)+8} = 2^{4(K-1)+4}x_0 + 32D_{K-1} + 6$$

and

$$y_{8K} = y_{8(K-1)+8} = 2^{4(K-1)+4}y_0 + 16D_{K-1} + 3.$$

Hence

$$\begin{aligned}
x_{8K+1} &= |x_{8K}| - y_{8K} - 1 \\
&= -2^{4K}x_0 - 32D_{K-1} - 6 - 2^{4K}y_0 - 16D_{K-1} - 3 - 1 \\
&= -2^{4K}x_0 - 2^{4K}y_0 - 48D_{K-1} - 10 \\
&= -2^{4K}x_0 - 2^{4K}y_0 - 48\left(\frac{2^{4(K-1)} - 1}{5}\right) - 10 \\
&= -2^{4K}x_0 - 2^{4K}y_0 - \frac{3 \cdot 2^{4K}}{5} + \frac{3}{5} - 1 \\
&= -2^{4K}x_0 - 2^{4K}y_0 - 3D_K - 1
\end{aligned}$$

and

$$\begin{aligned}
y_{8K+1} &= x_{8K} + |y_{8K}| \\
&= 2^{4K}x_0 + 32D_{K-1} + 6 - 2^{4K}y_0 - 16D_{K-1} - 3 \\
&= 2^{4K}x_0 - 2^{4K}y_0 + 16D_{K-1} + 3
\end{aligned}$$

$$\begin{aligned}
&= 2^{4K}x_0 - 2^{4K}y_0 + 16\left(\frac{2^{4(K-1)} - 1}{5}\right) + 3 \\
&= 2^{4K}x_0 - 2^{4K}y_0 + \frac{2^{4K}}{5} - \frac{2^4}{5} + 3 \\
&= 2^{4K}x_0 - 2^{4K}y_0 + D_K.
\end{aligned}$$

Hence

$$\begin{aligned}
x_{8K+2} &= |x_{8K+1}| - y_{8K+1} - 1 \\
&= 2^{4K}x_0 + 2^{4K}y_0 + 3D_K + 1 - 2^{4K}x_0 + 2^{4K}y_0 - D_K - 1 \\
&= 2^{4K+1}y_0 + 2D_K
\end{aligned}$$

and

$$\begin{aligned} y_{8K+2} &= x_{8K+1} + |y_{8K+1}| \\ &= -2^{4K}x_0 - 2^{4K}y_0 - 3D_K - 1 - 2^{4K}x_0 + 2^{4K}y_0 - D_K \\ &= -2^{4K+1}x_0 - 4D_K - 1. \end{aligned}$$

Recall that

$$\begin{aligned} (x_0, y_0) &\in [a_K, b_K] \times [c_K, d_K] \setminus [a_{K+1}, b_{K+1}] \times [c_{K+1}, d_{K+1}] \\ &= \left[\frac{-2^{4K-2}-1}{5\cdot 2^{4K-3}}, \frac{-2^{4K}+1}{5\cdot 2^{4K-1}}\right] \times \left[\frac{-2^{4K-2}-1}{5\cdot 2^{4K-2}}, \frac{-2^{4K}+1}{5\cdot 2^{4K}}\right] \\ &\quad \setminus \left[\frac{-2^{4(K+1)-2}-1}{5\cdot 2^{4(K+1)-3}}, \frac{-2^{4(K+1)}+1}{5\cdot 2^{4(K+1)-1}}\right] \times \left[\frac{-2^{4(K+1)-2}-1}{5\cdot 2^{4(K+1)-2}}, \frac{-2^{4(K+1)}+1}{5\cdot 2^{4(K+1)}}\right]. \end{aligned}$$

Suppose $(x_0, y_0) \in [a_K, a_{K+1}) \times [c_K, d_K]$.

Hence

$$\begin{aligned} y_{8K+2} &> -2^{4K+1}(a_{K+1}) - 4D_K - 1 \\ &= -2^{4K+1}\left(\frac{-2^{4(K+1)-2}-1}{5\cdot 2^{4(K+1)-3}}\right) - 4D_K - 1 \\ &= \frac{2^{8K+3}}{5\cdot 2^{4K+1}} + \frac{2^{4(K+1)}-1}{5\cdot 2^{4(K+1)-3}} - \frac{2^{4(K+2)}}{5} + \frac{4}{5} - 1 \\ &= 0 \end{aligned}$$

which is a contradiction.

Next suppose $(x_0, y_0) \in [a_{K+1}, b_K] \times [c_K, c_{K+1})$.

Then

$$\begin{aligned} x_{8K+3} &= |x_{8K+2}| - y_{8K+2} - 1 \\ &= -2^{4K+1}y_0 - 2D_K + 2^{4K+1}x_0 + 4D_K + 1 - 1 \\ &= 2^{4K+1}x_0 - 2^{4K+1}y_0 + 2D_K \end{aligned}$$

and

$$\begin{aligned} y_{8K+3} &= x_{8K+2} + |y_{8K+2}| \\ &= 2^{4K+1}y_0 + 2D_K + 2^{4K+1}x_0 + 4D_K + 1 \\ &= 2^{4K+1}x_0 + 2^{4K+1}y_0 + 6D_K + 1. \end{aligned}$$

Hence

$$\begin{aligned} x_{8K+4} &= |x_{8K+3}| - y_{8K+3} - 1 \\ &= -2^{4K+1}x_0 + 2^{4K+1}y_0 - 2D_K - 2^{4K+1}x_0 - 2^{4K+1}y_0 - 6D_K - 1 - 1 \\ &= -2^{4K+2}x_0 - 8D_K - 2 \end{aligned}$$

and

$$\begin{aligned} y_{8K+4} &= x_{8K+3} + |y_{8K+3}| \\ &= 2^{4K+1}x_0 - 2^{4K+1}y_0 + 2D_K - 2^{4K+1}x_0 - 2^{4K+1}y_0 - 6D_K - 1 \\ &= -2^{4K+2}y_0 - 4D_K - 1. \end{aligned}$$

Recall that $(x_0, y_0) \in [a_{K+1}, b_K] \times [c_K, c_{K+1})$.

Thus

$$\begin{aligned} y_{8K+4} &> -2^{4K+2}(c_{K+1}) - 4D_K - 1 \\ &= -2^{4K+2}\left(\frac{-2^{4K+2} - 1}{5 \cdot 2^{4K+2}}\right) - 4\left(\frac{2^{4K} - 1}{5}\right) - 1 \\ &= \frac{2^{8K+4}}{5 \cdot 2^{4K+2}} + \frac{2^{4K+2}}{5 \cdot 2^{4K+2}} - \frac{2^{4K+2}}{5} + \frac{4}{5} - 1 \\ &= 0 \end{aligned}$$

which is a contradiction.

Now suppose that $(x_0, y_0) \in (b_{K+1}, b_K] \times [c_{K+1}, d_K]$.

Hence

$$\begin{aligned} x_{8K+5} &= |x_{8K+4}| - y_{8K+4} - 1 \\ &= 2^{4K+2}x_0 + 8D_K + 2 + 2^{4K+2}y_0 + 4D_K + 1 - 1 \\ &= 2^{4K+2}x_0 + 2^{4K+2}y_0 + 12D_K + 2 \end{aligned}$$

and

$$\begin{aligned} y_{8K+5} &= x_{8K+4} + |y_{8K+4}| \\ &= -2^{4K+2}x_0 - 8D_K - 2 + 2^{4K+2}y_0 + 4D_K + 1 \\ &= -2^{4K+2}x_0 + 2^{4K+2}y_0 - 4D_K - 1. \end{aligned}$$

Then

$$\begin{aligned} x_{8K+6} &= |x_{8K+5}| - y_{8K+5} - 1 \\ &= -2^{4K+2}x_0 - 2^{4K+2}y_0 - 12D_K - 2 + 2^{4K+2}x_0 - 2^{4K+2}y_0 + 4D_K + 1 - 1 \\ &= -2^{4K+3}y_0 - 8D_K - 2 \end{aligned}$$

and

$$\begin{aligned} y_{8K+6} &= x_{8K+5} + |y_{8K+5}| \\ &= 2^{4K+2}x_0 + 2^{4K+2}y_0 + 12D_K + 2 + 2^{4K+2}x_0 - 2^{4K+2}y_0 + 4D_K + 1 \\ &= 2^{4K+3}x_0 + 16D_K + 3. \end{aligned}$$

Recall that $(x_0, y_0) \in (b_{K+1}, b_K] \times [c_{K+1}, d_K]$.

Thus

$$\begin{aligned} y_{8K+6} &> 2^{4K+3}(b_{K+1}) + 16\left(\frac{2^{4K}-1}{5}\right) + 3 \\ &= 2^{4K+3}\left(\frac{-2^{4(K+1)}+1}{5 \cdot 2^{4(K+1)-1}}\right) + 16\left(\frac{2^{4K}-1}{5}\right) + 3 \end{aligned}$$

$$\begin{aligned} &= \frac{-2^{4K+4}}{5} + \frac{1}{5} + \frac{2^{4K+4}}{5} - \frac{16}{5} + 3 \\ &= 0 \end{aligned}$$

which is a contradiction.

Finally, suppose $(x_0, y_0) \in [a_{K+1}, b_{K+1}] \times (d_{K+1}, d_K]$.

Thus

$$\begin{aligned} x_{8K+7} &= |x_{8K+6}| - y_{8K+6} - 1 \\ &= 2^{4K+3}y_0 + 8D_K + 2 - 2^{4K+3}x_0 - 16D_K - 3 - 1 \\ &= -2^{4K+3}x_0 + 2^{4K+3}y_0 - 8D_K - 2 \end{aligned}$$

and

$$\begin{aligned} y_{8K+7} &= x_{8K+6} + |y_{8K+6}| \\ &= -2^{4K+3}y_0 - 8D_K - 2 - 2^{4K+3}x_0 - 16D_K - 3 \\ &= -2^{4K+3}x_0 - 2^{4K+3}y_0 - 24D_K - 5. \end{aligned}$$

Hence

$$\begin{aligned} x_{8K+8} &= |x_{8K+7}| - y_{8K+7} - 1 \\ &= 2^{4K+3}x_0 - 2^{4K+3}y_0 + 8D_K + 2 + 2^{4K+3}x_0 + 2^{4K+3}y_0 + 24D_K + 5 - 1 \\ &= 2^{4K+3}x_0 + 32D_K + 6 \end{aligned}$$

and

$$\begin{aligned} y_{8K+8} &= x_{8K+7} + |y_{8K+7}| \\ &= -2^{4K+3}x_0 + 2^{4K+3}y_0 - 8D_K - 2 + 2^{4K+3}x_0 + 2^{4K+3}y_0 + 24D_K + 5 \\ &= 2^{4K+4}y_0 + 16D_K + 3. \end{aligned}$$

Recall that $(x_0, y_0) \in [a_{K+1}, b_{K+1}] \times (d_{K+1}, d_K]$.

Thus

$$\begin{aligned} y_{8K+8} &> 2^{4K+4}\left(d_{K+1}\right)+16\left(\frac{2^{4K}-1}{5}\right)+3 \\ &> 2^{4K+4}\left(\frac{-2^{4(K+1)}+1}{5\cdot 2^{4(K+1)}}\right)+16\left(\frac{2^{4K}-1}{5}\right)+3 \\ &= -\frac{2^{4K+4}}{5}+\frac{1}{5}+\frac{2^{4K+4}}{5}-\frac{16}{5}+3 \\ &= 0 \end{aligned}$$

which is a contradiction. The proof is complete.

□

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