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A New proof of the CR Pohožaev Identity and related Topics.

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ABSTRACT

In this paper we give a new proof for "the CR Pohožaev Identity" and deduce non existence results of positive solutions for semi-linear boundary value problems on star-shaped domains

$$(P)\begin{cases} -\Delta_H u &= g(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega, \end{cases}$$

where Δ_H is the sublaplacian of the Heisenberg group $\mathbb{H}^n$, g is a C^1 function on a star-shaped and bounded domain Ω of $\mathbb{H}^n$.

RESUMEN

En este artículo presentamos una nueva demostración de la identidad de CR Pohozaev sobre el grupo de Heisenberg y deducimos resultados sobre la no existencia de soluciones positivas para problemas semi-lineales con valores en la frontera sobre dominios estrellados

$$(P)\begin{cases} -\Delta_H u &= g(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega, \end{cases}$$

donde Δ_H es el sublaplaciano del grupo de Heisenberg $\mathbb{H}^n$, g es una función de clase C^1 sobre un dominio estrellado y acotado Ω de $\mathbb{H}^n$.

Keywords and Phrases: Analysis on CR manifolds, CR structure, CR Pohožaev Identity, Critical growth, Yamabe like problems.

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1 Introduction and Main Results

We are concerned with non existence results for the following semilinear boundary value problems on a bounded domain Ω of the Heisenberg group $\mathbb{H}^n$

$$(P)\begin{cases} -\Delta_H u &= g(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega, \end{cases}$$

where Δ_H is the sublaplacian of $\mathbb{H}^n$, g is a C^1 function.
Recall that the Heisenberg group $\mathbb{H}^n$ is the homogeneous Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ and group law given by

$$\tau_{\xi'}(\xi) = \xi' \cdot \xi = (x + x',\, y + y'\, t + t' + 2(< x, y' > - < x', y >))$$

where $< .,. >$ denotes the inner product in $\mathbb{R}^n$, $\xi = (x, y, t)$ and $\xi' = (x', y', t')$.
The homogeneous norm of the space $\mathbb{H}^n$ is

$$\rho(\xi) = \left((\mid x \mid^2 + \mid y \mid^2)^2 + t^2\right)^{\frac{1}{4}}$$

and the natural distance is accordingly defined by

$$d(\xi, \xi') = \rho(\xi^{-1} \cdot \xi').$$

The Koranyi ball of center ξ_0 and radius r for this distance is given by

$$B_r(\xi) = \{\xi \in \mathbb{H}^n / \ d(\xi_0, \xi) < \ r\}.$$

There are a remarkable families of transformations groups on $\mathbb{H}^n$, the group of parabolic dilations and the groups of left translations.
The parabolic $\mathbb{H}^n$-dilatations are the following transformations

$$\begin{aligned} \delta_\lambda : \mathbb{H}^n \quad &\longrightarrow \quad \mathbb{H}^n \\ (x, y, t) \quad &\longrightarrow \quad (\lambda x, \lambda y, \lambda^2 t)\,,\ \lambda > 0. \end{aligned}$$

The Jacobian determinant of δ_λ is λ^{2n+2}, it yields that the homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$.
For a given $\xi^{'} \in \mathbb{H}^n$, one can define a group of left translations by setting:

$$\tau_\alpha(\xi) = \tau_{\alpha\xi'}(\xi) = \alpha\xi^{'} \cdot \xi, \quad \forall \xi \in \mathbb{H}^n$$

The generators of the group of dilations $\{\delta_\lambda, \lambda > 0\}$ and the group of left translations $\{\tau_{\alpha\xi'}, \alpha \in \mathbb{R}\}$ are given respectively by the following smooth vector fields:

$$X = \sum_{i=1} \left(x_i \partial_{x_i} + y_i \partial_{y_i}\right) + 2t\partial_t \tag{1.1}$$

and

$$Y(\xi') = Y(x', y', t') = \sum_{i=1} \left(x_i' \partial_{x_i} + y_i' \partial_{y_i} \right) + (t' + 2(< x, y' > - < x', y >)) \partial_t. \tag{1.2}$$

We say that a function u is homogeneous of degree k with respect to the parabolic dilations $\{\delta_\lambda, \lambda > 0\}$ if and only if $u \circ \delta_\lambda = \lambda^k u$ for $\lambda > 0$, which implies that its Lie derivative with respect to X satisfies

$$L_X u = X\, u = k\, u.$$

For example, the naturel distance function is homogenous of degree 1. In the other hand a function u is homogeneous of degree k with respect to the group of left translations $\{\tau_{\alpha\xi'}, \alpha \in \mathbb{R}\}$ if and only if its Lie derivative with respect to Y satisfies

$$L_{Y(\xi')}\, u = Y(\xi')\, u = k\, u.$$

The subelliptic gradient is given by

$$\nabla_{\mathbb{H}^n} = (X_1, ..., X_n, Y_1, ..., Y_n)$$

where $X_i = \partial_{x_i} + 2y_i\partial_t,\ Y_i = \partial_{y_i} - 2x_i\partial_t,\ i \in \{1, 2...n\}$ span the horizontal subspace of the tangent space of $\mathbb{H}^n$ accordingly to the following decomposition

$$T\mathbb{H}^n = \mathcal{H} \oplus \mathbb{R}T,$$

where $\mathcal{H}$ is the horizontal subspace and T is the Reeb vector field given by $T = \partial_t$.
The Lie Algebra of left invariant vector fields is generated by $\{(X_i, Y_i)_{1 \le i \le n}, T\}$.

Since $[X_i,\ Y_i] = -4T$, the Heisenberg laplacian

$$\Delta_H = \sum_{i=1}^{n} (X_i^2 + Y_i^2),$$

is a second order degenerate elliptic operator of Hörmander type and hence it is hypoelliptic.
If we denote by $A = (a_{ij})$ the $(2n+1) \times (2n+1)$ symmetric matrix given by $a_{ij} = \delta_{ij}$ if $i, j = 1, ...2n,\ a_{(2n+1)j} = -2x_j$ if $j = n+1, ..., 2n$, and $a_{(2n+1)(2n+1)} = 4|z|^2$. We remark that the matrix A is related to Δ_H by the formula

$$\Delta_H = div(A\ \nabla)$$

where ∇ and div denote respectively the euclidian gradient and the euclidian divergence operator of $\mathbb{R}^{2n+1}$.
The canonical contact and volume forms of $\mathbb{H}^n$ are given by

$$\theta_0 = dt + 2 \sum_{1 \le i \le n} (x_i\, dy_i - \ y_i dx_i) \tag{1.3}$$

and

$$dv_{\theta_0} = \theta_0 \wedge d\theta_0^n. \tag{1.4}$$

A fundamental solution of $-\Delta_H$ with pole at zero is given by (one can see [7])

$$\Gamma(\xi) = \frac{c_Q}{d(\xi)^{Q-2}}, \text{ where } c_Q = \frac{\Gamma^2(n/2)}{2^{4-2n}\pi^{n+1}} \quad \text{and} \quad Q = 2n+2.$$

Moreover, a fundamental solution with pole at ξ is

$$\Gamma(\xi,\xi^{'}) = \frac{c_Q}{d(\xi,\xi')^{Q-2}}.$$

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality:

$$|\varphi|^2_{Q^*} \leq c|\nabla_{\mathbb{H}^n}\varphi|^2_2, \ \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{H}^n)$$

where $Q^* = \frac{2Q}{Q-2}$.
This inequality ensures in particular that for every domain Ω of $\mathbb{H}^n$, the function

$$|\varphi| = |\nabla_{\mathbb{H}^n}\varphi|_2$$

is a norm on $\mathcal{C}_0^\infty(\Omega)$. We denote by $S^{1,2}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to this norm, $S^{1,2}(\Omega)$ becomes a Hilbert space with the inner product:

$$< u, v >_{S^{1,2}} = \int_\Omega < \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} v > \ dv_{\theta_0}.$$

Define $S_0^{1,2}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm above.

The Pohožaev Identity is the principle tool used here to investigate the relation between domain geometry and solvability of equation (P). We seek u a positive solution to equation (P), where g has critical or supercritical growth, meaning, $g(u) \geq ku^{1+\frac{2}{n}}$ for some positive constant k. We ask the question " for a prescribed domain and a nonlinearity g, can we find a positive solution u?". For Euclidean domains $\Omega \subset \mathbb{R}^N$, S.Pohožaev in [19] proved that there is no solution for starlike ones, on the other hand, A.Bahri and J.M.Coron, W.Y.Ding in [1] and [6], have shown that a solution exists when $g(u) = u^{p^*}$, and the domain has nontrivial topology, here $p^* = (N+2)/(N-2)$ is the critical exponent for the compactness of the Sobolev inclusion $W_0^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, for $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{k}{n}$, $1 < p < q < \infty$ where $W_0^{k,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{k,p}(\Omega)} = \text{Sup}_{\ l(\alpha)\leq k}\|D^\alpha u\|_{L^p(\Omega)}$.

For the Heisenberg group and using arguments related to the topology of the domain, G.Citti and F.Uguzzoni [5] following the work of A. Bahri and Coron, gave the Kohn Laplacian counterpart of the celebrated theorem in [1], and proved an existence result for Yamabe type problem on domains which have a nontrivial homology group (with $\mathbb{Z}_2$-coefficients), I.Birendili, I.Capuzzo Dolcetta and A.Cutri in [3] used blow up techniques to prove existence results, while in [22] F.Uguzzoni gave a non-existence result for equation (P) involving the critical exponent on halfspaces of the Heisenberg group. We have also to mention the non existence results of E.Lanconelli and F.Uguzzoni on unbounded domains of the Heisenberg group in [14] and [15], and the existence of positives solutions on the Heisenberg group one can see [4] and[2].

For euclidian domains by strict-starlike, we mean that if $x \in R^n$ and ν is the boundary normal, then on the boundary of the domain $(x.\nu) > 0$ for all x. P.Pucci and J.Serrin noted that Pohožaev's result did not require strict starlikeness on the domain and what was needed was a domain with a vector function h that acted like the starlike vector field $h = x$. Several authors P.Pucci, J.Serrin, R.Schaaf, J.McGough, J.Mortesen, C.Rickett and G.Stubendieck in [20], [21], [16], [17] and [18] have examined this new class of h-starlike domains and the resulting extensions of the Pohožaev like results.
While for the Heisenberg group $\mathbb{H}^n$ using the geometry of the domain to give non existence and existence results for equation (P), N.Garofalo and E.Lanconelli in [11] have used the analogy with the hstarlike euclidean domains for a given vector field h. They defined for the Heisenberg group a notion of CR starlike domains for two special smooth vector fields, X and Y which are respectively the generator of the group of dilations and the generator of the group of left translations of $\mathbb{H}^n$ given by (1.1) and (1.2). Next we will introduce the definition given in [11] of domains starshapeness which will be used throughout the present work.

Given a piecewise C^1 bounded domain $\Omega \subset \mathbb{H}^n$, we say that it is $\delta-$starshaped with respect to a point $\xi_0 \in \Omega$, if denoting by N the outer unit normal to the boundary of $\tau_{\xi_0^{-1}}(\Omega)$, we have

$$X.N \geq 0 \tag{1.5}$$

at every point of $\partial(\tau_{\xi_0^{-1}}(\Omega))$.

For a bounded domain Ω of $\mathbb{H}^n$, we denote by $\mathcal{C}(\overline{\Omega})$ the space of all continuous functions $f : \overline{\Omega} \to \mathbb{R}$ such that $X_i f$, $Y_i f$, $X_i^2 f$ and $Y_i^2 f$ for $i \in \{1, 2, ...n\}$ are continuous functions on Ω and continuous up to the boundary of Ω.

Our main results are:

- **CR versions of the "Poho$\widetilde{z}$aev identity"**:

• Let $u \in C(\overline{\Omega})$ be a solution of the equation (P), then we have

$$\int_{\Omega} |\nabla_{\mathbb{H}^n} u|^2 X.N d\sigma = -(Q-2)\int_{\Omega} ug(u)du + 2Q\int_{\Omega} G(u)du.$$

where $G(u) = \int_0^u g(s)\, ds$.

• We replace in equation (P) $g(u)$ by $g(\xi, u) = u^{1+\frac{2}{n}} + h(\xi)\, u$, with $\xi \in \mathbb{H}^n$ and $h \in C^\infty(\mathbb{H}^n)$, set (P') the equation thus obtained. If $u \in C(\overline{\Omega})$ is a solution of (P'), then we have

$$\int_{\partial\Omega} |\nabla_{\mathbb{H}^n} u|^2 X.N d\sigma = -2\int_{\Omega} \Big(h + \frac{1}{2}(Xh)\Big)u^2\, dv_{\theta_0}.$$

- Pohožaev's non existence results:

Let $\Omega \subset \mathbb{H}^n$ be a bounded and connected domain such that $0 = (0,0,0) \in \Omega$ and Ω is $\delta-$starshaped with respect to this point.

•Then any positive solution u of equation (P) vanishes identically if

$$-(Q-2)ug(u) + 2QG(u) \leq 0. \tag{1.6}$$

• If $g(u) = u^{1+\frac{2}{n}} + \lambda\, u$, $\lambda \leq 0$, then (P) has no positive solution u different of the trivial solution $u \equiv 0$.

• Let the function h given in equation (P') satisfies

$$h + \frac{1}{2}(Xh) \leq 0. \tag{1.7}$$

Then there is no positive solution $u \in S_0^{1,2}(\Omega)$ of equation (P') unless $u \equiv 0$.

The paper is organized as follows. In section 2, we prove preliminary results and give the CR Pohožaev Identity. The section 3 is devoted to establish some non existence result for equation (P) based on the theory of unique continuation property proved by N. Garofallo and E. Lanconelli for solutions of semi linear equations on Heisenberg group domains, one can see [10] and [11]. In section 4, we study a Yamabe like problem on a bounded domain of the Heisenberg group and deduce a non existence result using a related CR Pohožaev Identity.

2 Description of the Problem

We will be interested on the existence of a positive solution to the following semilinear equation

$$(P)\begin{cases} -\Delta_H u &= g(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega, \end{cases}$$

where Δ_H is the sublaplacian of $\mathbb{H}^n$, g is a C^1 function on Ω a bounded domain of the Heisenberg group $\mathbb{H}^n$.

Lemma 2.1. If u is a solution for problem (P), then we have

$$-\int_\Omega \Delta_H u(Xu) = \int_\Omega g(u)(Xu) = \int_\Omega X(G(u)) = -(2n+2)\int_\Omega G(u)$$

where $G(u) = \int_0^u g(s)\, ds$.

Proof: We multiply equation (P) by Xu and integrate by parts, we obtain

$$-\int_\Omega \Delta_H u(Xu) = \int_\Omega g(u)(Xu).$$

Since $\frac{\partial}{\partial x_i}(x_i G(u)) = G(u) + x_i \frac{\partial}{\partial x_i} G(u)$ for $i \in \{1, ...n\}$, we have

$$\int_\Omega \frac{\partial}{\partial x_i}(x_i G(u)) = \int_\Omega G(u) + \int_\Omega x_i \frac{\partial}{\partial x_i} G(u),$$

thus it yields that $\int_\Omega G(u) + \int_\Omega x_i \frac{\partial}{\partial x_i} G(u) = 0$, since u is equal to zero on the boundary of Ω. In the same way we obtain $\int_\Omega G(u) + \int_\Omega y_i \frac{\partial}{\partial y_i} G(u) = 0$, for $i \in \{1, ...n\}$ and $\int_\Omega G(u) + \int_\Omega t \frac{\partial}{\partial t} G(u) = 0$, hence the proof of the lemma is complete.

In what follows, for a bounded domain Ω of $\mathbb{H}^n$, we denote by $\mathcal{C}(\overline{\Omega})$ the space of all continuous functions $f : \overline{\Omega} \to \mathbb{R}$ such that $X_i f$, $Y_i f$, $X_i^2 f$ and $Y_i^2 f$ for $i \in \{1, 2, ...n\}$ are continuous functions up to the boundary of Ω.

Next we will consider the following vector field on $\mathbb{H}^n$, $P = Xu(\nabla_{\mathbb{H}^n} u) = (P_1, P_2, ..., P_{2n})$, where u is in $\mathcal{C}(\overline{\Omega})$. If we denote by $\widetilde{div}$ the horizontal divergence operator on $\mathbb{H}^n$, we remark that

$$\widetilde{div} P := div_{\mathbb{H}^n} P = \sum_{i=1}^n (X_i P + Y_i P) = div \widetilde{P}. \tag{2.1}$$

where $\widetilde{P} = (\widetilde{P}_1, \widetilde{P}_2, ..., \widetilde{P}_{2n}, \widetilde{P}_{2n+1})$ is the vector field on $\mathbb{R}^{2n+1}$ obtained from P as

$$\widetilde{P}_j = P_j, \text{ for } j = 1, ...2n \text{ and } \widetilde{P}_{2n+1} = 2\sum_{j=1}^n (y_j P_j - x_j P_{n+j}).$$

Let Z be the vector field $|\nabla_{\mathbb{H}^n} u|^2 X$, since $divX = 2n+2$, it yields

$$\int_\Omega divZ = (2n+2)\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 + X < \nabla u, A\nabla u > . \tag{2.2}$$

Using (8) and (9), we obtain the following result:

Lemma 2.2. Let Ω be a bounded domain of $\mathbb{H}^n$ and $u \in \mathcal{C}(\overline{\Omega})$. Then

$$\int_\Omega \widetilde{div} P = \int_\Omega Xu\, \Delta_H u + \int_\Omega divZ - 2n\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 - \int_\Omega < A\nabla u, \nabla(Xu) > .$$

Proof:
We have

$$\widetilde{div} P = (Xu)\widetilde{div}(\nabla_{\mathbb{H}^n} u) + \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n}(Xu) = Xu\, \Delta_H u + \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n}(Xu).$$

A simple computation gives

$$\widetilde{P}_{2n+1} = 2\sum_{j=1}^{n} (Xu)\,(y_j X_j - x_j Y_j)$$

therefore, since $\nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n}(Xu) =< \nabla u, A\nabla Xu >$ and

$$\begin{aligned} < \nabla u, A\nabla Xu > &= X < \nabla u, A\nabla u > - < A\nabla u, \sum_{j=1}^{n}\Big(X(\frac{\partial u}{\partial x_i})\partial_{x_i} + X(\frac{\partial u}{\partial y_i})\partial_{y_i}\Big) + X(\frac{\partial u}{\partial t})\partial_t) > \\ &+ \ < \nabla u, A\nabla u > -2\frac{\partial u}{\partial t}\Big(\sum_{j=1}^{n}(y_j X_j(u) - x_j Y_j(u)\Big), \end{aligned}$$

we obtain

$$\begin{aligned} \int_\Omega \widetilde{div} P &= \int_\Omega Xu\, \Delta_H u + \int_\Omega divZ - (2n+2)\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 \\ &+ \int_\Omega < A\nabla u, \nabla u - \sum_{j=1}^{n}\Big(X(\frac{\partial u}{\partial x_i})\partial_{x_i} + X(\frac{\partial u}{\partial y_i})\partial_{y_i}\Big) + X(\frac{\partial u}{\partial t})\partial_t) > \\ &- 2\int_\Omega \frac{\partial u}{\partial t}(\sum_{j=1}^{n}(y_j X_j(u) - x_j Y_j(u)) \\ &= \int_\Omega Xu\, \Delta_H u + \int_\Omega divZ - 2n\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 - \int_\Omega < A\nabla u, \nabla(Xu) > . \end{aligned}$$

Denoting by N the euclidian unit outer normal to $\partial\Omega$ and $d\sigma$ the $2n$-dimensional Hausdorff measure on $\mathbb{R}^{2n+1}$, if u is in $\mathcal{C}(\overline{\Omega})$ the following holds

Theorem 2.1.

$$2\int_{\partial\Omega} X(u)(A\nabla u.N)d\sigma - \int_{\partial\Omega} |\nabla_{\mathbb{H}^n} u|^2 X.N d\sigma = 2\int_{\Omega} Xu\Delta_H u - 2n\int_{\Omega} |\nabla_{\mathbb{H}^n} u|^2.$$

Proof: We have

$$\int_{\Omega} divZ dv_{\theta_0} = \int_{\partial\Omega} Z.N d\sigma = \int_{\partial\Omega} < Z, N > d\sigma = \int_{\partial\Omega} |\nabla_{\mathbb{H}^n} u|^2 (X.N) d\sigma, \tag{2.3}$$

and

$$\int_{\Omega} \widetilde{div} P dv_{\theta_0} = \int_{\Omega} div\widetilde{P} dx = \int_{\partial\Omega} \widetilde{P}.N d\sigma,$$

where

$$\begin{aligned}\widetilde{P} &= (P, 2\sum X(u)(y_j X_j(u) - x_j Y_j(u))) = (Xu.\nabla_{\mathbb{H}^n} u, 2\sum_{i=1}^{n} (X(u) y_j X_j(u) - x_j Y_j(u)) \\ &= X(u)(\nabla_{\mathbb{H}^n} u, 2\sum_{i=1}^{n} (y_j X_j(u) - x_j Y_j(u)) = X(u)(A\nabla u).\end{aligned}$$

Therefore

$$\int_{\Omega} div\widetilde{P} dx = \int_{\partial\Omega} X(u)(A\nabla u.N) d\sigma. \tag{2.4}$$

On one hand, using Lemma 2.2 and (11), we obtain

$$\begin{aligned}\int_{\partial\Omega} X(u)(A\nabla u.N)d\sigma &= \int_{\Omega} Xu\Delta_H u dv_{\theta_0} + \int_{\partial\Omega} |\nabla_{\mathbb{H}^n} u|^2 X.N d\sigma \\ &- 2n\int_{\Omega} |\nabla_{\mathbb{H}^n} u|^2 dv_{\theta_0} - \int_{\Omega} < A\nabla u, \nabla(Xu) > dv_{\theta_0}.\end{aligned}$$

In the other hand, we have

$$\begin{aligned}\int_{\Omega} \widetilde{div} P = \int_{\Omega} div\widetilde{P} &= \int_{\Omega} div(X(u)A\nabla u) \\ &= \int_{\Omega} (X(u) div(A\nabla u) + DX(u)(A\nabla u) \\ &= \int_{\Omega} (X(u) div(A\nabla u) + \int_{\Omega} \nabla X(u).A\nabla u \\ &= \int_{\Omega} Xu.\Delta_H u + \int_{\Omega} < A\nabla u, \nabla(Xu) > .\end{aligned}$$

The result follows.

We are now ready to state a CR version of the "Pohoz̃aev identity". Let $g : \mathbb{R} \to \mathbb{R}$ be a C^1 function with primitive $G(u) = \int_0^u g(s)ds$ and let $u \in C(\overline{\Omega})$ be a solution of the equation

$$(P)\begin{cases} -\Delta_H u = g(u) & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{H}^n$. Then there hold

$$\int_\Omega (-\Delta_H u)Xu = \int_\Omega g(u)X(u) = -(2n+2)\int_\Omega G(u),$$

and

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 = \int_\Omega ug(u)du. \tag{2.5}$$

In the other hand $X.u =< X, \nabla u >$, since the unit outer normal $N = -\frac{\nabla u}{|\nabla u|}$, we obtain

$$X(u) = - < X, N > |\nabla u|.$$

Therefore

$$\begin{aligned} \int_{\partial\Omega} |\nabla_{\mathbb{H}^n} u|^2 X.N d\sigma &= \int_{\partial\Omega} < A\nabla u, \nabla u > .X.N d\sigma \\ &= \int_{\partial\Omega} < A|\nabla u|N, |\nabla u|N > X.N d\sigma \end{aligned}$$

and computing this product, one obtain

$$\begin{aligned} < A\nabla u, \nabla u >< X, N > &= |\nabla u|^2 < AN, N > . < X, N > \\ &= |\nabla u|^2 < AN, N >< X, \frac{-\nabla u}{|\nabla u|} > \\ &= -|\nabla u| < AN, N >< X, \nabla u > \\ &= -|\nabla u| < AN, N > X.u \\ &= < A.\nabla u, N > X(u). \end{aligned}$$

It yields

$$\int_{\partial\Omega} |\nabla_{\mathbb{H}^n} u|^2 X.N d\sigma = \int_{\partial\Omega} X(u) A\nabla u.N d\sigma. \tag{2.6}$$

Therefore using (2.5) and (2.6), Theorem 2.3 reads as

Theorem 2.2. Let $u \in C(\overline{\Omega})$ be a solution of the equation (P), then we have

$$\int_{\partial\Omega} |\nabla_{\mathbb{H}^n} u|^2 X.N d\sigma = -(Q-2)\int_\Omega ug(u)du + 2Q\int_\Omega G(u)du.$$

Theorem 2.4 is a CR version of the "Poho$\tilde{z}$aev identity".

3 Pohožaev's non existence results

We say that a family of functions has the unique continuation property, if no function besides possibly the zero function vanishes on a set of positive measure.
In this section we proceed to establish some non existence result based on the theory of unique continuation property proved by N. Garofallo and E. Lanconelli for solutions of semi linear equations on Heisenberg group domains, one can see [10] and [11].

We begin this section by introducing the notion of starshapeness which will be used throughout this paper.

Definition 3.1. [11] Given a piecewise C^1 domain $\Omega \subset \mathbb{H}^n$, we say that is $\delta-$starshaped with respect to a point $\xi_0 \in \Omega$, if denoting by N the outer unit normal to the boundary of $\tau_{\xi_0^{-1}}(\Omega)$, we have

$$X.N \geq 0 \tag{3.1}$$

at every point of $\partial(\tau_{\xi_0^{-1}}(\Omega))$.

We observe that if we left-translate ξ_0 to the origin then $\nu(\xi) = u(\tau_{\xi_0^{-1}}\xi)$ is in $C\overline{\tau_{\xi_0^{-1}}(\Omega)}$ and satisfies the same equation as u. Therefore we may assume without loss of generality that the origin belongs to the domain Ω.
By using the definition 3.1, we obtain as a consequence of theorem 2.4 the following non existence result for equation (P)

Theorem 3.2. Let $\Omega \subset \mathbb{H}^n$ be a connected and bounded domain containing $0 = (0,0,0)$, and assume that Ω is $\delta-$starshaped with respect to this point. Then any positive solution $u \in C(\overline{\Omega})$ of equation (P) vanishes identically if

$$-(Q-2)ug(u) + 2QG(u) \leq 0. \tag{3.2}$$

Proof:
The proof is similar to the one given by N.Garofallo and E.Lanconelli for solution of such example of semi linear equations on Heisenberg group domains, one can see [11]. The proof is based on the theory of the unique continuation property developed in [10].
Since the domain is δ-starshaped i.e $X.N \geq 0$ on the boundary of Ω, hence from theorem 2.4, we deduce that $|\nabla_{\mathbb{H}^n} u|^2$ is identically equal to 0 in $\partial\Omega \cap B_r(\bar{\xi})$ for some $\bar{\xi} \in \partial\Omega$ and $r > 0$. Therefore if we set $u \equiv 0$ in $(\mathbb{H}^n \setminus \bar{\Omega}) \cap B_r(\bar{\xi})$, we obtain a positive solution of

$$-\Delta_H u = Vu \quad \text{in} \quad B_r(\bar{\xi}) \tag{3.3}$$

where Δ_H is the sublaplacian of $\mathbb{H}^n$, $V \in L^\infty(B_r(\bar{\xi}))$, $V = \dfrac{g(u)}{u}$ when $u \neq 0$ and $V = 0$ when $u = 0$ in $B_r(\bar{\xi})$. In the appendix of [11] Corollary A.1, by using the method of the unique continuation property for the solution u of (16) the authors prove that $u \equiv 0$ in $B_r(\bar{\xi})$. We can reformulate the result of Corollary A.1 as follows, if we denote by D the maximal open set of $B_r(\bar{\xi})$ on which u vanishes then there exist a sphere S such its interior is entirely contained in D and there exist $\xi \in \partial N \cap S$. As u vanishes in one side of S, it follows that $\xi \in D$, and hence the maximal open set D of $B_r(\bar{\xi})$ on which u vanishes is the hole ball i.e $D = B_r(\bar{\xi})$. To complete the proof i.e to show that $u \equiv 0$ on Ω, we use the fact that Ω is connected.

Next we will focus on the special case where $g(u) = \lambda u + u^{p^*}$, $p^* = 1 + \dfrac{2}{n}$ is the critical exponent for the compactness of the Sobolev inclusion $S^{k,p}(\Omega) \hookrightarrow L^s(\Omega)$, for $\dfrac{1}{s} = \dfrac{1}{p} - \dfrac{k}{2n+2}$, $1 < p < s < \infty$; here $S^{k,p}(\Omega)$ is a Folland Stein space [12], the CR counterpart of The Sobolev space $W^{1,2}(\Omega)$ for euclidean domains. Define $S_0^{k,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{S^{k,p}(\Omega)} = \mathrm{Sup}_{\ l(\alpha)\le k}\|Z^\alpha u\|_{L^p(\Omega)}$, $Z^\alpha = (Z_{\alpha_1}, ...Z_{\alpha_k})$, where $\alpha = (\alpha_1, ..., \alpha_k)$, each α_j is an integer $1 \le \alpha_j \le 2n$, $l(\alpha) = \alpha_1 + ... + \alpha_k$ and

$$Z_{\alpha_j} = \begin{cases} X_{\alpha_j} & \text{for } 1 \le \alpha_j \le n \\ Y_{\alpha_j} & \text{for } n+1 \le \alpha_j \le 2n. \end{cases}$$

More precisely, given $\lambda \in \mathbb{R}$ we would like to solve the problem

$$E_{p^*}(\lambda) \begin{cases} -\Delta_H u &= u^{1+\frac{2}{n}} + \lambda u & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{in } \partial\Omega \end{cases}$$

We obtain in this case the following non existence result

Corollary 1. *Suppose Ω is a bounded domain in $\mathbb{H}^n$, which is $\delta-$starshaped with respect to the origin $0 = (0,0,0)$ and let $\lambda \le 0$. Then any solution $u \in S_0^{1,2}(\Omega)$ of the boundary value problem $E_{p^*}(\lambda)$ vanishes identically.*

Proof:
we will proceed by contradiction and suppose that there exist a nontrivial solution of $E_{p^*}(\lambda)$. A simple computation shows that

$$-(Q-2)ug(u) + 2QG(u) = 2\lambda u^2. \tag{3.4}$$

Therefore using the result of theorem 3.2, one deduce that $\lambda > 0$. The result follows.

Let us remark that one can obtain the above result for a strict-$\delta-$starshaped domain by a direct method, in fact two cases occur
-If $\lambda < 0$, from equality (17) and theorem 2.4, we deduce that there is no positive solutions of

$E_{p^*}(\lambda)$.
-If $\lambda = 0$, we use the Green formula for $u, v \in C(\overline{\Omega})$

$$\int_\Omega -\Delta_H u\, v\, dv_{\theta_0} = \int_\Omega \nabla_{\mathbb{H}^n} u\, \nabla_{\mathbb{H}^n} v\, dv_{\theta_0} - \int_{\partial\Omega} v\, A\nabla u.N d\sigma \tag{3.5}$$

and set $v \equiv 1$ in (18), since $N = \dfrac{-\nabla u}{|\nabla u|}$, we obtain for a solution u of (P)

$$\int_\Omega -\Delta_H\, u\, dv_{\theta_0} = \int_{\partial\Omega} \frac{|\nabla_{\mathbb{H}^n} u|^2}{|\nabla u|} d\sigma \tag{3.6}$$

Since Ω is strict-δ—starshaped with respect to $0 \in \mathbb{H}^n$, we have $X.N(\xi) > 0$ for all $\xi \in \partial\Omega$. Thus from theorem 2.4, we deduce that $|\nabla_{\mathbb{H}^n} u|^2$ is identically equal to 0 on the boundary of Ω, therefore

$$\int_\Omega -\Delta_H u = 0. \tag{3.7}$$

Hence $\displaystyle\int_\Omega u^{1+\frac{2}{n}} = 0$, which means $u = 0$, since $u \geq 0$.

Remarks

(1) The result of corollary 3.3 still hold true for supercritical value of the exponent p, i.e $p > p^*$, for any value of $\lambda < \lambda^* = \dfrac{n(p-1)-2}{p+1}$.

(2) If the domain Ω is not δ—starshaped then equation (E_p) can have solutions even if (15) holds. In fact, if we choose a pseudo annulus $\Omega = \{\xi = (x, y, t) \in \mathbb{H}^n / R_1 < x^2 + y^2 < R_2,\ |t| < T\}$ for fixed $R_1, R_2, T > 0$, then for every fixed $p > 1$ and $\lambda \geq 0$ the problem (E_p) has a positive solution $u \in S_0^{1,2}(\Omega) \cap C^\infty(\Omega)$, which is Hölder continuous up to the boundary one can see [11].

However we can approch problem $E_{p^*}(\lambda)$ by a direct method and attempt to obtain non-trivial solutions as relative minima of the functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla_{\mathbb{H}^n} u|^2 - \lambda u^2)\theta_0 \wedge d\theta_0^n, \tag{3.8}$$

on the unit sphere of $L^{2+\frac{2}{n}}(\Omega)$

$$\sum = \{u \in S_0^{1,2}(\Omega),\ \|u\|_{L^{2+\frac{2}{n}}}^{2+\frac{2}{n}} = 1\}. \tag{3.9}$$

Equivalently, one may seek to minimize the Sobolev quotient

$$S_\lambda(u) = \frac{\int_\Omega (|\nabla_{\mathbb{H}^n} u|^2 - \lambda u^2)\theta_0 \wedge d\theta_0^n}{\|u\|_{L^{2+\frac{2}{n}}}^{2+\frac{2}{n}}}, \quad u \neq 0. \tag{3.10}$$

Let us note that for $\lambda = 0$

$$S_0(\Omega) = \inf_{u \in S_0^{1,2}(\Omega),\ u \neq 0} S_\lambda(u) = \inf_{u \in S_0^{1,2}(\Omega),\ u \neq 0} \frac{\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 \theta_0 \wedge d\theta_0^n}{\|u\|^{2+\frac{2}{n}}_{L^{2+\frac{2}{n}}}}, \quad u \neq 0 \tag{3.11}$$

is related to the best constant for the Sobolev embedding $S_0^{1,2}(\Omega) \hookrightarrow L^{2+\frac{2}{n}}(\Omega)$.

4 Yamabe like problems

In the sequel we will consider the case where λ is a function. More precisely let h be a smooth function on $\mathbb{H}^n$, we are looking for solutions of the semilinear equation on a bounded domain Ω

$$E_{p^*}(h) \begin{cases} -\Delta_H u &= u^{1+\frac{2}{n}} + h\, u \quad \text{in } \Omega \\ u &> 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega \end{cases}$$

This problem arises naturally in CR geometry, in fact let $(M;\theta)$ be a CR manifold of dimension $2n+1$, $n \geq 1$. We ask the question on whether there exist a contact form $\widetilde{\theta}$ on M conformal to θ i.e $\widetilde{\theta} = u^{\frac{2}{n}}\theta, u > 0$ which has a constant Webster scalar curvature. If we denote by R_θ (respectively $R_{\widetilde{\theta}}$) the Webster scalar curvature of the contact form θ (respectively $\widetilde{\theta}$), we have the following relation:

$$(2 + \frac{2}{n})\Delta_b\, u + R_\theta u = R_{\widetilde{\theta}}\, u^{1+\frac{2}{n}} \tag{4.1}$$

where Δ_b is the sublaplacian (the real part of the Kohn Spencer laplacian) of the manifold M. The existence of such a conformal contact form of constant Webster scalar curvature is equivalent to the existence of a positive solution of (4.1). This problem is known to be the Yamabe problem, one can see [12], [13], [8] and [9].

We have the following result.

Lemma 4.1. If u is a solution of problem $E_{p^*}(h)$, then

$$\int_\Omega -\Delta_H u\, (Xu)\, dv_{\theta_0} = -\int_\Omega \Big((n+1)\, h + \frac{1}{2}\, X\, h\Big)\, u^2\, dv_{\theta_0} - n \int_\Omega u^{2+\frac{2}{n}}\, dv_{\theta_0}.$$

Proof:
We multiply equation $E_{p^*}(h)$ by Xu and integrate by parts, we obtain

$$\int_\Omega -\Delta_H u\,(Xu) = \int_\Omega h\,u(Xu) + \int_\Omega u^{1+\frac{2}{n}}\,(Xu).$$

on one hand, we have

$$2\,(h\,u)(X\,u) = X(h\,u^2) - \,(Xh)\,u^2, \tag{4.2}$$

and a simple computation as done in Lemma 2.1 gives

$$\int_\Omega X(h\,u^2) = -(2n+2)\int_\Omega h\,u^2. \tag{4.3}$$

On the other hand, we have

$$\int_\Omega u^{1+\frac{2}{n}}\,(Xu) = -n\,\int_\Omega u^{2+\frac{2}{n}}. \tag{4.4}$$

By using $(26), (27)$ and (28), we obtain the desired result.

Following the method used in section2, we obtain the CR version of the "Pohoẑaev identity" for the present case

Lemma 4.2. Let $u \in C(\overline{\Omega})$ be a solution of the equation $E_{p^*}(h)$, then we have

$$\int_{\partial\Omega} |\nabla_{\mathbb{H}^n} u|^2 X.N d\sigma \quad = \quad -2\int_\Omega \Big(h + \frac{1}{2}(Xh)\Big)u^2\,dv_{\theta_0}.$$

Proof:
Using theorem 2.3 and (13), we obtain

$$\int_\Omega -\Delta_H u\,(Xu) = -\frac{1}{2}\int_{\partial\Omega} |\nabla_{\mathbb{H}^n} u|^2 X.N d\sigma - n\int_\Omega |\nabla_{\mathbb{H}^n} u|^2. \tag{4.5}$$

By comparing the result of lemma 4.1 and (29), the proof of lemma 4.2 is completed.

We are now ready to state a non existence result for equation $E_{p^*}(h)$.

Corollary 2. *Suppose Ω is a connected and bounded domain in $\mathbb{H}^n$ containing 0. Suppose that Ω is δ−starshaped with respect to this point and let $h \in C^\infty(\mathbb{H}^n)$ satisfying*

$$h + \frac{1}{2}(Xh) \le \; 0. \tag{4.6}$$

Then there is no positive solution $u \in S_0^{1,2}(\Omega)$ of equation $E_{p^*}(h)$, $u \neq 0$.

Proof: The proof is similar to the one given for theorem 3.2 with $V = u^{\frac{2}{n}} + h$, when $u \neq 0$ and $V = 0$ when $u = 0$ in $B_r(\bar{\xi})$.

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