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Existence of Ψ-Bounded Solutions for Linear Matrix Difference Equations on $\mathbb{Z}^+$

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ABSTRACT

This paper deals with obtaining necessary and sufficient conditions for the existence of at least one Ψ-bounded solution for the linear matrix difference equation $X(n+1) = A(n)X(n)B(n) + F(n)$, where $F(n)$ is a Ψ-summable matrix valued function on $\mathbb{Z}^+$. Finally, we prove a result relating to the asymptotic behavior of the Ψ-bounded solutions of this equation on $\mathbb{Z}^+$.

RESUMEN

Este artículo se enfoca en obtener condiciones necesarias y suficientes para la existencia de la menos una solución Ψ-acotada para la ecuación lineal en diferencias matricial $X(n+1) = A(n)X(n)B(n) + F(n)$, donde $F(n)$ es una función Ψ-sumable con valores matriciales en $\mathbb{Z}^+$. Finalmente, probamos un resultado relacionado al comportamiento asintótico de las soluciones Ψ-acotadas de esta ecuación en $\mathbb{Z}^+$.

Keywords and Phrases: Difference Equations; Fundamental Matrix; Ψ-bounded; Ψ-summable, Kronecker product.

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1 Introduction

Difference equations serve as a natural description of observed evolution phenomena. The theory of difference equations is of immense use in the construction of dicrete mathematical models, which can explain better when compared to continuous models. One of the important fetures of difference equations is, they appear in the study of discretization methods for differential equations. Difference equations play an important role in many scientific fields such as numerical analysis, finite element techniques, control theory, discrete mathematical structures and several problems of mathematical modelling [1, 2, 14]. Due to the importance and rapid growth of research in this area, we confine our attention to the linear matrix difference equation

$$X(n+1) = A(n)X(n)B(n) + F(n), \tag{1}$$

where $A(n)$, $B(n)$,and $F(n)$ are $m \times m$ matrix-valued functions on $\mathbb{Z}^+ = \{1, 2, \ldots\}$. The basic problem under consideration is the determination of necessary and sufficient conditions for the existence of a solution with some specified boundedness condition. A Classical result of this type, for system of differential equations is given by Coppel [6, Theorem 2, Chapter V]. The problem of Ψ-boundedness of the solutions for systems of ordinary differential equations has been studied in many papers, [3, 4, 5, 7, 13, 16]. Recently [10, 18, 19], extended the concept of Ψ-boundedness of the solutions to Lyapunov matrix differential equations.

Recently , Han and Hong [15], Diamandescu [8, 11] extended the concept of Ψ-bounded solutions of system of differential equations to difference equations. The existence and uniqueness of solutions of matrix difference equation (1) was studied by Murty, Anand and Lakshmi Prasannam [17].

The aim of present paper is to give a necessary and sufficient condition for the existence of Ψ-bounded solution of the linear matrix difference equation (1) via Ψ-summable sequence. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions. Here, Ψ is a matrix-valued function. This paper include the results of Han and Hong [15] as a particular case when $B = I$, X and F are column vectors.

2 Preliminaries

In this section we present some basic definitions, notations and results which are useful for later discussion.

Let $\mathbb{R}^m$ be the Euclidean m-space. For $u = (u_1, u_2, u_3, \ldots, u_m)^T \in \mathbb{R}^m$, let $\|u\| = \max\{|u_1|, |u_2|, |u_3|, \ldots, |u_m|\}$ be the norm of u. Let $\mathbb{R}^{m\times m}$ be the linear space of all $m \times m$ real valued matrices. For a $m \times m$ real matrix $A = [a_{ij}]$, we define the norm $|A| = \sup_{\|u\|\leq 1} \|Au\|$. It is well-known that

$$|A| = \max_{1\leq i\leq m}\{\sum_{j=1}^{m} |a_{ij}|\}.$$

Let $\Psi_k : \mathbb{Z}^+ \to \mathbb{R} - \{0\}$ ($\mathbb{R} - \{0\}$ is the set of all nonzero real numbers), $k = 1, 2, \dots m$, and let

$$\Psi = \text{diag}[\Psi_1, \Psi_2, \dots, \Psi_m].$$

Then the matrix $\Psi(n)$ is an invertible square matrix of order m, for all $n \in \mathbb{Z}^+$.

Definition 2.1. *[12] Let $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & . & . & . & a_{1q}B \\ a_{21}B & a_{22}B & . & . & . & a_{2q}B \\ . & . & . & . & . & . \\ a_{p1}B & a_{p2}B & . & . & . & a_{pq}B \end{bmatrix}$$

is an $pr \times qs$ matrix and is in $\mathbb{R}^{pr\times qs}$.

Definition 2.2. *[12] Let $A = [a_{ij}] \in \mathbb{R}^{p\times q}$, then the vectorization operator $\mathsf{Vec} : \mathbb{R}^{p\times q} \to \mathbb{R}^{pq}$, defined and denote by*

$$\hat{A} = \mathsf{Vec}A = \begin{bmatrix} A_{.1} \\ A_{.2} \\ . \\ . \\ A_{.q} \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{pj} \end{bmatrix} (1 \le \ j \le \ q).$$

Lemma 2.3. *The vectorization operator $\mathsf{Vec} : \mathbb{R}^{m\times m} \to \mathbb{R}^{m^2}$, is a linear and one-to-one operator. In addition, Vec and Vec^{-1} are continuous operators.*

Proof. The fact that the vectorization operator is linear and one-to-one is immediate. Now, for$A = [a_{ij}] \in \mathbb{R}^{m\times m}$, we have

$$\|\mathsf{Vec}(A)\| = \max_{1\le i,j\le m} \{|a_{ij}|\} \le \max_{1\le i\le m} \left\{ \sum_{j=1}^{m} |a_{ij}| \right\} = |A|.$$

Thus, the vectorization operator is continuous and $\|\mathsf{Vec}\| \le 1$.

In addition, for $A = I_m$ (the identity $m \times m$ matrix) we have $\|\mathsf{Vec}(I_m)\| = 1 = |I_m|$ and then, $\|\mathsf{Vec}\| = 1$.

Obviously, the inverse of the vectorization operator, $\mathsf{Vec}^{-1} : \mathbb{R}^{m^2} \to \mathbb{R}^{m\times m}$, is defined by

$$\mathsf{Vec}^{-1}(u) = \begin{bmatrix} u_1 & u_{m+1} & . & . & . & u_{m^2-m+1} \\ u_2 & u_{m+2} & . & . & . & u_{m^2-m+2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ u_m & u_{2m} & . & . & . & u_{m^2} \end{bmatrix}.$$

Where $u = (u_1, u_2, u_3, ..., u_{m^2})^T \in \mathbb{R}^{m^2}$. We have

$$\left|\mathsf{Vec}^{-1}(u)\right| = \max_{1\le i\le m}\left\{\sum_{j=0}^{m-1}|u_{mj+i}|\right\} \le m.\max_{1\le i\le m}\{|u_i|\} = m.\,\|u\|.$$

Thus, Vec^{-1} is a continuous operator. Also, if we take $u = \mathsf{Vec}A$ in the above inequality, then the following inequality holds

$$|A| \le m\|\mathsf{Vec}A\|,$$

for every $A \in \mathbb{R}^{m\times m}$. □

Regarding properties and rules for Kronecker product of matrices we refer to [12].

Now by applying the Vec operator to the linear nonhomogeneous matrix difference equation (1) and using Kronecker product properties, we have

$$\hat{X}(n+1) = G(n)\hat{X}(n) + \hat{F}(n), \tag{2}$$

where $G(n) = B^T(n) \otimes A(n)$ is a $m^2 \times m^2$ matrix and $\hat{F}(n) = \mathsf{Vec}F(n)$ is a column matrix of order m^2. The equation (2) is called the Kronecker product difference equation associated with (1). It is clear that, if $X(n)$ is a solution of (1) if and only if $\hat{X}(n) = \mathsf{Vec}X(n)$ is a solution of (2).

The corresponding homogeneous difference equation of (2) is

$$\hat{X}(n+1) = G(n)\hat{X}(n). \tag{3}$$

Definition 2.4. *[15] A sequence $\phi : \mathbb{Z}^+ \to \mathbb{R}^m$ is said to be Ψ- bounded on $\mathbb{Z}^+$ if $\Psi(n)\phi(n)$ is bounded on $\mathbb{Z}^+$ (i.e., there exists $L > 0$ such that $\|\Psi(n)\phi(n)\| \le L$, for all $n \in \mathbb{Z}^+$).*

Extend this definition to matrix functions.

Definition 2.5. *A matrix sequence $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$ is said to be Ψ-bounded on $\mathbb{Z}^+$ if ΨF is bounded on $\mathbb{Z}^+$ (i.e., there exists $L > 0$ such that $|\Psi(n)F(n)| \le L$, for all $n \in \mathbb{Z}^+$).*

Definition 2.6. *[15] A sequence $\phi : \mathbb{Z}^+ \to \mathbb{R}^m$ is said to be Ψ-summable on $\mathbb{Z}^+$ if $\phi(n) \in l^1$ and $\Psi(n)\phi(n) \in l^1$. $\left(\text{i.e., } \lim_{p\to\infty}\sum_{n=1}^{p}\|\Psi(n)\phi(n)\| < \infty\right)$.*

Extend this definition to matrix functions.

Definition 2.7. *A matrix sequence $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$ is called Ψ-summable on $\mathbb{Z}^+$ if $\sum_{n=1}^{\infty}\Psi(n)F(n)$ is convergent $\left(\text{i.e., } \lim_{p\to\infty}\sum_{n=1}^{p}|\Psi(n)F(n)| < \infty\right)$.*

Now we shall assume that $A(n)$ and $B(n)$ are bounded $m \times m$ matrices on $\mathbb{Z}^+$ and $F(n)$ is a Ψ-summable matrix function on $\mathbb{Z}^+$.

By a solution of (1), we mean a matrix function $W(n)$ satisfying the equation (1) for all most all $n \in \mathbb{Z}^+$.

The following lemmas play a vital role in the proof of main result.

Lemma 2.8. *The matrix function* $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$ *is* Ψ*-summable on* $\mathbb{Z}^+$ *if and only if the vector function* $\mathrm{Vec}F(n)$ *is* $I_m \otimes \Psi$*-summable on* $\mathbb{Z}^+$*.*

Proof. From the proof of Lemma 2.3, it follows that

$$\frac{1}{m}\,|A| \leq \|\mathrm{Vec}A\|_{\mathbb{R}^{m^2}} \leq |A|\,,$$

for every $A \in \mathbb{R}^{m\times m}$.

Put $A = \Psi(n)F(n)$ in the above inequality, we have

$$\frac{1}{m}\,|\Psi(n)F(n)| \leq \|(I_m \otimes \Psi(n)).\mathrm{Vec}F(n)\|_{\mathbb{R}^{m^2}} \leq |\Psi(n)F(n)|\,, \tag{4}$$

$n \in \mathbb{Z}^+$, for all matrix functions $F(n)$.

Suppose that $F(n)$ is Ψ-summable on $\mathbb{Z}^+$. From (4), we get

$$\|(I_m \otimes \Psi(n)).\mathrm{Vec}F(n)\|_{\mathbb{R}^{m^2}} \leq |\Psi(n)F(n)|\,,$$

which implies

$$\sum_{n=1}^{\infty} \|(I_m \otimes \Psi(n)).\mathrm{Vec}F(n)\|_{\mathbb{R}^{m^2}} \leq \sum_{n=1}^{\infty} |\Psi(n)F(n)|\,.$$

From comparison test, Definitions 2.6 and 2.7, $\hat{F}(n)$ is $I_m \otimes \Psi$-summable on $\mathbb{Z}^+$.

Conversely suppose that $\hat{F}(n)$ is $I_m \otimes \Psi$-summable on $\mathbb{Z}^+$. Again from (4), we get

$$|\Psi(n)F(n)| \leq m\,\|(I_m \otimes \Psi(n)).\mathrm{Vec}F(n)\|_{\mathbb{R}^{m^2}}\,,$$

which implies

$$\sum_{n=1}^{\infty} |\Psi(n)F(n)| \leq m \sum_{n=1}^{\infty} \|(I_m \otimes \Psi(n)).\mathrm{Vec}F(n)\|_{\mathbb{R}^{m^2}}\,.$$

From comparison test, Definitions 2.6 and 2.7, $F(n)$ is Ψ-summable on $\mathbb{Z}^+$. Now the proof is complete. □

Lemma 2.9. *The matrix function* $F(n)$ *is* Ψ *- bounded on* $\mathbb{Z}^+$ *if and only if the vector function* $\mathrm{Vec}F(n)$ *is* $I_m \otimes \Psi$ *- bounded on* $\mathbb{Z}^+$*.*

Proof. The proof easily follows from the inequality (4). □

Lemma 2.10. *If* $A(n)$, $B(n)$ *are invertible matrix functions and* $F(n)$ *is a matrix function on* $\mathbb{Z}^+$*. Let* $Y(n)$ *and* $Z(n)$ *be the fundamental matrices for the matrix difference equations*

$$X(n+1) = A(n)X(n), \quad n \in \mathbb{Z}^+ \tag{5}$$

and

$$X(n+1) = B^T(n)X(n), \quad n \in \mathbb{Z}^+ \tag{6}$$

respectively. Then the matrix $Z(n) \otimes Y(n)$ *is a fundamental matrix of* (3).

Proof. Consider

$$\begin{aligned} Z(n+1) \otimes Y(n+1) &= B^T(n)Z(n) \otimes A(n)Y(n) \\ &= (B^T(n) \otimes A(n))(Z(n) \otimes Y(n)) \\ &= G(n)(Z(n) \otimes Y(n)), \end{aligned}$$

for all $n \in \mathbb{Z}^+$.

On the other hand, the matrix $Z(n) \otimes Y(n)$ is an invertible matrix for all $n \in \mathbb{Z}^+$ (because $Z(n)$ and $Y(n)$ are invertible matrices for all $n \in \mathbb{Z}^+$). □

Let $\mathbf{X}_1$ denote the subspace of $\mathbb{R}^{m\times m}$ consisting of all matrices which are values of Ψ-bounded solution of $X(n+1) = A(n)X(n)B(n)$ on $\mathbb{Z}^+$ at $n = 1$ and let $\mathbf{X}_2$ an arbitrary fixed subspace of $\mathbb{R}^{m\times m}$, supplementary to $\mathbf{X}_1$. Let P_1, P_2 denote the corresponding projections of $\mathbb{R}^{m\times m}$ onto $\mathbf{X}_1$, $\mathbf{X}_2$ respectively.

Then $\overline{\mathbf{X}}_1$ denote the subspace of $\mathbb{R}^{m^2}$ consisting of all vectors which are values of $I_m \otimes \Psi$-bounded solution of (3) on $\mathbb{Z}^+$ at $n = 1$ and $\overline{\mathbf{X}}_2$ a fixed subspace of $\mathbb{R}^{m^2}$, supplementary to $\overline{\mathbf{X}}_1$. Let Q_1, Q_2 denote the corresponding projections of $\mathbb{R}^{m^2}$ onto $\overline{\mathbf{X}}_1$, $\overline{\mathbf{X}}_2$ respectively.

Theorem 2.11. *Let* $Y(n)$ *and* $Z(n)$ *be the fundamental matrices for the systems* (5) *and* (6)*. If*

$$\begin{aligned} \hat{X}(n) = \sum_{k=1}^{n-1} &(Z(n) \otimes Y(n))Q_1(Z^{-1}(k+1) \otimes Y^{-1}(k+1))\hat{F}(k) \\ &- \sum_{k=1}^{\infty}(Z(n) \otimes Y(n))Q_2(Z^{-1}(k+1) \otimes Y^{-1}(k+1))\hat{F}(k) \end{aligned} \tag{7}$$

is convergent, then it is a solution of (2) *on* $\mathbb{Z}^+$.

Proof. It is easily seen that $\hat{X}(n)$ is the solution of (2) on $\mathbb{Z}^+$. □

The following theorems are useful in the proofs of our main results.

Theorem 2.12. *[15] Let* $\{A(n)\}$ *be bounded. Then*

$$x(n+1) = A(n)x(n) + f(n) \tag{8}$$

has at least one Ψ*-bounded solution on* $\mathbb{Z}^+$ *for every* Ψ*-summable sequence* $\{f(n)\}$ *on* $\mathbb{Z}^+$ *if and only if there is a positive constant* K *such that*

$$\begin{aligned} |\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)| \le K, \quad 1 \le k+1 \le n \\ |\Psi(n)Y(n)P_2Y^{-1}(k+1)\Psi^{-1}(k)| \le K, \quad 1 \le n < k+1. \end{aligned} \tag{9}$$

Theorem 2.13. *[15] Suppose that:*

(1) The fundamental matrix $Y(n)$ *of* $x(n+1) = A(n)x(n)$ *satisfies conditions*

(a) $\lim_{n\to\infty} |\Psi(n)Y(n)P_1| = 0,$

(b) condition (9) *holds, where* K *is a positive constant,* P_1 *and* P_2 *are suplementary projections.*

(2) The sequence $f : \mathbb{Z}^+ \to \mathbb{R}^m$ *is* Ψ*-summable on* $\mathbb{Z}^+$*.*

Then, every Ψ*-bounded solution* $x(n)$ *of* (8) *satisfies*

$$\lim_{n\to\infty} \|\Psi(n)x(n)\| = 0.$$

3 Main result

The main theorems of this paper are proved in this section.

Theorem 3.1. *Let* $A(n)$ *and* $B(n)$ *be bounded matrices on* $\mathbb{Z}^+$*, then* (1) *has at least one* Ψ*-bounded solution on* $\mathbb{Z}^+$ *for every* Ψ*-summable matrix function* $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$ *on* $\mathbb{Z}^+$ *if and only if there exists a positive constant* K *such that*

$$\begin{aligned} |(Z(n) \otimes \Psi(n)Y(n))Q_1(Z^{-1}(k+1) \otimes Y^{-1}(k+1)\Psi^{-1}(k))| \le K, \quad 1 \le k+1 \le n, \\ |(Z(n) \otimes \Psi(n)Y(n))Q_2(Z^{-1}(k+1) \otimes Y^{-1}(k+1)\Psi^{-1}(k))| \le K, \quad 1 \le n < k+1. \end{aligned} \tag{10}$$

Proof. Suppose that the equation (1) has at least one Ψ-bounded solution on $\mathbb{Z}^+$ for every Ψ-summable matrix function $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$.

Let $\hat{F} : \mathbb{Z}^+ \to \mathbb{R}^{m^2}$ be $I_m \otimes \Psi$-summable function on $\mathbb{Z}^+$. From Lemma 2.8, it follows that the matrix function $F(n) = \mathrm{Vec}^{-1}\hat{F}(n)$ is Ψ - summable matrix function on $\mathbb{Z}^+$. From the hypothesis, the system (1) has at least one Ψ - bounded solution $X(n)$ on $\mathbb{Z}^+$. From Lemma 2.9, it follows that the vector valued function $\hat{X}(n) = \mathrm{Vec}X(n)$ is a $I_m \otimes \Psi$-bounded solution of (2) on $\mathbb{Z}^+$.

Thus, equation (2) has at least one $I_m \otimes \Psi$-bounded solution on $\mathbb{Z}^+$ for every $I_m \otimes \Psi$-summable function $\hat{F}$ on $\mathbb{Z}^+$.

From Theorem 2.12, there is a positive constant K such that the fundamental matrix $T(n) = Z(n) \otimes Y(n)$ of the system (3) satisfies the condition

$$|(I_m \otimes \Psi(n))T(n)Q_1T^{-1}(k+1)(I_m \otimes \Psi^{-1}(k))| \leq K, \quad 1 \leq k+1 \leq n,$$
$$|(I_m \otimes \Psi(n))T(n)Q_2T^{-1}(k+1)(I_m \otimes \Psi^{-1}(k))| \leq K, \quad 1 \leq n < k+1.$$

Putting $T(n) = Z(n) \otimes Y(n)$ and using Kronecker product properties, (10) holds.

Conversely suppose that (10) holds for some $K > 0$.

Let $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$ be a Ψ-summable matrix function on $\mathbb{Z}^+$. From Lemma 2.8, it follows that the vector valued function $\hat{F}(n) = \mathsf{Vec}F(n)$ is a $I_m \otimes \Psi$-summable function on $\mathbb{Z}^+$.

Since $A(n)$, $B(n)$ are bounded, then $G(n) = B^T(n) \otimes A(n)$ is also bounded. Now from Theorem 2.12, the difference equation (2) has at least one $I_m \otimes \Psi$ - bounded solution on $\mathbb{Z}^+$. Let $x(n)$ be this solution.

From Lemma 2.9, it follows that the matrix function $X(n) = \mathsf{Vec}^{-1}x(n)$ is a Ψ-bounded solution of the equation (1) on $\mathbb{Z}^+$ (because $F(n) = \mathsf{Vec}^{-1}\hat{F}(n)$).

Thus, the matrix difference equation (1) has at least one Ψ-bounded solution on $\mathbb{Z}^+$ for every Ψ-summable matrix function F on $\mathbb{Z}^+$. $\square$

Theorem 3.2. *Suppose that:*

(1) The fundamental matrices $Y(n)$ *and* $Z(n)$ *of* (5) *and* (6) *satisfies:*

- *(a)* $\lim_{n\to\infty} |(Z(n) \otimes \Psi(n)Y(n))Q_1| = 0$*;*
- *(b) condition* (10) *holds, for some* $K > 0$*.*

(2) The matrix function $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$ *is* Ψ*-summable on* $\mathbb{Z}^+$*.*

Then, every Ψ*-bounded solution* X *of* (1) *is such that*

$$\lim_{n\to\infty} |\Psi(n)X(n)| = 0.$$

Proof. Let $X(n)$ be a Ψ-bounded solution of (1). From Lemma 2.9, it follows that the function $\hat{X}(n) = \mathsf{Vec}X(n)$ is a $I_m \otimes \Psi$- bounded solution on $\mathbb{Z}^+$ of the difference equation (2). Also from Lemma 2.8, the function $\hat{F}(n)$ is $I_m \otimes \Psi$-summable on $\mathbb{Z}^+$. From the Theorem 2.13, it follows that

$$\lim_{n\to\infty} \left\|(I_m \otimes \Psi(n))\, \hat{X}(n)\right\| = 0.$$

Now, from the inequality (4) we have

$$|\Psi(n)X(n)| \leq m \left\|(I_m \otimes \Psi(n))\, \hat{X}(n)\right\|, n \in \mathbb{Z}^+$$

and, then

$$\lim_{n\to\infty} |\Psi(n)X(n)| = 0.$$

$\square$

The following example illustrates the above theorems.

Example 3.1.

Consider the matrix difference equation (1) with

$$A(n) = \begin{bmatrix} \frac{n+1}{n} & 0 \\ 0 & 2 \end{bmatrix}, \quad B(n) = \begin{bmatrix} \left(\frac{n+2}{n+3}\right)^{\frac{1}{4}} & 0 \\ 0 & \frac{n+3}{n+1} \end{bmatrix} \text{ and } F(n) = \begin{bmatrix} \frac{n}{3^n} & 0 \\ 0 & \frac{n^2 2^{n-1}}{3^n} \end{bmatrix}.$$

Then,

$$Y(n) = \begin{bmatrix} n & 0 \\ 0 & 2^{n-1} \end{bmatrix} \text{ and } Z(n) = \begin{bmatrix} \left(\frac{3}{n+2}\right)^{\frac{1}{4}} & 0 \\ 0 & \frac{(n+1)(n+2)}{6} \end{bmatrix}$$

are the fundamental matrices for (5) and (6) respectively. Consider

$$\Psi(n) = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & 2^{1-n} \end{bmatrix}, \text{ for all } n \in \mathbb{Z}^+.$$

There exist projections

$$Q_1 = \begin{bmatrix} I_2 & O_2 \\ O_2 & O_2 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} O_2 & O_2 \\ O_2 & I_2 \end{bmatrix}$$

such that conditions in (10) are satisfied with $K = 2$.

In addition, the hypothesis (1a) and (2) of Theorem 3.2 are satisfied. Because

$$|(Z(n) \otimes \Psi(n)Y(n))Q_1| = \left(\frac{3}{n+2}\right)^{\frac{1}{4}},$$

$$\Psi(n)F(n) = \begin{bmatrix} \frac{n^2}{3^n} & 0 \\ 0 & \frac{n^2}{3^n} \end{bmatrix},$$

and

$$\sum_{n=1}^{\infty} |\Psi(n)F(n)| = \sum_{n=1}^{\infty} \frac{n^2}{3^n} < \infty \text{ (ratio test)},$$

the matrix function F is Ψ-summable on $\mathbb{Z}^+$. From Theorems 3.1 and 3.2, the difference equation has at least one Ψ-bounded solution and every Ψ-bounded solution X of (1) is such that $\lim\limits_{n\to\infty} |\Psi(n)X(n)| = 0$.

Remark 3.1.

Theorem 3.2 is no longer true if we require that the matrix function F be Ψ-bounded on $\mathbb{Z}^+$, instead of the condition (2) in the above Theorem.

This is shown in the following example.

Example 3.2.

Consider the matrix difference equation (1) with

$$A(n)=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 2\end{bmatrix},\quad B(n)=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\quad\text{and}\quad F(n)=\begin{bmatrix}1 & 0\\ 0 & 3^n\end{bmatrix}.$$

Then,

$$Y(n)=\begin{bmatrix}2^{1-n} & 0\\ 0 & 2^{n-1}\end{bmatrix}\quad\text{and}\quad Z(n)=\begin{bmatrix}1 & 0\\ 0 & 3^{n-1}\end{bmatrix}$$

are the fundamental matrices for (5) and (6) respectively. Consider

$$\Psi(n)=\begin{bmatrix}1 & 0\\ 0 & 3^{1-n}\end{bmatrix},\quad\text{for all}\ n\in\mathbb{Z}^+.$$

There exist projections

$$Q_1=\begin{bmatrix}I_2 & O_2\\ O_2 & O_2\end{bmatrix}\quad\text{and}\quad Q_2=\begin{bmatrix}O_2 & O_2\\ O_2 & I_2\end{bmatrix},$$

such that conditions in hypothesis (1) are satisfied with $K=2$. Also $|\Psi(n)F(n)|=3$, for $n\in\mathbb{Z}^+$. Therefore, F is Ψ-bounded on $\mathbb{Z}^+$. Clearly, the function F is not Ψ-summable on $\mathbb{Z}^+$.

The solutions of the equation (1) are

$$X(n)=\begin{bmatrix}2^{1-n}(c_1-2)+2 & (\frac{3}{2})^{n-1}c_2\\ 2^{n-1}c_3 & 6^{n-1}(c_4+1)-3^{n-1}\end{bmatrix},$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. It is easily seen that, there is no solution $X(n)$ of (1) for $\lim\limits_{n\to\infty}|\Psi(n)X(n)|=0$.

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