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Fractional Voronovskaya type asymptotic expansions for quasi-interpolation neural network operators

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ABSTRACT

Here we study further the quasi-interpolation of sigmoidal and hyperbolic tangent types neural network operators of one hidden layer. Based on fractional calculus theory we derive fractional Voronovskaya type asymptotic expansions for the error of approximation of these operators to the unit operator.

RESUMEN

Estudiamos la cuasi-interpolación de los operadores de redes neuronales de tipo tangencial hiperbólico y sigmoidal de una capa oculta. Basados en la Teoría del Cálculo Fraccional, obtenemos expansiones asintóticas del tipo Voronovskaya para el error en la aproximación de estos operadores hacia el operador unitario.

Keywords and Phrases: Neural Network Fractional Approximation, Voro- novskaya Asymptotic Expansion, fractional derivative.

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1 Background

We need

Definition 1. Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [13], pp. 49-52) the function

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{1}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a,b]$.

Definition 2. (see also [3], [14], [15]). Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{2}$$

$\forall\, x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$. Notice $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$.

Convention 3. We assume that

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{3}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{4}$$

for all $x, x_0 \in [a,b]$.

We mention

Proposition 4. (by [5]) Let $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.

Also we have

Proposition 5. (by [5]) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in [a,b]$.

Theorem 6. ([5]) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.

We mention the left Caputo fractional Taylor formula with integral remainder.

Theorem 7. ([13], p. 54) Let $f \in AC^m([a,b])$, $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-J)^{\alpha-1} D_{*x_0}^{\alpha} f(J)\, dJ, \tag{5}$$

$\forall\, x \geq x_0$; $x, x_0 \in [a,b]$.

Also we mention the right Caputo fractional Taylor formula.

Theorem 8. ([3]) Let $f \in AC^m([a,b])$, $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then

$$f(x) = \sum_{j=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_x^{x_0} (J - x)^{\alpha - 1} D_{x_0 -}^{\alpha} f(J)\, dJ, \tag{6}$$

$\forall\, x \le x_0$; $x, x_0 \in [a,b]$.

For more on fractional calculus related to this work see [2], [4] and [7].

We consider here the sigmoidal function of logarithmic type

$$s(x) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}.$$

It has the properties $\lim_{x \to +\infty} s(x) = 1$ and $\lim_{x \to -\infty} s(x) = 0$.

This function plays the role of an activation function in the hidden layer of neural networks.

As in [12], we consider

$$\Phi(x) := \frac{1}{2}\left(s(x+1) - s(x-1)\right), \quad x \in \mathbb{R}. \tag{7}$$

We notice the following properties:

i) $\Phi(x) > 0$, $\forall\, x \in \mathbb{R}$,

ii) $\sum_{k=-\infty}^{\infty} \Phi(x - k) = 1$, $\forall\, x \in \mathbb{R}$,

iii) $\sum_{k=-\infty}^{\infty} \Phi(nx - k) = 1$, $\forall\, x \in \mathbb{R}$; $n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi(x)\, dx = 1$,

v) Φ is a density function,

vi) Φ is even: $\Phi(-x) = \Phi(x)$, $x \ge 0$.

We see that ([12])

$$\Phi(x) = \left(\frac{e^2 - 1}{2e}\right) \frac{e^{-x}}{(1 + e^{-x-1})(1 + e^{-x+1})} = 8 \tag{1.1}$$
$$\left(\frac{e^2 - 1}{2e^2}\right) \frac{1}{(1 + e^{x-1})(1 + e^{-x-1})}.$$

vii) By [12] Φ is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$.

viii) By [11] for $n \in \mathbb{N}$, $0 < \beta < 1$, we get

$$\sum_{\substack{k=-\infty \\ : |nx-k| > n^{1-\beta}}}^{\infty} \Phi(nx-k) < \left(\frac{e^2-1}{2}\right) e^{-n^{(1-\beta)}} = 3.1992 e^{-n^{(1-\beta)}}. \tag{9}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of a number. Consider $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

ix) By [11] it holds

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < \frac{1}{\Phi(1)} = 5.250312578, \ \forall\, x \in [a,b]. \tag{10}$$

x) By [11] it holds $\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \neq 1$, for at least some $x \in [a,b]$.

Let $f \in C([a,b])$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

We study further (see also [11]) the quasi-interpolation positive linear neural network operator

$$G_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}, \quad x \in [a,b]. \tag{11}$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

We also consider here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$. Furthermore it holds $\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} > 0$.

This function plays also the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}\left(\tanh(x+1) - \tanh(x-1)\right) > 0, \quad \forall\, x \in \mathbb{R}. \tag{12}$$

We easily see that $\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Here we follow [8]

Proposition 9. $\Psi(x)$ for $x \geq 0$ is strictly decreasing.

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x\to-\infty} \Psi(x) = 0 = \lim_{x\to\infty} \Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is at zero, $\Psi(0) = 0.3809297$.

Theorem 10. We have that $\sum_{i=-\infty}^{\infty} \Psi(x-i) = 1, \ \forall\, x \in \mathbb{R}$.

Thus

$$\sum_{i=-\infty}^{\infty} \Psi(nx-i) = 1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

Furthermore we get:

Since Ψ is even it holds $\sum_{i=-\infty}^{\infty} \Psi(i-x) = 1, \ \forall x \in \mathbb{R}$.

Hence $\sum_{i=-\infty}^{\infty} \Psi(i+x) = 1, \ \forall\, x \in \mathbb{R}$, and $\sum_{i=-\infty}^{\infty} \Psi(x+i) = 1, \ \forall\, x \in \mathbb{R}$.

Theorem 11. It holds $\int_{-\infty}^{\infty} \Psi(x)\, dx = 1$.

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 12. Let $0 < \beta < 1$ and $n \in \mathbb{N}$. It holds

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} \Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\beta)}}. \tag{13}$$

Theorem 13. Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} < 4.1488766 = \frac{1}{\Psi(1)}. \tag{14}$$

Also by [8], we obtain

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) \neq 1, \tag{15}$$

for at least some $x \in [a,b]$.

Definition 14. Let $f \in C([a,b])$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

We further study, as in [8], the quasi-interpolation positive linear neural network operator

$$F_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)}, \quad x \in [a,b]. \tag{16}$$

We find here fractional Voronovskaya type asymptotic expansions for $G_n(f,x)$ and $F_n(f,x)$, $x \in [a,b]$.

For related work on neural networks also see [1], [6], [9] and [10]. For neural networks in general see [16], [17] and [18].

2 Main Results

We present our first main result

Theorem 15. Let $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $f \in AC^N([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$ large enough. Assume that $\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}$, $\left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \leq M$, $M > 0$. Then

$$G_n(f,x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} G_n\left((\cdot - x)^j\right)(x) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{17}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (17) collapses.

The last (17) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[G_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} G_n\left((\cdot - x)^j\right)(x) \right] \to 0, \tag{18}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, or $f^{(j)}(x) = 0$, $j = 1, ..., N-1$, then we derive that

$$n^{\beta(\alpha-\varepsilon)} \left[G_n(f,x) - f(x) \right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. Of great interest is the case of $\alpha = \frac{1}{2}$.

Proof. From [13], p. 54; (5), we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \frac{1}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ, \tag{19}$$

for all $x \leq \frac{k}{n} \leq b$.

Also from [3]; (6), using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \frac{1}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ, \tag{20}$$

for all $a \leq \frac{k}{n} \leq x$.

We call

$$V(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx - k). \tag{21}$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right) \Phi(nx-k)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\Phi(nx-k)}{V(x)} \left(\frac{k}{n} - x\right)^j + \tag{22}$$

$$\frac{\Phi(nx-k)}{V(x)\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ,$$

all $x \le \frac{k}{n} \le b$, iff $\lceil nx\rceil \le k \le \lfloor nb\rfloor$, and

$$\frac{f\left(\frac{k}{n}\right)\Phi(nx-k)}{V(x)} = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\Phi(nx-k)}{V(x)}\left(\frac{k}{n}-x\right)^{j} + \tag{23}$$

$$\frac{\Phi(nx-k)}{V(x)\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ,$$

for all $a \le \frac{k}{n} \le x$, iff $\lceil na\rceil \le k \le \lfloor nx\rfloor$.

We have that $\lceil nx\rceil \le \lfloor nx\rfloor + 1.$

Therefore it holds

$$\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{f\left(\frac{k}{n}\right)\Phi(nx-k)}{V(x)} = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j}}{V(x)} + \tag{24}$$

$$\frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Phi(nx-k)}{V(x)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right),$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}f\left(\frac{k}{n}\right)\frac{\Phi(nx-k)}{V(x)} = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\Phi(nx-k)}{V(x)}\left(\frac{k}{n}-x\right)^{j} + \tag{25}$$

$$\frac{1}{\Gamma(\alpha)}\left(\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\Phi(nx-k)}{V(x)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right).$$

Adding the last two equalities (24) and (25) we obtain

$$G_n(f,x) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\frac{\Phi(nx-k)}{V(x)} = \tag{26}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\Phi(nx-k)}{V(x)}\left(\frac{k}{n}-x\right)^{j} +$$

$$\frac{1}{\Gamma(\alpha)V(x)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Phi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ + \right.$$

$$\left.\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Phi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)\right)dJ\right\}.$$

So we have derived

$$T(x) := G_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} G_n\left((\cdot - x)^j\right)(x) = \theta_n^*(x), \tag{27}$$

where

$$\theta_n^*(x) := \frac{1}{\Gamma(\alpha) V(x)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right.$$

$$\left. + \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k) \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right\}. \tag{28}$$

We set

$$\theta_{1n}^*(x) := \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k)}{V(x)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right), \tag{29}$$

and

$$\theta_{2n}^* := \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k)}{V(x)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right), \tag{30}$$

i.e.

$$\theta_n^*(x) = \theta_{1n}^*(x) + \theta_{2n}^*(x). \tag{31}$$

We assume $b - a > \frac{1}{n^{\beta}}$, $0 < \beta < 1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\beta}} \right\rceil$. It is always true that either $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\beta}}$ or $\left|\frac{k}{n} - x\right| > \frac{1}{n^{\beta}}$.

For $k = \lceil na \rceil, ..., \lfloor nx \rfloor$, we consider

$$\gamma_{1k} := \left| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right| \leq \tag{32}$$

$$\int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left|D_{x-}^{\alpha} f(J)\right| dJ$$

$$\leq \left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} \frac{\left(x - \frac{\kappa}{n}\right)^{\alpha}}{\alpha} \leq \left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}. \tag{33}$$

That is

$$\gamma_{1k} \leq \left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \tag{34}$$

for $k = \lceil na \rceil, ..., \lfloor nx \rfloor$.

Also we have in case of $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{1k} \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left|D_{x-}^{\alpha} f(J)\right| dJ \tag{35}$$

$$\leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{\left(x-\frac{\kappa}{n}\right)^{\alpha}}{\alpha} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{1}{n^{\alpha\beta}\alpha}.$$

So that, when $\left(x-\frac{k}{n}\right) \leq \frac{1}{n^{\beta}}$, we get

$$\gamma_{1k} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{1}{\alpha n^{a\beta}}. \tag{36}$$

Therefore

$$|\theta_{1n}^{*}(x)| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k)}{V(x)} \gamma_{1k} \right) = \frac{1}{\Gamma(\alpha)} \cdot$$

$$\left\{ \frac{\sum_{\substack{k=\lceil na \rceil \\ :\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\beta}}}}^{\lfloor nx \rfloor} \Phi(nx-k)}{V(x)} \gamma_{1k} + \frac{\sum_{\substack{k=\lceil na \rceil \\ :\left|\frac{k}{n}-x\right| > \frac{1}{n^{\beta}}}}^{\lfloor nx \rfloor} \Phi(nx-k)}{V(x)} \gamma_{1k} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \left(\frac{\sum_{\substack{k=\lceil na \rceil \\ :\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\beta}}}}^{\lfloor nx \rfloor} \Phi(nx-k)}{V(x)} \right) \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{1}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{V(x)} \left(\sum_{\substack{k=\lceil na \rceil \\ :\left|\frac{k}{n}-x\right| > \frac{1}{n^{\beta}}}}^{\lfloor nx \rfloor} \Phi(nx-k) \right) \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha} \right\} \overset{(\text{by } (9),\ (10))}{\leq} \tag{37}$$

$$\frac{\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + (5.250312578)(3.1992)\, e^{-n^{(1-\beta)}} (x-a)^{\alpha} \right\}.$$

Therefore we proved

$$|\theta_{1n}^{*}(x)| \leq \frac{\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + (16.7968)\, e^{-n^{(1-\beta)}} (x-a)^{\alpha} \right\}. \tag{38}$$

But for large enough $n \in \mathbb{N}$ we get

$$|\theta_{1n}^{*}(x)| \leq \frac{2\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)\, n^{\alpha\beta}}. \tag{39}$$

Similarly we have

$$\gamma_{2k} := \left| \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right| \leq$$

$$\int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \left| D_{*x}^{\alpha} f(J) \right| dJ \leq$$

$$\left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{\left(\frac{k}{n} - x \right)^{\alpha}}{\alpha} \leq \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}. \tag{40}$$

That is

$$\gamma_{2k} \leq \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}, \tag{41}$$

for $k = \lfloor nx \rfloor + 1, ..., \lfloor nb \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{2k} \leq \frac{\left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}}. \tag{42}$$

Consequently it holds

$$\left| \theta_{2n}^{*}(x) \right| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k)}{V(x)} \gamma_{2k} \right) =$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \left(\frac{\sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx-k)}{V(x)} \right) \frac{\left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{V(x)} \left(\sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx-k) \right) \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha} \right\} \leq$$

$$\frac{\left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + (16.7968)\, e^{-n^{(1-\beta)}} (b-x)^{\alpha} \right\}. \tag{43}$$

That is

$$\left| \theta_{2n}^{*}(x) \right| \leq \frac{\left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + (16.7968)\, e^{-n^{(1-\beta)}} (b-x)^{\alpha} \right\}. \tag{44}$$

But for large enough $n \in \mathbb{N}$ we get

$$\left| \theta_{2n}^{*}(x) \right| \leq \frac{2 \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)\, n^{\alpha\beta}}. \tag{45}$$

Since $\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}$, $\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]} \leq M$, $M > 0$, we derive

$$\left|\theta_n^*(x)\right| \leq \left|\theta_{1n}^*(x)\right| + \left|\theta_{2n}^*(x)\right| \overset{\text{(by (39), (45))}}{\leq} \frac{4M}{\Gamma(\alpha+1)\, n^{\alpha\beta}}. \tag{46}$$

That is for large enough $n \in \mathbb{N}$ we get

$$\left|T(x)\right| = \left|\theta_n^*(x)\right| \leq \left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\alpha\beta}}\right), \tag{47}$$

resulting to

$$\left|T(x)\right| = O\left(\frac{1}{n^{\alpha\beta}}\right), \tag{48}$$

and

$$\left|T(x)\right| = o(1). \tag{49}$$

And, letting $0 < \varepsilon \leq \alpha$, we derive

$$\frac{\left|T(x)\right|}{\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right)} \leq \left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\beta\varepsilon}}\right) \to 0, \tag{50}$$

as $n \to \infty$.

I.e.

$$\left|T(x)\right| = o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{51}$$

proving the claim. ∎

We present our second main result

Theorem 16. Let $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $f \in AC^N([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$ large enough. Assume that $\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}$, $\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]} \leq M$, $M > 0$. Then

$$F_n(f,x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} F_n\left((\cdot - x)^j\right)(x) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{52}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (52) collapses.

The last (52) implies that

$$n^{\beta(\alpha-\varepsilon)}\left[F_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} F_n\left((\cdot - x)^j\right)(x)\right] \to 0, \tag{53}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, or $f^{(j)}(x) = 0$, $j = 1, \ldots, N-1$, then we derive that

$$n^{\beta(\alpha-\varepsilon)}\left[F_n(f,x) - f(x)\right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. Of great interest is the case of $\alpha = \frac{1}{2}$.

Proof. Similar to Theorem 15, using (13) and (14). ∎

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