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Squares in Euler triples from Fibonacci and Lucas numbers

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ABSTRACT

In this paper we shall continue to study from [4], for $k = -1$ and $k = 5$, the infinite sequences of triples $A = (F_{2n+1},\ F_{2n+3},\ F_{2n+5})$, $B = (F_{2n+1},\ 5F_{2n+3},\ F_{2n+5})$, $C = (L_{2n+1}, L_{2n+3}, L_{2n+5})$, $D = (L_{2n+1}, 5L_{2n+3}, L_{2n+5})$ with the property that the product of any two different components of them increased by k are squares. The sequences A and B are built from the Fibonacci numbers F_n while the sequences C and D from the Lucas numbers L_n. We show some interesting properties of these sequences that give various methods how to get squares from them.

RESUMEN

En este artículo continuaremos el estudio de [4], para $k = -1$ y $k = 5$, las secuencias infinitas de tripletas $A = (F_{2n+1},\ F_{2n+3},\ F_{2n+5})$, $B = (F_{2n+1},\ 5F_{2n+3},\ F_{2n+5})$, $C = (L_{2n+1}, L_{2n+3}, L_{2n+5})$, $D = (L_{2n+1}, 5L_{2n+3}, L_{2n+5})$ con la propiedad que el producto de dos componentes diferentes que se aumenta en k son cuadrados. Las secuencias A y B se construyen con los números de Fibonacci F_n mientras que las secuencias C y D se construyen con los números de Lucas L_n. Mostramos algunas propiedades interesantes de estas secuencias que entregan muchos métodos de cómo conseguir los cuadrados de ellos.

Keywords and Phrases: $D(k)$-triple, Fibonacci numbers, Lucas numbers, square, symmetric sum, alternating sum, product, component

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1 Introduction

For integers a, b and c, let us write $a \overset{b}{\sim} c$ provided $a + b = c^2$. For the triples $A = (a, b, c)$, $D = (d, e, f)$ and $\widetilde{A} = (\widetilde{a}, \widetilde{b}, \widetilde{c})$ the notation $A \overset{D}{\sim} \widetilde{A}$ means that $b\,c \overset{d}{\sim} \widetilde{a}$, $c\,a \overset{e}{\sim} \widetilde{b}$ and $a\,b \overset{f}{\sim} \widetilde{c}$. When $D = (k, k, k)$, let us write $A \overset{k}{\sim} \widetilde{A}$ for $A \overset{D}{\sim} \widetilde{A}$. Hence, A is the $D(k)$-triple (see [1]) if and only if there is a triple $\widetilde{A}$ such that $A \overset{k}{\sim} \widetilde{A}$.

In the paper [4] we constructed infinite sequences $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ and $\beta = \{\beta(n)\}_{n=0}^{\infty}$ of $D(-1)$-triples and $\gamma = \{\gamma(n)\}_{n=0}^{\infty}$ and $\delta = \{\delta(n)\}_{n=0}^{\infty}$ of $D(5)$-triples. Here, $\alpha(n) = A = (F_{2n+1},\ F_{2n+3},\ F_{2n+5})$, $\beta(n) = B = (F_{2n+1},\ 5F_{2n+3},\ F_{2n+5})$ and $\gamma(n) = C = (L_{2n+1},\ L_{2n+3},\ L_{2n+5})$, $\delta(n) = D = (L_{2n+1},\ 5L_{2n+3},\ L_{2n+5})$, where the Fibonacci and Lucas sequences of natural numbers F_n and L_n are defined by the recurrence relations $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$ and $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$.

The numbers F_k make the integer sequence $A000045$ from [6] while the numbers L_k make $A000032$.

The goal of this article is to further explore the properties of the sequences α, β, γ and δ. Each member of these sequences is an Euler $D(-1)$- or $D(5)$-triple (see [2] and [3]) so that many of their properties follow from the properties of the general (pencils of) Euler triples. It is therefore interesting to look for those properties in which at least two of the sequences appear. This paper presents several results of this kind giving many squares from the components, various sums and products of the sequences α, β, γ and δ. Most of our theorems have also versions for the associated sequences $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ and $\widetilde{\delta}$, where $\widetilde{\alpha}(n) = \widetilde{A} = (F_{2n+4}, F_{2n+3}, F_{2n+2})$, $\widetilde{\beta}(n) = \widetilde{B} = (L_{2n+4}, F_{2n+3}, L_{2n+2})$, $\widetilde{\gamma}(n) = \widetilde{C} = (L_{2n+4}, L_{2n+3}, L_{2n+2})$, $\widetilde{\delta}(n) = \widetilde{D} = (5F_{2n+4}, L_{2n+3},\ 5F_{2n+2})$ satisfy $A \overset{-1}{\sim} \widetilde{A}$, $B \overset{-1}{\sim} \widetilde{B}$, $C \overset{5}{\sim} \widetilde{C}$ and $D \overset{5}{\sim} \widetilde{D}$.

2 Squares from products of components

The relations $A \overset{-1}{\sim} \widetilde{A}$ and $C \overset{5}{\sim} \widetilde{C}$ imply that the components of A and C satisfy $A_2A_3 \overset{-1}{\sim} \widetilde{A}_1$ and $C_2C_3 \overset{5}{\sim} \widetilde{C}_1$. Our first theorem shows that the product $A_2A_3C_2C_3$ is in a similar relation with respect to 1. Of course, the other products $A_3A_1C_3C_1$, $A_1A_2C_1C_2$ as well as $B_2B_3D_2D_3$, $B_3B_1D_3D_1$ and $B_1B_2D_1D_2$ exhibit a similar property.

Theorem 1. *The following hold for the products of components:*

$$A_2\,A_3\,C_2\,C_3 \overset{1}{\sim} F_{4n+8}, \qquad \tfrac{1}{5}B_2\,B_3\,D_2\,D_3 \overset{9}{\sim} L_{4n+8},$$

$$A_3\,A_1\,C_3\,C_1 \overset{9}{\sim} F_{4n+6}, \qquad B_3\,B_1\,D_3\,D_1 \overset{9}{\sim} F_{4n+6},$$

$$A_1\,A_2\,C_1\,C_2 \overset{1}{\sim} F_{4n+4}, \qquad \tfrac{1}{5}B_1\,B_2\,D_1\,D_2 \overset{9}{\sim} L_{4n+4}.$$

Proof. Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi}$. Since $F_j = \frac{\varphi^j - \psi^j}{\varphi - \psi}$ and $L_j = \varphi^j + \psi^j$, it follows that $A_2 = \frac{\varphi^{2n+3} - \psi^{2n+3}}{\varphi - \psi}$, $A_3 = \frac{\varphi^{2n+5} - \psi^{2n+5}}{\varphi - \psi}$ and $C_2 = \varphi^{2n+3} + \psi^{2n+3}$, $C_3 = \varphi^{2n+5} + \psi^{2n+5}$. After the substitutions $\psi = -\frac{1}{\varphi}$ and $M = \varphi^n$, the sum of $A_2 A_3 C_2 C_3$ and 1 becomes $\frac{\varphi^{16}(M^8 - \psi^{16})^2}{20\, M^8}$. However, this is precisely the square of F_{4n+8}. This shows the first relation. The other relations have similar proofs. □

The version of the previous theorem for the sequences $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ and $\widetilde{\delta}$ is the following result.

Theorem 2. *The products of components of* $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ *and* $\widetilde{D}$ *satisfy:*

$$\widetilde{A}_2\widetilde{A}_3\widetilde{C}_2\widetilde{C}_3 \overset{1}{\sim} F_{4n+5}, \qquad \widetilde{B}_2\widetilde{B}_3\widetilde{D}_2\widetilde{D}_3 \overset{1}{\sim} L_{4n+5},$$

$$\widetilde{A}_3\widetilde{A}_1\widetilde{C}_3\widetilde{C}_1 \overset{1}{\sim} F_{4n+6}, \qquad \tfrac{1}{25}\widetilde{B}_3\widetilde{B}_1\widetilde{D}_3\widetilde{D}_1 \overset{1}{\sim} F_{4n+6},$$

$$\widetilde{A}_1\widetilde{A}_2\widetilde{C}_1\widetilde{C}_2 \overset{1}{\sim} F_{4n+7}, \qquad \widetilde{B}_1\widetilde{B}_2\widetilde{D}_1\widetilde{D}_2 \overset{1}{\sim} L_{4n+7}.$$

Proof. Since $\widetilde{A}_2 = \frac{\varphi^{2n+3} - \psi^{2n+3}}{\varphi - \psi}$, $\widetilde{A}_3 = \frac{\varphi^{2n+2} - \psi^{2n+2}}{\varphi - \psi}$, $\widetilde{C}_2 = \varphi^{2n+3} + \psi^{2n+3}$ and $\widetilde{C}_3 = \varphi^{2n+2} + \psi^{2n+2}$, the sum of $\widetilde{A}_2\widetilde{A}_3\widetilde{C}_2\widetilde{C}_3$ and 1, after the substitutions $\psi = -\frac{1}{\varphi}$ and $M = \varphi^n$, becomes $\frac{\varphi^{10}(M^8 + \psi^{10})^2}{5\, M^8}$. However, the square of F_{4n+5} has the same value. This proves the first relation $\widetilde{A}_2\widetilde{A}_3\widetilde{C}_2\widetilde{C}_3 \overset{1}{\sim} F_{4n+5}$. The remaining five relations in this theorem have similar proofs. □

The same kind of relations hold also for the products of components from all four sequences α, β, γ and δ.

Theorem 3. *The following relations for products of components hold:*

$$A_2B_3C_2D_3 \overset{1}{\sim} F_{4n+8}, \qquad \tfrac{1}{25}A_3B_2C_3D_2 \overset{1}{\sim} F_{4n+8},$$

$$A_3B_1C_3D_1 \overset{9}{\sim} F_{4n+6}, \qquad A_1B_3C_1D_3 \overset{9}{\sim} F_{4n+6},$$

$$\tfrac{1}{25}A_1B_2C_1D_2 \overset{1}{\sim} F_{4n+4}, \qquad A_2B_1C_2D_1 \overset{1}{\sim} F_{4n+4}.$$

Proof. Since $B_3 = A_3$ and $D_3 = C_3$, the first relation is the consequence of the first relation in Theorem 1.

In order to prove the second relation, notice that $B_2 = 5\,A_2$ and $D_2 = 5\,C_2$ so that the multiplication of the identity behind the first relation in Theorem 1 with 25 we conclude that the second relation holds. The other relations in this theorem have similar proofs. □

There is again the version of the previous theorem for the products of components from all four sequences $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ and $\widetilde{\delta}$.

Theorem 4. *The products of components of* $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ *and* $\widetilde{D}$ *satisfy:*

$$\widetilde{A}_2\widetilde{B}_3\widetilde{C}_2\widetilde{D}_3 \overset{1}{\sim} L_{4n+5}, \qquad \widetilde{A}_3\widetilde{B}_2\widetilde{C}_3\widetilde{D}_2 \overset{1}{\sim} F_{4n+5},$$

$$\widetilde{A}_3\widetilde{B}_1\widetilde{C}_3\widetilde{D}_1 \overset{9}{\sim} L_{4n+6}, \qquad \widetilde{A}_1\widetilde{B}_3\widetilde{C}_1\widetilde{D}_3 \overset{9}{\sim} L_{4n+6},$$

$$\widetilde{A}_1\widetilde{B}_2\widetilde{C}_1\widetilde{D}_2 \overset{1}{\sim} F_{4n+7}, \qquad \widetilde{A}_2\widetilde{B}_1\widetilde{C}_2\widetilde{D}_1 \overset{1}{\sim} L_{4n+7}.$$

Proof. Since $\widetilde{A}_2 = \widetilde{B}_2$ and $\widetilde{C}_2 = \widetilde{D}_2$, the first, the second, the fifth and the sixth relations are the consequence of the second, the first, the fifth and the sixth relations in Theorem 2.

In order to prove the third relation, note that the components $\widetilde{A}_3$, $\widetilde{B}_1$ and $\widetilde{C}_3$, $\widetilde{D}_1$ are $\frac{\varphi^{2n+2}-\psi^{2n+2})}{\varphi-\psi}$, $\varphi^{2n+4}+\psi^{2n+4}$, $\varphi^{2n+2}+\psi^{2n+2}$ and $\frac{5(\varphi^{2n+4}-\psi^{2n+4})}{\varphi-\psi}$. It is now clear from the proof of Theorem 1 that the sum of $\widetilde{A}_3\widetilde{B}_1\widetilde{C}_3\widetilde{D}_1$ and 9 is precisely the square of L_{4n+6}. This shows the third relation. The fourth relation has a similar proof. □

Nice relationships of the same kind hold also for the products of components with other choices of indices.

Theorem 5.

$$\tfrac{1}{5}A_2\,B_2\,C_3\,D_3 \overset{0}{\sim} F_{2n+3}L_{2n+5}, \qquad \tfrac{1}{5}A_3\,B_3\,C_2\,D_2 \overset{0}{\sim} F_{2n+5}L_{2n+3},$$

$$A_3\,B_3\,C_1\,D_1 \overset{0}{\sim} F_{2n+5}L_{2n+1}, \qquad A_1\,B_1\,C_3\,D_3 \overset{0}{\sim} F_{2n+1}L_{2n+5},$$

$$\tfrac{1}{5}A_1\,B_1\,C_2\,D_2 \overset{0}{\sim} F_{2n+1}L_{2n+3}, \qquad \tfrac{1}{5}A_2\,B_2\,C_1\,D_1 \overset{0}{\sim} F_{2n+3}L_{2n+1}.$$

Proof. Since $B_2 = 5A_2$, $A_2 = F_{2n+3}$ and $C_3 = D_3 = L_{2n+5}$, the product $\frac{1}{5}A_2\,B_2\,C_3\,D_3$ is the square of $F_{2n+3}L_{2n+5}$. The other claims in this theorem have similar proofs. □

The version of the previous theorem for the products of components from all four associated sequences is the following result.

Theorem 6. *The products of components of* $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ *and* $\widetilde{D}$ *satisfy:*

$$\frac{\widetilde{A}_2\widetilde{B}_2\widetilde{C}_3\widetilde{D}_3}{5\,F_{4n+4}} \overset{0}{\sim} F_{2n+3}, \qquad \frac{\widetilde{A}_3\widetilde{B}_3\widetilde{C}_2\widetilde{D}_2}{F_{4n+4}} \overset{0}{\sim} L_{2n+3},$$

$$\tfrac{1}{5}\widetilde{A}_3\widetilde{B}_3\widetilde{C}_1\widetilde{D}_1 \overset{1}{\sim} F_{4n+6}, \qquad \tfrac{1}{5}\widetilde{A}_1\widetilde{B}_1\widetilde{C}_3\widetilde{D}_3 \overset{1}{\sim} F_{4n+6},$$

$$\frac{\widetilde{A}_1\widetilde{B}_1\widetilde{C}_2\widetilde{D}_2}{F_{4n+8}} \overset{0}{\sim} L_{2n+3}, \qquad \frac{\widetilde{A}_2\widetilde{B}_2\widetilde{C}_1\widetilde{D}_1}{5\,F_{4n+8}} \overset{0}{\sim} F_{2n+3}.$$

Proof. Since $\widetilde{A}_2 = \widetilde{B}_2 = F_{2n+3}$, $\widetilde{C}_3 = L_{2n+2}$, $\widetilde{D}_3 = 5F_{2n+2}$, we see that the first relation clearly holds. The others in this theorem are proved similarly. □

This time the pairs (A, D) and (B, C) have equal indices.

Theorem 7. *The following hold for the products of components:*

$$\tfrac{1}{5}A_2\, B_3\, C_3\, D_2 \overset{1}{\sim} F_{4n+8}, \qquad \tfrac{1}{5}A_3\, B_2\, C_2\, D_3 \overset{1}{\sim} F_{4n+8},$$

$$A_3\, B_1\, C_1\, D_3 \overset{9}{\sim} F_{4n+6}, \qquad A_1\, B_3\, C_3\, D_1 \overset{9}{\sim} F_{4n+6},$$

$$\tfrac{1}{5}A_1\, B_2\, C_2\, D_1 \overset{1}{\sim} F_{4n+4}, \qquad \tfrac{1}{5}A_2\, B_1\, C_1\, D_2 \overset{1}{\sim} F_{4n+4}.$$

Proof. Since $A_2 = F_{2n+3}$, $B_3 = F_{2n+5}$, $C_3 = L_{2n+5}$ and $D_2 = 5L_{2n+3}$, the sum of $\frac{1}{5}A_2\, B_3\, C_3\, D_2$ and 1 is $F_{4n+6}F_{4n+10} + 1 = F_{4n+8}^2$. The other claims in this theorem have similar proofs. □

Once again the version of the previous theorem for the associated sequences includes interesting relations.

Theorem 8. *The products of components of* $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ *and* $\widetilde{D}$ *satisfy:*

$$\frac{\widetilde{A}_2\widetilde{B}_3\widetilde{C}_3\widetilde{D}_2}{F_{4n+6}} \overset{0}{\sim} L_{2n+2}, \qquad \frac{\widetilde{A}_3\widetilde{B}_2\widetilde{C}_2\widetilde{D}_3}{5\,F_{4n+6}} \overset{0}{\sim} F_{2n+2},$$

$$\tfrac{1}{5}\widetilde{A}_3\widetilde{B}_1\widetilde{C}_1\widetilde{D}_3 \overset{0}{\sim} F_{2n+2}L_{2n+4}, \qquad \tfrac{1}{5}\widetilde{A}_1\widetilde{B}_3\widetilde{C}_3\widetilde{D}_1 \overset{0}{\sim} F_{2n+4}L_{2n+2},$$

$$\frac{\widetilde{A}_1\widetilde{B}_2\widetilde{C}_2\widetilde{D}_1}{5\,F_{4n+6}} \overset{0}{\sim} F_{2n+4}, \qquad \frac{\widetilde{A}_2\widetilde{B}_1\widetilde{C}_1\widetilde{D}_2}{F_{4n+6}} \overset{0}{\sim} L_{2n+4}.$$

Proof. Since $\widetilde{A}_2 = F_{2n+3}$ $\widetilde{B}_3 = \widetilde{C}_3 = L_{2n+2}$, $\widetilde{D}_2 = L_{2n+3}$ and $F_{2n+3}L_{2n+3} = F_{4n+6}$ we see that the first relation clearly holds. The other relations in this theorem are proved similarly. □

It is interesting that in some cases we can even mix components of the triples A, B, C, D and $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ as the relations $\widetilde{A}_2\, B_3\, C_3\, \widetilde{D}_2 \overset{1}{\sim} F_{4n+8}$, $\widetilde{A}_2\, \widetilde{B}_2\, C_3\, D_3 \overset{0}{\sim} F_{2n+3}L_{2n+5}$ and $\widetilde{A}_2\, \widetilde{B}_2\, C_1\, D_1 \overset{0}{\sim} F_{2n+3}L_{2n+1}$ show, but we do not see a general pattern here.

3 Squares from symmetric sums

Let $\sigma_1, \sigma_2, \sigma_3 : \mathbb{Z}^3 \to \mathbb{Z}$ be the basic symmetric functions defined for $x = (a, b, c)$ by $x_{\sigma_1} = a + b + c$, $x_{\sigma_2} = b\,c + c\,a + a\,b$, $x_{\sigma_3} = a\,b\,c$. Let σ_2^*, $\sigma_1^* : \mathbb{Z}^3 \to \mathbb{Z}$ be defined by $x_{\sigma_2^*} = b\,c - c\,a + a\,b$ and $x_{\sigma_1^*} = a - b + c$. Note that $x_{\sigma_1^*}$ is the determinant of the 1×3 matrix $[a, b, c]$ (see [5]).

For the sums σ_2 and σ_2^* of the components the following relations are true.

Theorem 9. *The following is true for the sums* σ_2 *of the components:*

$$A_{\sigma_2} C_{\sigma_2} \overset{21}{\sim} 4\, F_{4n+6}, \qquad B_{\sigma_2} D_{\sigma_2} \overset{69}{\sim} 16\, F_{4n+6},$$

$$\widetilde{A}_{\sigma_2} \widetilde{C}_{\sigma_2} \overset{5}{\sim} 2\, F_{4n+7}, \qquad \widetilde{B}_{\sigma_2} \widetilde{D}_{\sigma_2} \overset{45}{\sim} 10\, F_{4n+6}.$$

Proof. Since $A_{\sigma_2} = \frac{1}{5}(4L_{4n+6} + 13)$ and $C_{\sigma_2} = 4L_{4n+6} - 13$, the sum $A_{\sigma_2} C_{\sigma_2} + 21$ is $\frac{1}{5}[(4L_{4n+6})^2 - 64]$ that we recognize as the square of $4F_{4n+6}$. This proves the first relation $A_{\sigma_2} C_{\sigma_2} \overset{21}{\sim} 4\, F_{4n+6}$. The other relations in this theorem have similar proofs. ☐

The sums $\widetilde{B}_{\sigma_2^*}$ and $\widetilde{D}_{\sigma_2^*}$ have constant values -1 and 5. On the other hand, we have the following result.

Theorem 10. *The following is true for the sums* σ_2^* *of the components:*

$$A_{\sigma_2^*} C_{\sigma_2^*} \overset{-3}{\sim} 2\, F_{4n+6}, \qquad B_{\sigma_2^*} D_{\sigma_2^*} \overset{-51}{\sim} 14\, F_{4n+6}, \qquad \widetilde{A}_{\sigma_2^*} \widetilde{C}_{\sigma_2^*} \overset{5}{\sim} 2\, F_{4n+6}.$$

Proof. Since $B_{\sigma_2^*} = \frac{1}{5}(14L_{4n+6} + 23)$ and $D_{\sigma_2^*} = 14L_{4n+6} - 23$, the sum $B_{\sigma_2^*} D_{\sigma_2^*} - 51$ is the quotient $\frac{196(L_{4n+6}^2 - 4)}{5}$. It is now easy to check that this is the square of $14\, F_{4n+6}$. This proves the second relation $B_{\sigma_2^*} C_{\sigma_2} \overset{-51}{\sim} 14\, F_{4n+6}$. The other relations in this theorem have similar proofs. ☐

Some similar relations where all four letters A, B, C and D appear make the following result.

Theorem 11. *The following is true for the sums* σ_2^* *of the components:*

$$\widetilde{A}_{\sigma_2^*} \widetilde{D}_{\sigma_2^*} + \widetilde{B}_{\sigma_2^*} \widetilde{C}_{\sigma_2^*} = 6, \qquad \widetilde{A}_{\sigma_2^*} \widetilde{C}_{\sigma_2^*} + \widetilde{B}_{\sigma_2^*} \widetilde{D}_{\sigma_2^*} \overset{10}{\sim} 2\, F_{4n+5},$$

$$-\widetilde{A}_{\sigma_2^*} \widetilde{B}_{\sigma_2^*} \widetilde{C}_{\sigma_2^*} \widetilde{D}_{\sigma_2^*} \overset{9}{\sim} 2\, L_{4n+5}.$$

Proof. Since $\widetilde{B}_{\sigma_2^*} = -1$, $\widetilde{A}_{\sigma_2^*} = \frac{1}{5}(2\, L_{4n+5} + 3)$, $\widetilde{C}_{\sigma_2^*} = 2\, L_{4n+5} - 3$ and $\widetilde{D}_{\sigma_2^*} = 5$, it follows that $\widetilde{A}_{\sigma_2^*} \widetilde{D}_{\sigma_2^*} + \widetilde{B}_{\sigma_2^*} \widetilde{C}_{\sigma_2^*} = 6$. The second and the third relations in this theorem have similar proofs. ☐

Here are two relations which contains both sums σ_2 and σ_2^*.

Theorem 12. *The following is true for the sums* σ_2 *and* σ_2^**:*

$$\tfrac{1}{36}(A_{\sigma_2} D_{\sigma_2} - B_{\sigma_2^*} C_{\sigma_2^*}) \overset{2}{\sim} F_{4n+6}, \qquad 3B_{\sigma_2^*} C_{\sigma_2^*} - A_{\sigma_2} D_{\sigma_2} \overset{-74}{\sim} 2\, L_{4n+6} + 6.$$

Proof. Since the sums $B_{\sigma_2^*}$, A_{σ_2}, $C_{\sigma_2^*}$ and D_{σ_2} are equal $\frac{1}{5}(14\, L_{4n+6} + 23)$, $\frac{1}{5}(4\, L_{4n+6} + 13)$, $2\, L_{4n+6} + 1$ and $16\, L_{4n+6} - 37$, we infer that the sum $\frac{1}{36}(A_{\sigma_2} D_{\sigma_2} - B_{\sigma_2^*} C_{\sigma_2^*}) + 2$ is the square of F_{4n+6}. The second relation in this theorem has analogous proof. ☐

In the next result we consider the products of the same components of the triples A, B, C and D and the product of their components.

Theorem 13. *The following relations hold:*

$$A_1B_1C_1D_1 \overset{0}{\sim} F_{4n+2}, \qquad A_2B_2C_2D_2 \overset{0}{\sim} 5\,F_{4n+6},$$

$$A_3B_3C_3D_3 \overset{0}{\sim} F_{4n+10}, \qquad A_{\sigma_3}\,B_{\sigma_3}\,C_{\sigma_3}\,D_{\sigma_3} \overset{0}{\sim} 5\,F_{4n+2}\,F_{4n+6}\,F_{4n+10}.$$

Proof. Since the product $F_{2n+5}L_{2n+5}$ is F_{4n+10}, $A_3 = B_3 = F_{2n+5}$ and $C_3 = D_3 = L_{2n+5}$, it follows that $A_3B_3C_3D_3 = (F_{2n+5}\,L_{2n+5})^2 = F^2_{4n+10}$. This proves the third relation. The other relations have similar proofs. □

The products of the same components of the triples $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ and the product of their components appear in the following result.

Theorem 14. *The following relations are true:*

$$\tfrac{1}{5}\widetilde{A}_1\widetilde{B}_1\widetilde{C}_1\widetilde{D}_1 \overset{0}{\sim} F_{4n+8}, \qquad \tfrac{1}{5}\widetilde{A}_3\widetilde{B}_3\widetilde{C}_3\widetilde{D}_3 \overset{0}{\sim} F_{4n+4},$$

$$\widetilde{A}_2\widetilde{B}_2\widetilde{C}_2\widetilde{D}_2 \overset{0}{\sim} F_{4n+6}, \qquad \widetilde{A}_{\sigma_3}\,\widetilde{B}_{\sigma_3}\,\widetilde{C}_{\sigma_3}\,\widetilde{D}_{\sigma_3} \overset{0}{\sim} 5\,F_{4n+8}F_{4n+6}F_{4n+4}.$$

Proof. Since $\widetilde{A}_3 = F_{2n+2}$, $\widetilde{B}_3 = L_{2n+2}$, $\widetilde{C}_3 = L_{2n+2}$, $\widetilde{D}_3 = 5\,F_{2n+2}$ and $F_{2n+2}L_{2n+2} = F_{4n+4}$, it follows that the product $\frac{1}{5}\widetilde{A}_3\widetilde{B}_3\widetilde{C}_3\widetilde{D}_3$ is the square of F_{4n+4}. This proves the second relation. The other relations in this theorem have similar proofs. □

The products of the sums σ_1 and σ^* of the components of the triples A, B, C and D show the same kind of relations. This is also true for the associated triples $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$.

Theorem 15. *The following relations hold for the sums* σ_1 *and* σ_1^**:*

$$A_{\sigma_1}\,B_{\sigma_1}\,C_{\sigma_1}\,D_{\sigma_1} \overset{0}{\sim} 32\,F_{4n+6}, \qquad \tfrac{1}{144}\widetilde{A}_{\sigma_1}\,\widetilde{B}_{\sigma_1}\,\widetilde{C}_{\sigma_1}\,\widetilde{D}_{\sigma_1} \overset{1}{\sim} F_{4n+7},$$

$$A_{\sigma_1^*}\,B_{\sigma_1^*}\,C_{\sigma_1^*}\,D_{\sigma_1^*} \overset{0}{\sim} 4\,F_{4n+6}, \qquad \tfrac{1}{64}\widetilde{A}_{\sigma_1^*}\,\widetilde{B}_{\sigma_1^*}\,\widetilde{C}_{\sigma_1^*}\,\widetilde{D}_{\sigma_1^*} \overset{1}{\sim} F_{4n+5}.$$

Proof. The sums of the components A_{σ_1}, B_{σ_1}, C_{σ_1} and D_{σ_1} are equal $4\,F_{2n+3}$, $8F_{2n+3}$, $4\,L_{2n+3}$ and $8\,L_{2n+3}$. Hence, the product $A_{\sigma_1}\,B_{\sigma_1}\,C_{\sigma_1}\,D_{\sigma_1}$ is the square of $32\,F_{4n+6}$ since $F_{2n+3}L_{2n+3} = F_{4n+6}$. This proves the above first relation. The other relations in this theorem have similar proofs. □

In the next result we combine the sums σ_1 and σ_1^* in each product.

Theorem 16. *The following relations hold for the sums* σ_1 *and* σ_1^**:*

$$A_{\sigma_1}\,B_{\sigma_1^*}\,C_{\sigma_1}\,D_{\sigma_1^*} \overset{0}{\sim} 8\,F_{4n+6}, \qquad A_{\sigma_1^*}\,B_{\sigma_1}\,C_{\sigma_1^*}\,D_{\sigma_1} \overset{0}{\sim} 16\,F_{4n+6},$$

$$\tfrac{1}{24}\widetilde{A}_{\sigma_1}\,\widetilde{B}_{\sigma_1^*}\,\widetilde{C}_{\sigma_1^*}\,\widetilde{D}_{\sigma_1} \overset{1}{\sim} 2\,F_{4n+6} + 1, \qquad \tfrac{1}{24}\widetilde{A}_{\sigma_1^*}\,\widetilde{B}_{\sigma_1}\,\widetilde{C}_{\sigma_1}\,\widetilde{D}_{\sigma_1^*} \overset{1}{\sim} 2\,F_{4n+6} - 1,$$

$$\tfrac{1}{64}\widetilde{A}_{\sigma_1}\,\widetilde{B}_{\sigma_1^*}\,\widetilde{C}_{\sigma_1}\,\widetilde{D}_{\sigma_1^*} \overset{1}{\sim} F_{4n+7}, \qquad \tfrac{1}{144}\widetilde{A}_{\sigma_1^*}\,\widetilde{B}_{\sigma_1}\,\widetilde{C}_{\sigma_1^*}\,\widetilde{D}_{\sigma_1} \overset{1}{\sim} F_{4n+5}.$$

Proof. The sums of the components A_{σ_1}, $B_{\sigma_1^*}$, C_{σ_1} and $D_{\sigma_1^*}$ are equal $4\,F_{2n+3}$, $-2\,F_{2n+3}$, $4\,L_{2n+3}$ and $-2\,L_{2n+3}$. The product $A_{\sigma_1} B_{\sigma_1^*} C_{\sigma_1} D_{\sigma_1^*}$ is therefore the square of $8\,F_{4n+6}$ since $F_{2n+3}L_{2n+3} = F_{4n+6}$. This proves the first relation. The other relations in this theorem have analogous proofs. $\square$

4 Squares from the sums of squares

For a natural number $k > 1$, let the sums ν_k, $\nu_k^* : \mathbb{Z}^3 \to \mathbb{Z}$ of powers be defined for $x = (a, b, c)$ by $x_{\nu_k} = a^k + b^k + c^k$ and $x_{\nu_k^*} = a^k - b^k + c^k$.

We proceed with the version of the Theorem 9 for the sums ν_2 of the squares of components.

Theorem 17. *The following relations are true for the sums ν_2:*

$$\tfrac{1}{4} A_{\nu_2} C_{\nu_2} \overset{-11}{\sim} 4\,F_{4n+6}, \qquad \tfrac{1}{4} B_{\nu_2} D_{\nu_2} \overset{-59}{\sim} 16\,F_{4n+6},$$

$$\tfrac{1}{4} \widetilde{A}_{\nu_2} \widetilde{C}_{\nu_2} \overset{-3}{\sim} 2\,F_{4n+6}, \qquad \tfrac{1}{4} \widetilde{B}_{\nu_2} \widetilde{D}_{\nu_2} \overset{-27}{\sim} 8\,F_{4n+6}.$$

Proof. Since A_{ν_2} and C_{ν_2} are $\frac{2}{5}(4\,L_{4n+6} + 3)$ and $2(4\,L_{4n+6} - 3)$, the difference of $\frac{1}{4} A_{\nu_2} C_{\nu_2}$ and 11 is equal $\frac{16(L_{4n+6}^2 - 4)}{5}$. But, one can easily check that $\frac{L_{4n+6}^2 - 4}{5} = F_{4n+6}^2$ so that the above quotient is the square of $4\,F_{4n+6}$. This concludes the proof of the first relation. The other relations in this theorem have similar proofs. $\square$

The next is the version of the Theorem 10 for the alternating sums ν_2^* of the squares of components.

Theorem 18. *The following relations are true for the sums ν_2^*:*

$$\tfrac{1}{4} A_{\nu_2^*} C_{\nu_2^*} \overset{-7}{\sim} 3\,F_{4n+6}, \qquad \tfrac{1}{4} B_{\nu_2^*} D_{\nu_2^*} \overset{41}{\sim} 9\,F_{4n+6},$$

$$\tfrac{1}{4} \widetilde{A}_{\nu_2^*} \widetilde{C}_{\nu_2^*} \overset{1}{\sim} F_{4n+6}, \qquad \tfrac{1}{4} \widetilde{B}_{\nu_2^*} \widetilde{D}_{\nu_2^*} \overset{-23}{\sim} 7\,F_{4n+6}.$$

Proof. Notice that the alternating sums of squares of components $A_{\nu_2^*}$ and $C_{\nu_2^*}$ are $\frac{2}{5}(3\,L_{4n+6} + 1)$ and $2(3\,L_{4n+6} + 1)$. Hence, the sum of $\frac{1}{4} A_{\nu_2^*} C_{\nu_2^*}$ and -7 is equal to the following quotient $\frac{9(L_{4n+6}^2 - 4)}{5}$. This quotient is in fact the square of $3\,F_{4n+6}$. This proves the first relation. The remaining three relations in this theorem have similar proofs. $\square$

Certain sums of products of the sums ν_2^* of components show the same behavior.

Theorem 19. *The following relations are true for the sums ν_2^*:*

$$\tfrac{1}{8}(A_{\nu_2^*} D_{\nu_2^*} + B_{\nu_2^*} C_{\nu_2^*}) \overset{17}{\sim} \sqrt{-27}\,F_{4n+6},$$

$$\tfrac{1}{8}(\widetilde{A}_{\nu_2^*} \widetilde{D}_{\nu_2^*} + \widetilde{B}_{\nu_2^*} \widetilde{C}_{\nu_2^*}) \overset{-11}{\sim} \sqrt{7}\,F_{4n+6}.$$

Proof. Notice that the alternating sums of squares of components $\widetilde{A}_{v_2^*}$, $\widetilde{B}_{v_2^*}$, $\widetilde{C}_{v_2^*}$ and $\widetilde{D}_{v_2^*}$ are $2\,F_{2n+2}F_{2n+4}$, $\frac{2}{5}(7\,L_{4n+6}+9)$, $2\,L_{2n+2}L_{2n+4}$ and $2(7\,L_{4n+6}-9)$. Hence, the sum of $\frac{1}{8}(\widetilde{A}_{v_2^*}\widetilde{D}_{v_2^*}+\widetilde{B}_{v_2^*}\widetilde{C}_{v_2^*})$ and -11 is equal to the square of $\sqrt{7}\,F_{4n+6}$. This proves the second relation. The first relation has a similar proof. □

5 Squares from the products ⊙, ▷ and ◁

Let us introduce three binary operations $\odot$, $\triangleright$ and $\triangleleft$ on the set $\mathbb{Z}^3$ of triples of integers by the rules $(a,\,b,\,c)\odot(u,\,v,\,w)=(a\,u,\,b\,v,\,c\,w)$, $(a,\,b,\,c)\triangleright(u,\,v,\,w)=(a\,v,\,b\,w,\,c\,u)$, and

$$(a,\,b,\,c)\triangleleft(u,\,v,\,w)=(a\,w,\,b\,u,\,c\,v).$$

This section contains four theorems which show that the operations $\odot$, $\triangleright$ and $\triangleleft$ are also the source of squares from components of the eight sequences.

Theorem 20. *The following relations for the sequences* A, B, C *and* D *hold:*
$(A\odot B)_{\sigma_1}(C\odot D)_{\sigma_1}\overset{-76}{\sim}12\,F_{4n+6}$, $(A\triangleright B)_{\sigma_1}(C\triangleright D)_{\sigma_1}\overset{61}{\sim}4\,L_{4n+5}$ *and*
$(A\triangleleft B)_{\sigma_1}(C\triangleleft D)_{\sigma_1}\overset{61}{\sim}4\,L_{4n+7}$.

Proof. Since $(A\triangleright B)_{\sigma_1}=4\,F_{4n+5}+5$ and $(C\triangleright D)_{\sigma_1}=5(4\,F_{4n+5}-5)$, it follows that the sum of $(A\triangleright B)_{\sigma_1}(C\triangleright D)_{\sigma_1}$ and 61 is the product $16(5\,F_{4n+5}^2-4)$, i. e., the square of $4\,L_{4n+5}$. This proves the second relation. The first and the third could be established similarly. □

Theorem 21. *The following relations for the sequences* A, B, C *and* D *hold:*
$\frac{1}{4}(A\odot B)_{\sigma_1^*}(C\odot D)_{\sigma_1^*}\overset{1}{\sim}F_{4n+6}$, $(A\triangleright B)_{\sigma_1^*}(C\triangleright D)_{\sigma_1^*}\overset{69}{\sim}2\,F_{4n+2}$ *and*
$(A\triangleleft B)_{\sigma_1^*}(C\triangleleft D)_{\sigma_1^*}\overset{69}{\sim}2\,F_{4n+10}$.

Proof. Since the sums $(A\triangleright B)_{\sigma_1^*}$ and $(C\triangleright D)_{\sigma_1^*}$ are $\frac{1}{5}(2\,L_{4n+2}+19)$ and $2\,L_{4n+2}-19$, it follows that the sum of $(A\triangleright B)_{\sigma_1^*}(C\triangleright D)_{\sigma_1^*}$ and 69 is the square of $2\,F_{4n+2}$. This is the outline of the proof of the second relation. The similar proofs of the first and the third relation are left to the reader. □

Theorem 22. *The following relations for the triples* $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ *and* $\widetilde{D}$ *hold:*
$(\widetilde{A}\odot\widetilde{B})_{\sigma_1}(\widetilde{C}\odot\widetilde{D})_{\sigma_1}\overset{36}{\sim}31\,F_{4n+3}+7\,F_{4n}$, $(\widetilde{A}\triangleright\widetilde{B})_{\sigma_1}(\widetilde{C}\triangleright\widetilde{D})_{\sigma_1}\overset{-3}{\sim}2\,F_{4n+8}$, *and*
$(\widetilde{A}\triangleleft\widetilde{B})_{\sigma_1}(\widetilde{C}\triangleleft\widetilde{D})_{\sigma_1}\overset{29}{\sim}4\,F_{4n+7}$.

Proof. Since the sums $(\widetilde{A}\triangleleft\widetilde{B})_{\sigma_1}$ and $(\widetilde{C}\triangleleft\widetilde{D})_{\sigma_1}$ are $\frac{1}{5}(2\,L_{4n+8}+1)$ and $2\,L_{4n+8}-1$, it follows that the sum of $(\widetilde{A}\triangleleft\widetilde{B})_{\sigma_1}(\widetilde{C}\triangleleft\widetilde{D})_{\sigma_1}$ and -3 is the square of $2\,F_{4n+8}$. This is the outline of the proof of the third relation. The similar proofs of the first and the second relation are left to the reader. □

Theorem 23. *The following relations hold for the triples* $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ *and* $\widetilde{D}$*:*
$(\widetilde{A} \odot \widetilde{B})_{\sigma_1^*}(\widetilde{C} \odot \widetilde{D})_{\sigma_1^*} \overset{36}{\sim} 23\,F_{4n+3} + 5\,F_{4n}$, $\quad (\widetilde{A} \triangleright \widetilde{B})_{\sigma_1^*}(\widetilde{C} \triangleright \widetilde{D})_{\sigma_1^*} \overset{29}{\sim} 4\,F_{4n+5}$ *and*
$(\widetilde{A} \triangleleft \widetilde{B})_{\sigma_1^*}(\widetilde{C} \triangleleft \widetilde{D})_{\sigma_1^*} \overset{-3}{\sim} 2\,F_{4n+4}$.

Proof. Since the sums $(\widetilde{A} \triangleleft \widetilde{B})_{\sigma_1^*}$ and $(\widetilde{C} \triangleleft \widetilde{D})_{\sigma_1^*}$ are $-\frac{1}{5}(2\,L_{4n+4} + 1)$ and $1 - 2\,L_{4n+4}$, it follows that the difference of $(\widetilde{A} \triangleleft \widetilde{B})_{\sigma_1^*}(\widetilde{C} \triangleleft \widetilde{D})_{\sigma_1^*}$ and 3 is the square of $2\,F_{4n+4}$. This is the outline of the proof of the third relation. The similar proofs of the first and the second relation are left to the reader. □

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