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On certain functional equation in semiprime rings and standard operator algebras

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ABSTRACT

The main purpose of this paper is to prove the following result, which is related to a classical result of Chernoff. Let X be a real or complex Banach space, let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on X and let $\mathcal{A}(X) \subseteq \mathcal{L}(X)$ be a standard operator algebra. Suppose there exists a linear mapping $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relation $2D(A^n) = D(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1})$ for all $A \in \mathcal{A}(X)$, where $n \geq 2$ is some fixed integer. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and some fixed $B \in \mathcal{L}(X)$, which means that D is a linear derivation. In particular, D is continuous.

RESUMEN

El propósito principal de este artículo es probar el siguiente resultado, el cual se relaciona a un resultado clásico de Chernoff. Sea X un espacio de Banach real o complejo, sea $\mathcal{L}(X)$ el álgebra de todos los operadores lineales acotados en X y sea $\mathcal{A}(X) \subseteq \mathcal{L}(X)$ una álgebra de operadores estándar. Supongamos que existe una aplicación lineal $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfaciendo la relación $2D(A^n) = D(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1})$ para todo $A \in \mathcal{A}(X)$, donde $n \geq 2$ es algún entero fijo. En este caso D es de la forma $D(A) = [A, B]$ para todo $A \in \mathcal{A}(X)$ y algún $B \in \mathcal{L}(X)$ fijo, lo que significa que D es una derivación lineal. En particular, D es continua.

Keywords and Phrases: Prime ring, semiprime ring, Banach space, standard operator algebra, derivation, Jordan derivation.

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This research has been motivated by the work of Vukman [19]. Throughout, R will represent an associative ring with center $Z(R)$. As usual we write $[x, y]$ for $xy - yx$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$.

Let A be an algebra over the real or complex field and let B be a subalgebra of A. A linear mapping $D : B \to A$ is called a linear derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in B$. In case we have a ring R, an additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists such $a \in R$ that $D(x) = [x, a]$ holds for all $x \in R$.

Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [9] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein theorem can be found in [2]. Cusack [7] generalized Herstein theorem to 2-torsion free semiprime rings (see [3] for an alternative proof). Herstein theorem has been fairly generalized by Beidar, Brešar, Chebotar and Martindale [1]. For results concerning derivations in rings and algebras we refer to [5, 11, 16, 17, 18, 19], where further references can be found. Let X be a real or complex Banach space and let $\mathcal{L}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathcal{L}(X)$, respectively. An algebra $\mathcal{A}(X) \subseteq \mathcal{L}(X)$ is said to be standard in case $\mathcal{F}(X) \subset \mathcal{A}(X)$. Let us point out that any standard operator algebra is prime.

Motivated by the work of Brešar [4], Vukman [19] has recently conjectured that in case we have an additive mapping $D : R \to R$, where R is a 2-torsion free semiprime ring satisfying the relation

$$2D(xyx) = D(xy)x + xyD(x) + D(x)yx + xD(yx) \tag{1}$$

for all pairs $x, y \in R$, then D is a derivation. Note that in case a ring has the identity element, the proof of Vukman's conjecture is immediate. Namely, in this case the substitution $y = e$ in the relation (1), where e stands for the identity element, gives that D is a Jordan derivation and then it follows from Cusack's generalization of Herstein theorem that D is a derivation. The substitution $y = x^{n-2}$ in the relation (1) gives

$$2D(x^n) = D(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1}),$$

which leads to the following conjecture.

Conjecture 0.1. *Let R be a semiprime ring with suitable torsion restrictions and let $D : R \to R$ be an additive mapping. Suppose that*

$$2D(x^n) = D(x^{n-1})x + x^{n-1}D(x) \; + \; D(x)x^{n-1} + xD(x^{n-1})$$

holds for all $x \in R$ and some fixed integer $n \geq 2$. In this case D is a derivation.

It is our aim in this paper to prove the conjecture above in case a ring has the identity element.

Theorem 0.2. *Let* $n \geq 2$ *be some fixed integer, let* R *be a* $n!$*-torsion free semiprime ring with the identity element and let* $D : R \to R$ *be an additive mapping satisfying the relation*

$$2D(x^n) = D(x^{n-1})x + x^{n-1}D(x) \ + \ D(x)x^{n-1} + xD(x^{n-1})$$

for all $x \in R$*. In this case* D *is a derivation.*

Proof. We have the relation

$$2D(x^n) = D(x^{n-1})x + x^{n-1}D(x) \ + \ D(x)x^{n-1} + xD(x^{n-1}) \tag{2}$$

and let us denote the identity element of R by e. Putting e for x in the above relation, we obtain

$$D(e) = 0. \tag{3}$$

Let y be any element of the center $Z(R)$. Putting $x + y$ in the above relation, we obtain

$$\begin{aligned} 2\sum_{i=0}^{n} \binom{n}{i} D(x^{n-i}y^i) \quad &= \quad \left(\sum_{i=0}^{n-1} \binom{n-1}{i} D(x^{n-1-i}y^i)\right)(x+y) \\ &+ \quad \left(\sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}y^i\right) D(x+y) \\ &+ \quad D(x+y)\left(\sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}y^i\right) \\ &+ \quad (x+y)\left(\sum_{i=0}^{n-1} \binom{n-1}{i} D(x^{n-1-i}y^i)\right). \end{aligned}$$

Using (2) in the above relation and rearranging it in sense of collecting together terms involving equal number of factors of y, we obtain

$$\sum_{i=1}^{n-1} f_i(x,y) = 0,$$

where $f_i(x,y)$ stands for the expression of terms involving i factors of y. Replacing x by $x + 2y$, $x + 3y, \ldots, x + (n-1)y$ in turn in the relation (2) and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(x,y)$, $i = 1, 2, \ldots, n-1$, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of this matrix is different from zero, it follows that the system has only a trivial solution. In particular,

$$\begin{aligned} f_{n-2}(x,e) &= 2\binom{n}{n-2}D(x^2) - \binom{n-1}{n-2}D(x)x - \binom{n-1}{n-3}D(x^2) \\ &- \binom{n-1}{n-2}xD(x) - \binom{n-1}{n-3}x^2a - \binom{n-1}{n-2}D(x)x \\ &- \binom{n-1}{n-3}ax^2 - \binom{n-1}{n-2}xD(x) - \binom{n-1}{n-3}D(x^2), \end{aligned}$$

where a denotes $T(e)$. After some calculation and considering the relation (3), we obtain

$$(n(n-1)-(n-1)(n-2))D(x^2) = 2(n-1)(D(x)x + xD(x)).$$

Since R is $2(n-1)$-torsion free, the above relation reduces to

$$D(x^2) = D(x)x + xD(x)$$

for all $x \in R$. In other words, D is a Jordan derivation and Cusack's generalization of Herstein theorem now implies that D is a derivation, which completes the proof. □

In the proof of Theorem 0.2 we used methods similar to those used by Vukman and Kosi-Ulbl in [10]. We proceed with the following result in the spirit of Conjecture 0.1.

Theorem 0.3. *Let X be a real or complex Banach space and let $\mathcal{A}(X)$ be a standard operator algebra on X. Suppose there exists a linear mapping $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relation*

$$2D(A^n) = D(A^{n-1})A + A^{n-1}D(A) \ + \ D(A)A^{n-1} + AD(A^{n-1})$$

for all $A \in \mathcal{A}(X)$ and some fixed integer $n \geq 2$. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and some fixed $B \in \mathcal{L}(X)$, which means that D is a linear derivation.

In case $n = 3$ the above relation reduces to Theorem 4 in [19]. Let us point out that in Theorem 0.3 we obtain as a result the continuity of D under purely algebraic assumptions concerning D, which means that Theorem 0.3 might be of some interest from the automatic continuity point of view. For results concerning automatic continuity we refer the reader to [8] and [13]. In the proof of Theorem 0.3 we use Herstein theorem, the result below and methods that are similar to those used by Kosi-Ulbl and Vukman in [12].

Theorem 0.4. *Let X be a real or complex Banach space, let $\mathcal{A}(X)$ be a standard operator algebra on X and let $D : \mathcal{A}(X) \to \mathcal{L}(X)$ be a linear derivation. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and some fixed $B \in \mathcal{L}(X)$.*

Theorem 0.4 has been proved by Chernoff [6] (see also [14, 15]).

Proof of the Theorem 0.3. We have the relation

$$2D(A^n) = D(A^{n-1})A + A^{n-1}D(A) \ + \ D(A)A^{n-1} + AD(A^{n-1}) \tag{4}$$

for all $A \in \mathcal{A}(X)$. Let us first restrict our attention on $\mathcal{F}(X)$. Let A be from $\mathcal{F}(X)$ and let $P \in \mathcal{F}(X)$ be a projection with $AP = PA = A$. Putting P for A in the relation (4), we obtain

$$D(P) = D(P)P + PD(P). \tag{5}$$

Putting $A+P$ for A in the relation (4), we obtain, similary as in the proof of Theorem 0.2, the relation

$$\begin{aligned} 2\sum_{i=0}^{n}\binom{n}{i}D(A^{n-i}P^i) &= \left(\sum_{i=0}^{n-1}\binom{n-1}{i}D(A^{n-1-i}P^i)\right)(A+P) \\ &+ \left(\sum_{i=0}^{n-1}\binom{n-1}{i}A^{n-1-i}P^i\right)D(A+P) \\ &+ D(A+P)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}A^{n-1-i}P^i\right) \\ &+ (A+P)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}D(A^{n-1-i}P^i)\right). \end{aligned}$$

Using (4) and (5) in the above relation and rearranging it in sense of collecting together terms involving equal number of factors of P, we obtain

$$\sum_{i=1}^{n-1} f_i(A,P) = 0,$$

where $f_i(A,P)$ stands for the expression of terms involving i factors of P. Replacing A by $A+2P$, $A+3P$, ... , $A+(n-1)P$ in turn in the relation (4) and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(A,P)$, $i = 1,2,\ldots,n-1$, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of this matrix is different from zero, it follows that the system has only a trivial solution. In particular,

$$\begin{aligned} f_{n-1}(A,P) &= 2\binom{n}{n-1}D(A) - \binom{n-1}{n-1}D(P)A - \binom{n-1}{n-2}D(A)P \\ &- \binom{n-1}{n-1}PD(A) - \binom{n-1}{n-2}AD(P) - \binom{n-1}{n-1}D(A)P \\ &- \binom{n-1}{n-2}D(P)A - \binom{n-1}{n-1}AD(P) - \binom{n-1}{n-2}PD(A). \end{aligned}$$

The above relation reduces to

$$2D(A) = D(A)P + AD(P) + D(P)A + PD(A) \tag{6}$$

and putting A^2 for A in the above relation, we obtain

$$2D(A^2) = D(A^2)P + A^2D(P) + D(P)A^2 + PD(A^2). \tag{7}$$

As the previously mentioned system of $n-1$ homogeneous equations has only a trivial solution, we also obtain

$$\begin{aligned} f_{n-2}(A,P) &= 2\binom{n}{n-2}D(A^2) - \binom{n-1}{n-2}D(A)A - \binom{n-1}{n-3}D(A^2)P \\ &- \binom{n-1}{n-2}AD(A) - \binom{n-1}{n-3}A^2D(P) - \binom{n-1}{n-2}D(A)A \\ &- \binom{n-1}{n-3}D(P)A^2 - \binom{n-1}{n-2}AD(A) - \binom{n-1}{n-3}PD(A^2). \end{aligned}$$

The above relation now reduces to

$$\begin{aligned} n(n-1)D(A^2) &= 2(n-1)(D(A)A + AD(A)) + \\ &+ \binom{n-1}{n-3}(D(A^2)P + A^2D(P) + D(P)A^2 + PD(A^2)). \end{aligned}$$

Applying the relation (7) in the above relation, we obtain

$$n(n-1)D(A^2) = 2(n-1)(D(A)A + AD(A)) + (n-1)(n-2)D(A^2),$$

which reduces to

$$D(A^2) = D(A)A + AD(A). \tag{8}$$

From the relation (6) one can conclude that D maps $\mathcal{F}(X)$ into itself. We therefore have a linear mapping D, which maps $\mathcal{F}(X)$ into itself and satisfies the relation (8) for all $A \in \mathcal{F}(X)$. In other words, D is a Jordan derivation on $\mathcal{F}(X)$ and since $\mathcal{F}(X)$ is prime, it follows, according to Herstein theorem, that D is a derivation on $\mathcal{F}(X)$. Applying Theorem 0.4 one can conclude that D is of the form

$$D(A) = [A, B] \tag{9}$$

for all $A \in \mathcal{F}(X)$ and some fixed $B \in \mathcal{L}(X)$. It remains to prove that (9) holds for all $A \in \mathcal{A}(X)$ as well. For this purpose we introduce $D_1 : \mathcal{A}(X) \to \mathcal{L}(X)$ by $D_1(A) = [A, B]$ and consider the mapping $D_0 = D - D_1$. The mapping D_0 is obviously linear, satisfies the relation (4) and vanishes on $\mathcal{F}(X)$. It is our aim to prove that D_0 vanishes on $\mathcal{A}(X)$ as well. Let $A \in \mathcal{A}(X)$, let P be a one-dimensional projection and let us introduce $S \in \mathcal{A}(X)$ by $S = A + PAP - (AP + PA)$. We have $SP = PS = 0$. Obviously, $D_0(S) = D_0(A)$. By the relation (4) we now have

$$\begin{aligned} &D_0(S^{n-1})S + S^{n-1}D_0(S) + D_0(S)S^{n-1} + SD_0(S^{n-1}) \\ &= 2D_0(S^n) = 2D_0(S^n + P) = 2D_0((S+P)^n) \\ &= D_0((S+P)^{n-1})(S+P) + (S+P)^{n-1}D_0(S+P) \\ &+ D_0(S+P)(S+P)^{n-1} + (S+P)D_0((S+P)^{n-1}) \\ &= D_0(S^{n-1})S + D_0(S^{n-1})P + S^{n-1}D_0(S) + PD_0(S) \\ &+ D_0(S)S^{n-1} + D_0(S)P + SD_0(S^{n-1}) + PD_0(S^{n-1}). \end{aligned}$$

From the above relation it follows that

$$D_0(S^{n-1})P + PD_0(S) + D_0(S)P + PD_0(S^{n-1}) = 0.$$

Since $D_0(S) = D_0(A)$, we can rewrite the above relation as

$$D_0(A^{n-1})P + PD_0(A) + D_0(A)P + PD_0(A^{n-1}) = 0. \tag{10}$$

Putting $2A$ for A in the above relation, we obtain

$$2^{n-1}D_0(A^{n-1})P + 2PD_0(A) + 2D_0(A)P + 2^{n-1}PD_0(A^{n-1}) = 0. \tag{11}$$

In case $n = 2$, the relation (10) implies that

$$PD_0(A) + D_0(A)P = 0. \tag{12}$$

In case $n > 2$, the relations (10) and (11) give the above relation (12). Multiplying the above relation from both sides by P, we obtain

$$PD_0(A)P = 0.$$

Right multiplication by P in the relation (12) gives $PD_0(A)P + D_0(A)P = 0$, which is reduced by the above relation to

$$D_0(A)P = 0.$$

Since P is an arbitrary one-dimensional projection, it follows from the above relation that $D_0(A) = 0$ for all $A \in \mathcal{A}(X)$, which completes the proof of the theorem. □

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