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An other uncertainty principle for the Hankel transform

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ABSTRACT

We use the Hausdorff-Young inequality for the Hankel transform to prove the uncertainly principle in terms of entropy. Next, we show that we can obtain the Heisenberg-Pauli-Weyl inequality related to the same Hankel transform.

RESUMEN

Usamos la desigualdad de Hausdorff-Young para la transformada de Hankel para probar el principio de incertidumbre en términos de la entropía. Además probamos que podemos obtener la desigualdad de Heisenberg-Pauli-Weyl relacionada con la misma transformada de Hankel.

Keywords and Phrases: Uncertainty principle, Hausdorff-Young inequality, entropy, Hankel transform

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1 Introduction:

The uncertainly principles play an import role in harmonic analysis. They state that a function f and its Fourier transform $\widehat{f}$ can not be simultaneously sharply localized in the sense that it is impossible for a nonzero function and its Fourier transform to be simultaneously small.
Many mathematical formulations of this fact can be found in [2, 5, 6, 11, 16, 17].
For a probability density function f on $\mathbb{R}^n$, the entropy of f is defined according to [18] by

$$E(f) = -\int_{\mathbb{R}^n} f(x)\ln(f(x))dx.$$

The entropy $E(f)$ is closely related to quantum mechanics [4] and constitutes one of the important way to measure the concentration of f.
The uncertainly principle in terms of entropy consists to compare the entropy of $|f|^2$ with that of $|\widehat{f}|^2$.
A first result has been given in [13], where Hirschman established a weak version of this uncertainly principle by showing that for every square integrable function f on $\mathbb{R}^n$ with respect to the Lebesgue measure, such that $||f||_2 = 1$, we have

$$E(|f|^2) + E(|\widehat{f}|^2) \geqslant 0.$$

Later, in [1, 2], Beckner proved the following stronger inequality, that is for every square integrable function f on $\mathbb{R}^n$; $||f||_2 = 1$,

$$E(|f|^2) + E(|\widehat{f}|^2) \geqslant n(1 - \ln 2).$$

The last inequality constituted a very powerful result which implies in particular the well known Heisenberg-Pauli-Weyl uncertainly principle [17].

In this paper, we consider the singular differential operator defined on $]0, +\infty[$ by

$$\ell_\alpha = \frac{d^2}{dr^2} + \frac{2\alpha+1}{r}\frac{d}{dr} = \frac{1}{r^{2\alpha+1}}\frac{d}{dr}[r^{2\alpha+1}\frac{d}{dr}]; \ \alpha \geqslant \frac{-1}{2}.$$

The Hankel transform associated with the operator ℓ_α is defined by

$$\mathscr{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(r)j_\alpha(\lambda r)d\mu_\alpha(r),$$

where
. $d\mu_\alpha(r)$ is the measure defined on $[0, +\infty[$ by

$$d\mu_\alpha(r) = \frac{r^{2\alpha+1}dr}{2^\alpha\,\Gamma(\alpha+1)}.$$

. j_α is the modified Bessel-function that will be defined in the second section .
Many harmonic analysis results have been establish for the Hankel transform $\mathscr{H}_\alpha$ [14, 19, 20].
Also, many uncertainly principles have been proved for the transform $\mathscr{H}_\alpha$ [17, 21].

Our investigation in this work consists to establish the uncertainly principle in terms of entropy for the Hankel transform $\mathscr{H}_\alpha$.
For a nonnegative measurable function f on $[0,+\infty[$, the entropy of f is defined by

$$E_{\mu_\alpha}(f) = -\int_0^{+\infty} f(r)\ln(f(r))d\mu_\alpha(r).$$

Then using the Hausdorff-Young inequality for $\mathscr{H}_\alpha$ [9], we establish the main result of this work.
. *Let* $f \in L^2(d\mu_\alpha)$; $||f||_{2,\mu_\alpha} = 1$ *such that*

$$\int_0^{+\infty} |f(r)|^2 \Big|\ln(|f(r)|)\Big| d\mu_\alpha(r) < +\infty,$$

and

$$\int_0^{+\infty} \Big|\mathscr{H}_\alpha(f)(\lambda)\Big|^2 \Big|\ln\left(|\mathscr{H}_\alpha(f)(\lambda)|\right)\Big| d\mu_\alpha(\lambda) < +\infty.$$

Then

$$E_{\mu_\alpha}\left(|f|^2\right) + E_{\mu_\alpha}\left(\Big|\mathscr{H}_\alpha(f)\Big|^2\right) \geqslant (2\alpha+1)(1-\ln 2),$$

where $L^p(d\mu_\alpha)$; $p \in [1,+\infty]$, is the Lebesgue space of measurable functions on $[0,+\infty[$ such that

$$||f||_{p,\mu_\alpha} < +\infty,$$

with

$$||f||_{p,\mu_\alpha} = \begin{cases} \left(\int_0^{+\infty} \Big|f(r)\Big|^p d\mu_\alpha(r)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[, \\ \underset{r\in [0,+\infty[}{\text{ess sup}} \Big|f(r)\Big|, & \text{if } p = +\infty. \end{cases}$$

Using this result, we prove that we can derive the Heisenberg -Pauli-Weyl inequality for $\mathscr{H}_\alpha$, that is
. For every $f \in L^2(d\mu_\alpha)$; we have

$$||rf||_{2,\mu_\alpha}||\lambda\mathscr{H}_\alpha(f)||_{2,\mu_\alpha} \geqslant (\alpha+1)||f||^2_{2,\mu_\alpha}.$$

2 The Hankel operator

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with Hankel operator.
Let ℓ_α be the Bessel operator defined on $]0 + \infty[$ by

$$\ell_\alpha u = \frac{d^2}{dr^2}u + \frac{2\alpha+1}{r}\frac{du}{dr}.$$

Then, for every $\lambda \in \mathbb{C}$, the following system

$$\begin{cases} \ell_\alpha u(x) = -\lambda^2 u(x), \\ u(0) = 1, \\ u'(0) = 0, \end{cases}$$

admits a unique solution given by $j_\alpha(\lambda .)$, where

$$\begin{aligned} j_\alpha(z) &= \frac{2^\alpha \Gamma(\alpha+1)}{z^\alpha} J_\alpha(z) \\ &= \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\alpha+k+1)} \left(\frac{z}{2}\right)^{2k}, \end{aligned} \tag{2.1}$$

with J_α is the Bessel function of first kind and index α [7, 8, 15, 22].
The modified Bessel function j_α satisfies the following properties

for every $\alpha \geqslant -\frac{1}{2}$, the modified Bessel function j_α has the Mehler integral representation, for every $z \in \mathbb{C}$,

$$j_\alpha(z) = \begin{cases} \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \displaystyle\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos zt dt, \text{ if } \alpha > \dfrac{-1}{2}, \\ \cos z, \text{ if } \alpha = \dfrac{-1}{2}. \end{cases}$$

Consequently, for every $k \in \mathbb{N}$ and $z \in \mathbb{C}$; we have

$$\left|j_\alpha^{(k)}(z)\right| \leqslant \exp\big(|\operatorname{Im} z|\big). \tag{2.2}$$

The eigenfunction $j_\alpha(\lambda .)$ satisfies the following product formula [22],
for all $r, s \in [0, +\infty[$

$$j_\alpha(\lambda r) j_\alpha(\lambda s) = \begin{cases} \dfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \displaystyle\int_0^\pi j_\alpha\big(\lambda\sqrt{r^2+s^2+2rs\cos\theta}\big) \sin^{2\alpha}\theta d\theta; \text{ if } \alpha > \dfrac{-1}{2}, \\ \dfrac{j_\alpha\Big(\lambda(r+s)\Big) + j_\alpha\Big(\lambda\big((r-s)\big)\Big)}{2}, \text{ if } \alpha = \dfrac{-1}{2}. \end{cases} \tag{2.3}$$

The previous product formula allows us to define the Hankel translation operator and the convolution product related to the operator ℓ_α as follows

Definition 2.1. i) For every $r \in [0, +\infty[$, the Hankel translation operator τ_r^α is defined on $L^p(d\mu_\alpha)$; $p \in [1, +\infty]$, by

$$\tau_r^\alpha f(s) =$$

$$\begin{cases} \dfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \displaystyle\int_0^\pi f\big(\sqrt{r^2+s^2+2rs\cos\theta}\big) \sin^{2\alpha}\theta d\theta; \text{ if } \alpha > \dfrac{-1}{2}, \\ \dfrac{f(r+s) + f(|r-s|)}{2}, \text{ if } \alpha = \dfrac{-1}{2}. \end{cases}$$

ii) The convolution product of $f, g \in L^1(d\mu_\alpha)$ is defined for every $r \in [0, +\infty[$, by

$$f *_\alpha g(r) = \int_0^{+\infty} \tau_r^\alpha(f)(s) g(s) d\mu_\alpha(s). \tag{2.4}$$

Then the product formula (2.3) can be written

$$\tau_r^\alpha(j_\alpha(\lambda.))(s) = j_\alpha(\lambda r) j_\alpha(\lambda s). \tag{2.5}$$

We have the properties

Proposition 2.2. *i. For every* $f \in L^p(d\mu_\alpha)$; $1 \leqslant p \leqslant +\infty$, *and for every* $r \in [0, +\infty[$, *the function* $\tau_r^\alpha(f)$ *belongs to* $L^p(d\mu_\alpha)$ *and we have*

$$\left|\left|\tau_r^\alpha(f)\right|\right|_{p,\mu_\alpha} \leqslant ||f||_{p,\mu_\alpha}. \tag{2.6}$$

ii. For $f,\ g \in L^1(d\mu_\alpha)$, *the function* $f *_\alpha g$ *belongs to* $L^1(d\mu_\alpha)$; *the convolution product is commutative, associative and we have*

$$||f *_\alpha g||_{1,\mu_\alpha} \leqslant ||f||_{1,\mu_\alpha} ||g||_{1,\mu_\alpha}. \tag{2.7}$$

Moreover, if $1 \leqslant p, q, r \leqslant +\infty$ *are such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ *and if* $f \in L^p(d\mu_\alpha), g \in L^q(d\mu_\alpha)$, *then the function* $f *_\alpha g$ *belongs to* $L^r(d\mu_\alpha)$, *and we have the Young's inequality*

$$||f *_\alpha g||_{r,\mu_\alpha} \leqslant ||f||_{p,\mu_\alpha} ||g||_{q,\mu_\alpha}. \tag{2.8}$$

iii. For every $f \in L^1(d\mu_\alpha)$, *and* $r \in [0, +\infty[$ *the function* $\tau_r^\alpha(f)$ *belongs to* $L^1(d\mu_\alpha)$ *and we have*

$$\int_0^{+\infty} \tau_r^\alpha(f)(s) d\mu_\alpha(s) = \int_0^{+\infty} f(r) d\mu_\alpha(r). \tag{2.9}$$

We denoted by

. $\mathscr{C}_{*,0}(\mathbb{R})$ the space of even continuous functions f on $\mathbb{R}$ such that

$$\lim_{|r| \to +\infty} f(r) = 0.$$

. $\mathscr{S}_e(\mathbb{R})$ the space of even infinitely differentiable functions on $\mathbb{R}$; rapidly decreasing together with all their derivatives.

Now, we shall define the Hankel transform and we give the most important properties.

Definition 2.3. The Hankel transform $\mathscr{H}_\alpha$ is defined on $L^1(d\mu_\alpha)$ by

$$\forall \lambda \in \mathbb{R};\ \mathscr{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(r) j_\alpha(\lambda r) d\mu_\alpha(r), \tag{2.10}$$

where j_α is the modified Bessel function defined by the relation(2.1).

Proposition 2.4. *i. For every* $f \in L^1(d\mu_\alpha)$*, the function* $\mathscr{H}_\alpha(f)$ *belongs to the space* $\mathscr{C}_{*,0}(\mathbb{R})$ *and*

$$\big|\big|\mathscr{H}_\alpha(f)\big|\big|_{\infty,\mu_\alpha} \leqslant \|f\|_{1,\mu_\alpha}. \tag{2.11}$$

ii. For every $f \in L^1(d\mu_\alpha)$ *and* $r \in [0,+\infty[$,

$$\mathscr{H}_\alpha(\tau_r^\alpha(f))(\lambda) = j_\alpha(\lambda r)\mathscr{H}_\alpha(f)(\lambda). \tag{2.12}$$

iii. For all $f, g \in L^1(d\mu_\alpha)$,

$$\mathscr{H}_\alpha(f *_\alpha g)(\lambda) = \mathscr{H}_\alpha(f)(\lambda)\mathscr{H}_\alpha(g)(\lambda). \tag{2.13}$$

Theorem 2.5. *(Inversion formula) Let* $f \in L^1(d\mu_\alpha)$ *such that* $\mathscr{H}_\alpha(f) \in L^1(d\mu_\alpha)$*, then for almost every* $r \in [0,+\infty[$*, we have*

$$f(r) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda) j_\alpha(\lambda r) d\mu_\alpha(\lambda) = \mathscr{H}_\alpha\big(\mathscr{H}_\alpha(f)\big)(r). \tag{2.14}$$

Theorem 2.6. *(Plancherel) The Hankel transform* $\mathscr{H}_\alpha$ *can be extented to an isometric isomorphism from* $L^2(d\mu_\alpha)$ *onto itself. In particular, for all* f
and $g \in L^2(d\mu_\alpha)$*, we have (Parseval equality)*

$$\int_0^{+\infty} f(r)\overline{g(r)} d\mu_\alpha(r) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)} d\mu_\alpha(\lambda). \tag{2.15}$$

Proposition 2.7. *i. Let* $f \in L^1(d\mu_\alpha)$ *and* $g \in L^2(d\mu_\alpha)$*; by the relation (2.8), the function* $f *_\alpha g$ *belongs to* $L^2(d\mu_\alpha)$*, moreover*

$$\mathscr{H}_\alpha(f *_\alpha g)(\lambda) = \mathscr{H}_\alpha(f)(\lambda)\mathscr{H}_\alpha(g)(\lambda). \tag{2.16}$$

ii. For all f *and* $g \in L^2(d\mu_\alpha)$*, the function* $f *_\alpha g$ *belongs to* $\mathscr{C}_{*,0}(\mathbb{R})$ *and we have*

$$f *_\alpha g = \mathscr{H}_\alpha\big(\mathscr{H}_\alpha(f).\mathscr{H}_\alpha(g)\big). \tag{2.17}$$

iii. The Hankel transform $\mathscr{H}_\alpha$ *is a topological isomorphism from* $\mathscr{S}_e(\mathbb{R})$ *onto itself and we have*

$$\mathscr{H}_\alpha^{-1} = \mathscr{H}_\alpha. \tag{2.18}$$

iv. For every $f \in \mathscr{S}(\mathbb{R})$ *and* $g \in L^2(d\mu_\alpha)$*, we have*

$$\mathscr{H}_\alpha(fg)(\lambda) = \mathscr{H}_\alpha(f)(\lambda) *_\alpha \mathscr{H}_\alpha(g)(\lambda). \tag{2.19}$$

Definition 2.8. The Gaussian kernel associated with the Hankel operator is defined by

$$\forall t > 0,\ g_t(r) = \frac{e^{\frac{-r^2}{2t^2}}}{t^{2\alpha+2}}. \tag{2.20}$$

Thus, one can easily see that the family $(g_t)_{t>0}$ is an approximation of the identity, in particular, for every $f \in L^2(d\mu_\alpha)$ we have

$$\lim_{t\to 0^+} \big|\big|g_t *_\alpha f - f\big|\big|_{2,\mu_\alpha} = 0. \tag{2.21}$$

3 Uncertainty principle in terms of entropy for the Hankel transform

This section contains the main result of this paper that is the uncertainty principle in terms of entropy for the Hankel transform $\mathscr{H}_\alpha$. We start this section by the following Hausdorff-Young inequality.

Theorem 3.1. *[9] Let* $p \in]1, 2]$, *for every* $f \in L^p(d\mu_\alpha)$, *the function* $\mathscr{H}_\alpha(f)$ *belongs to* $L^{p'}(d\mu_\alpha)$; $p' = \frac{p}{p-1}$, *and we have*

$$\|\mathscr{H}_\alpha(f)\|_{p',\mu_\alpha} \leqslant A_{p,\alpha}\|f\|_{p,\mu_\alpha}, \tag{3.1}$$

where $A_{p,\alpha}$ *is the Babenko-Beckner constant,*

$$A_{p,\alpha} = \Big(\frac{p^{\frac{1}{p}}}{(\frac{p}{p-1})^{\frac{p-1}{p}}}\Big)^{\alpha+1}.$$

Lemma 3.2. *Let* x *be a positive real number. For every* $p \in [1, 2[$,

$$x^2 - x \leqslant \frac{x^p - x^2}{p-2} \leqslant x^2 \ln x. \tag{3.2}$$

Proof. Let $x > 0$ and let us put

$$g(p) = \frac{x^p - x^2}{p-2}.$$

g is differentiable on $[1, 2[$ and we have $g'(p) = \dfrac{h(p)}{(p-2)^2}$,
where $h(p) = (p-2)\ln(x).x^p - x^p + x^2$.
On the other hand,

$$\forall p \in [1, 2[;\ h'(p) = (p-2)x^p(\ln(x))^2 < 0$$

and $h(2) = 0$.
Thus, for every $p \in [1, 2]$, $h(p) \geqslant 0$ and the function g is increasing on $[1, 2[$, hence

$$g(1) \leqslant g(p) \leqslant \lim_{p\to 2^-} g(p).$$

□

In the following, we shall prove the uncertainty principle in terms of entropy for $f \in L^1(d\mu_\alpha) \cap L^2(d\mu_\alpha)$ such that $\|f\|_{2,\mu_\alpha} = 1$.

Theorem 3.3. *Let* $f \in L^1(d\mu_\alpha) \cap L^2(d\mu_\alpha)$; $\|f\|_{2,\mu_\alpha} = 1$, *such that*

$$\int_0^\infty |f(r)|^2 \; \big|\ln(|f(r)|)\big| \; d\mu_\alpha(r) < +\infty,$$

and

$$\int_0^\infty |\mathscr{H}(f)(\lambda)|^2 \; \big|\ln(|\mathscr{H}(f)(\lambda)|)\big| \; d\mu_\alpha(\lambda) < +\infty.$$

Then,

$$E_{\mu_\alpha}(|f|^2) + E_{\mu_\alpha}(|\mathscr{H}_\alpha(f)|^2) \quad \geqslant \quad (2\alpha+2)(1-\ln 2). \tag{3.3}$$

Proof. Let $f \in L^1(d\mu_\alpha) \cap L^2(d\mu_\alpha)$; $\|f\|_{2,\mu_\alpha} = 1$ and let φ be the function defined on $]1,2]$ by

$$\varphi(p) = \int_o^{+\infty} |\mathscr{H}_\alpha(f)(\lambda)|^{\frac{p}{p-1}} d\mu_\alpha(\lambda) - \Big(\frac{\frac{1}{p}^{\frac{1}{p}}}{(\frac{p}{p-1})^{\frac{p-1}{p}}}\Big)^{\frac{p(\alpha+1)}{p-1}} \Big(\int_o^{+\infty} |f(x)|^p d\mu_\alpha(x)\Big)^{\frac{1}{p-1}}.$$

Then, by relation (3.1),

$$\forall p \in]1,2]; \;\; \varphi(p) \leqslant 0.$$

On the other hand, Theorem 2.6 means that $\varphi(2) = 0$. This implies that

$$\frac{d\varphi}{dp}(2^-) \geqslant 0. \tag{3.4}$$

Since $f \in L^1(d\mu_\alpha) \cap L^2(d\mu_\alpha)$, then by a standard interpolation argument, f belongs to $L^p(d\mu_\alpha)$; $p \in [1,2]$.
Using Lemma 3.2, the hypothesis and Lebesgue dominated convergence theorem, we deduce that

$$\frac{d}{dp}\Big[\int_o^{+\infty} |f(r)|^p d\mu_\alpha(r)\Big](2^-) \quad = \quad \int_o^{+\infty} \lim_{p\to 2^-} \frac{|f(r)|^p - |f(r)|^2}{p-2} d\mu_\alpha(r). \tag{3.5}$$

Thus

$$\frac{d}{dp}\Big[\int_o^{+\infty} |f(r)|^p d\mu_\alpha(r)\Big](2^-) \quad = \quad \frac{1}{2}\int_o^{+\infty} \ln|f(r)|^2 |f(r)|^2 d\mu_\alpha(r)]. \tag{3.6}$$

Now, since $f \in L^1(d\mu_\alpha) \cap L^2(d\mu_\alpha)$, by using again the Lebesgue dominated convergence theorem, we get

$$\lim_{p\to 2}\int_o^{+\infty} |f(r)|^p d\mu_\alpha(r) \quad = \quad \int_o^{+\infty} |f(r)|^2 d\mu_\alpha(r) = 1. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\frac{d}{dp}\Big[\Big(\int_o^{+\infty} |f(r)|^p d\mu_\alpha(r)\Big)^{\frac{1}{p-1}}\Big](2^-) \quad = \quad -\frac{1}{2} E_{\mu_\alpha}(|f|^2). \tag{3.8}$$

As the same way, one can see that

$$\frac{d}{dp}\Big[\int_0^{+\infty}|\mathscr{H}_\alpha(f)(\lambda)|^{\frac{p}{p-1}}d\mu_\alpha(\lambda)\Big](2^-) = -\frac{1}{2}E_{\mu_\alpha}(|\mathscr{H}_\alpha(f)|^2). \tag{3.9}$$

Finally, basic calculations show that

$$\frac{d}{dp}\Big[(\frac{\frac{1}{p}^{\frac{1}{p}}}{(\frac{p}{p-1})^{\frac{p-1}{p}}})^{\frac{p}{p-1}(\alpha+1)}\Big](2^-) = (\alpha+1)(1-\ln 2). \tag{3.10}$$

Thus according to relations(3.8), (3.9) and (3.10), it follows that

$$\frac{d\varphi}{dp}(2^-) = \frac{1}{2}E_{\mu_\alpha}(|f|^2)+\frac{1}{2}E_{\mu_\alpha}(|\mathscr{H}_\alpha(f)|^2)-(\alpha+1)(1-\ln 2). \tag{3.11}$$

The proof is complete by using (3.4).

□

Lemma 3.4. *Let* f *be a measurable function on* $[0,+\infty[$ *and let*

$$\omega:[0,+\infty[\longmapsto[0,+\infty[$$

be a convex function such that $\omega(|f|)$ *belongs to* $L^1(d\mu_\alpha)$.
Let $(f_k)_{k\in\mathbb{N}}$ *be a sequence of nonnegative measurable functions on* $\mathbb{R}_+$ *such that for every* $k\in\mathbb{N}$; $||f_k||_{1,\mu_\alpha}=1$ *and the sequence* $(f_k*_\alpha f)_{k\in\mathbb{N}}$ *converges pointwise to* f.
Then, for every $k\in\mathbb{N}$, *the function* $\omega(|f_k*_\alpha f|$ *belongs to* $L^1(d\mu_\alpha)$, *and we have*

$$\lim_{k\to+\infty}\int_0^{+\infty}\omega(|f_k*_\alpha f|)(r)d\mu_\alpha(r) = \int_0^{+\infty}\omega(|f(r)|)d\mu_\alpha(r). \tag{3.12}$$

Proof. We have

$$||\omega\circ|f|||_{1,\mu_\alpha} = \int_0^{+\infty}\liminf_{k\to+\infty}\omega(|f_k*_\alpha f|)(r)d\mu_\alpha(r), \tag{3.13}$$

and by using Fatou's lemma, we get

$$||\omega\circ|f|||_{1,\alpha} \leqslant \liminf_{p\to+\infty}\int_0^{+\infty}\omega(|f_k*_\alpha f(r)|)d\mu_\alpha(r). \tag{3.14}$$

Conversely, according to relation (2.9), we have

$$\forall k\in\mathbb{N};\ \forall\lambda\in\mathbb{R}_+,\ \int_0^{+\infty}\tau_\lambda^\alpha(f_k)(r)d\mu_\alpha(r) = ||f_k||_{1,\mu_\alpha}=1, \tag{3.15}$$

which means that for every $\lambda\in\mathbb{R}_+$, $\tau_\lambda^\alpha(f_k)(r)d\mu_\alpha(r)$ is a probability measure on $\mathbb{R}_+$.
Therefore, by using Jensen's inequality for convex functions, we get

$$\begin{aligned}
\forall r\in\mathbb{R}_+,\ \omega(|f_k *_\alpha f(r)|) &= \omega(|\int_0^{+\infty} f(x)\tau_r^\alpha(f_k)(x)d\mu_\alpha(x))\\
&\leqslant \omega(\int_0^{+\infty} |f(x)\tau_r^\alpha(f_k)(x)|d\mu_\alpha(x)\\
&\leqslant \int_0^{+\infty} \omega(|f(x)|)\tau_r^\alpha(f_k)(x)d\mu_\alpha(x)\\
&= f_k *_\alpha \big(\omega\circ|f|\big)(r). \qquad (3.16)
\end{aligned}$$

In particular, $\omega\circ|f_k *_\alpha f|\in L^1(d\mu_\alpha)$. Hence by relations (2.9) and (3.16), we deduce that

$$\begin{aligned}
\limsup_{k\to+\infty}\int_0^{+\infty}\omega(|f_k *_\alpha f(r)|)d\mu_\alpha(r) &\leqslant \limsup_{k\to+\infty}\int_0^{+\infty} f_k *_\alpha \big(\omega\circ|f(r)|\big)d\mu_\alpha(r)\\
&= \limsup_{k\to+\infty}\|f_k *_\alpha \big(\omega\circ|f|\big)\|_{1,\alpha}\\
&\leqslant \|\omega\circ|f|\|_{1,\alpha}. \qquad (3.17)
\end{aligned}$$

The relations (3.14) and (3.17) show that

$$\lim_{k\to+\infty}\int_0^{+\infty}\omega(|f_k *_\alpha f|)(r)d\mu_\alpha(r)=\int_0^{+\infty}\omega(|f(r)|)d\mu_\alpha(r).$$

□

Now, e are able to prove the main result.

Theorem 3.5. *Let* $f\in L^2(d\mu_\alpha)$*;* $\|f\|_{2,\mu_\alpha}=1$, *such that*

$$\int_0^\infty |f(r)|^2\ \big|\ln(|f(r)|)\big|\ d\mu_\alpha(r)<+\infty,$$

and

$$\int_0^\infty |\mathscr{H}_\alpha(f)(\lambda)|^2\ \big|\ln(|\mathscr{H}_\alpha(f)(\lambda)|)\big|\ d\mu_\alpha(\lambda)<+\infty.$$

Then, we have

$$E_{\mu_\alpha}(|f|^2)+E_{\mu_\alpha}(|\mathscr{H}_\alpha(f)|^2)\ \geqslant\ (2\alpha+2)(1-\ln 2). \qquad (3.18)$$

Proof. The main idea of this proof is to combine Theorem 3.3 with the standard density argument. Indeed, we will show that for every $f\in L^2(d\mu_\alpha)$; there exists a sequence $(f_k)_{k\in\mathbb{N}}\in L^1(d\mu_\alpha)\cap L^2(d\mu_\alpha)$ such that

$$\lim_{k\to+\infty}\|f_k\|_{2,\mu_\alpha}\ =\ \|f\|_{2,\mu_\alpha}, \qquad (3.19)$$

$$\lim_{k\to+\infty} E_{\mu_\alpha}(|f_k|^2) = E_{\mu_\alpha}(|f|^2), \tag{3.20}$$

$$\lim_{k\to+\infty} E_\alpha(|\mathscr{H}_\alpha(f_k)|^2) = E_\alpha(|\mathscr{H}_\alpha(f)|^2). \tag{3.21}$$

Let $(h_k)_{k\in\mathbb{N}}$ be the sequence of functions defined by

$$h_k(r) = 2^{\alpha+1}k^{2\alpha+2}e^{-k^2r^2} = g_{\frac{1}{k\sqrt{2}}}(r). \tag{3.22}$$

Then, by relation(2.21), we have

$$\forall f \in L^2(d\mu_\alpha);\ \lim_{k\to+\infty} \|h_k *_\alpha f - f\|_{2,\mu_\alpha} = 0. \tag{3.23}$$

Furthermore, according to Weber's formula [15], we know that for all $k \in \mathbb{N}^*$, $\alpha \geqslant \frac{-1}{2}$,

$$\int_0^{+\infty} e^{-k^2r^2} j_\alpha(xr) r^{2\alpha+1} dr = \Gamma(\alpha+1)\frac{e^{-\frac{x^2}{4k^2}}}{2k^{2\alpha+2}}. \tag{3.24}$$

Hence, by relation (3.24), we deduce that

$$\begin{aligned}\mathscr{H}_\alpha^{-1}(h_k)(\lambda) &= \frac{2k^{2\alpha+2}}{\Gamma(\alpha+1)}\int_0^{+\infty} e^{-k^2r^2} j_\alpha(\lambda r) r^{2\alpha+1} d\mu_\alpha(r) \\ &= e^{-\frac{\lambda^2}{4k^2}}.\end{aligned} \tag{3.25}$$

Let $(\psi_k)_{k\in\mathbb{N}}$ be the sequence of functions defined on $\mathbb{R}_+$ by

$$\psi_k(\lambda) = e^{-\frac{\lambda^2}{4k^2}} = \mathscr{H}_\alpha^{-1}(h_k)(\lambda). \tag{3.26}$$

Let $f \in L^2(d\mu_\alpha)$; $\|f\|_{2,\mu_\alpha} = 1$, then according to relation(3.23), we have

$$\lim_{k\to+\infty} \|\mathscr{H}_\alpha(\psi_k) *_\alpha \mathscr{H}_\alpha(f) - \mathscr{H}_\alpha(f)\|_{2,\mu_\alpha} = 0.$$

In particular, there is a subsequence $(\psi_{\sigma(k)})_{k\in\mathbb{N}}$ such that
$\mathscr{H}_\alpha(\psi_{\sigma(k)}) *_\alpha \mathscr{H}_\alpha(f) = h_{\sigma(k)} *_\alpha \mathscr{H}_\alpha(f)$ converges pointwise to $\mathscr{H}_\alpha(f)$.
Let $(f_k)_{k\in\mathbb{N}}$ be the sequence of measurable functions on $\mathbb{R}_+$ defined by

$$f_k = \psi_{\sigma(k)} f. \tag{3.27}$$

Since $\psi_{\sigma(k)} \in L^2(d\mu_\alpha) \cap C_{*,0}(\mathbb{R}_+)$, then f_k belongs to the space $L^1(d\mu_\alpha) \cap L^2(d\mu_\alpha)$.
Replacing f by $\frac{f_k}{\|f_k\|_{2,\mu_\alpha}}$ in Theorem 3.3 and using the fact that

$$\|f\|_{2,\mu_\alpha} = \|\mathscr{H}_\alpha(f)\|_{2,\mu_\alpha};\ f \in L^2(d\mu_\alpha),$$

we deduce that

$$-\int_{0}^{+\infty} \ln\left(|f_k(r)|^2\right)|f_k(r)|^2 d\mu_\alpha(r) - \int_{0}^{+\infty} \ln\left(|\mathscr{H}_\alpha(f_k)(\lambda)|^2\right)|\mathscr{H}_\alpha(f_k)(\lambda)|^2 d\mu_\alpha(\lambda) \tag{3.28}$$

$$\geqslant (2\alpha+2)(1-\ln 2)\|f_k\|^2_{2,\mu_\alpha} - 2\|f_k\|^2_{2,\alpha}\ln\left(\|f_k\|^2_{2,\mu_\alpha}\right). \tag{3.29}$$

Now, by using the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to+\infty}\|f_k\|_{2,\mu_\alpha} = \|f\|_{2,\mu_\alpha}. \tag{3.30}$$

On the other hand, one can see that for all $k \in \mathbb{N}$, and for almost every $r \in [0, +\infty[$, we have

$$\ln|f_k(r)|^2|f_k(r)|^2 \leqslant C|f(r)|^2 + \ln|f(r)|^2|f(r)|^2, \tag{3.31}$$

Hence, using again the Lebesgue dominated convergence theorem, we get

$$-\lim_{k\to+\infty}\int_{0}^{+\infty} \ln\left(|f_k(r)|^2\right)|f_k(r)|^2 d\mu_\alpha(r) = E_{\mu_\alpha}(|f|^2). \tag{3.32}$$

Now, let us show that

$$\lim_{k\to+\infty} E_{\mu_\alpha}(|\mathscr{H}_\alpha(f_k)|^2) = E_{\mu_\alpha}(\mathscr{H}_\alpha(f)|^2).$$

For this, we denote by ω_1 , ω_2 the functions defined on $\mathbb{R}$ by

$$\omega_1(t)=\begin{cases} t^2\ln|t|, & \text{if } |t|>1 \\ 0, & \text{if } |t|\leqslant 1, \end{cases}$$

and

$$\omega_2(t)=\begin{cases} 2t^2, & \text{if } |t|\geqslant 1 \\ -t^2\ln|t| + 2t^2, & \text{if } |t|\leqslant 1\ , t\neq 0 \\ 0, & \text{if } t=0. \end{cases}$$

Then ω_1 and ω_2 are both nonnegative and convex, moreover; we have

$$\forall t>0,\ t^2\ln|t| = \omega_1(t) - \omega_2(t) + 2t^2. \tag{3.33}$$

From the hypothesis, for each $i \in \{1, 2\}$, the function $\omega_i(|\mathscr{H}_\alpha(f)|)$ belongs to $L^1(d\mu_\alpha)$. Now, from Proposition 2.7 iv), for every $k \in \mathbb{N}^*$; we have

$$\mathscr{H}_\alpha(f_k) = h_{\sigma(k)} *_\alpha \mathscr{H}_\alpha(f)$$

and we know that $h_{\sigma(k)} *_\alpha \mathscr{H}_\alpha(f)$ converges pointwise to $\mathscr{H}_\alpha(f)$ and $\|h_{\sigma(k)}\|_{1,\mu_\alpha} = 1$. So, the hypothesis of Lemma 3.4 are satisfies and we get

$$\lim_{k\to+\infty}\int_{0}^{+\infty} \omega_i(|\mathscr{H}_\alpha(f_k)|)(r)d\mu_\alpha(r) = \int_{0}^{+\infty} \omega_i(|\mathscr{H}_\alpha(f)|)(r)d\mu_\alpha(r), \tag{3.34}$$

and therefore, by relations(3.30) and (3.33) we get

$$\lim_{k\to+\infty}\int_{0}^{+\infty} \ln|\mathscr{H}_\alpha(f_k)|^2|\mathscr{H}_\alpha(f_k)(r)|^2 d\mu_\alpha(r) = E_{\mu_\alpha}(|\mathscr{H}_\alpha(f)|^2). \tag{3.35}$$

The proof is complete by combining relations (3.29), (3.30), (3.32)and(3.35).

□

4 Heisenberg-Pauli-Weyl inequality for the Hankel transform

Lemma 4.1. *Let* $f \in L^2(d\mu_\alpha)$ *such that* $||f||_{2,\mu_\alpha} = 1$*.Then, for every* $t > 0$*,*

$$\int_0^{+\infty} |f(r)|^2 \ln\Big(\frac{|f(r)|^2}{g_t(r)}\Big) d\mu_\alpha(r) \geqslant 0, \tag{4.1}$$

where $g_t(r)$*is given by(2.20).*

Proof. Since the function $\omega(t) = t\ln t$ is convex on $]0, +\infty[$, and $d\nu_\alpha(r) = g_t(r)d\mu_\alpha(r)$ is a probability measure on $]0, +\infty[$ then, applying Jensen's inequality, we get

$$\begin{aligned}
\int_0^{+\infty} |f(r)|^2 \ln\Big(\frac{|f(r)|^2}{g_t(r)}\Big) d\mu_\alpha(r) &= \int_0^{+\infty} \omega\Big(\frac{|f(r)|^2}{g_t(r)}\Big) d\nu_\alpha(r) \\
&\geqslant \omega\Big(\int_0^{+\infty} \frac{|f(r)|^2}{g_t(r)} d\nu_\alpha(r)\Big) \\
&= \omega(||f||^2_{2,\mu_\alpha}) \\
&= \omega(1) \\
&= 0.
\end{aligned}$$

□

Theorem 4.2. *(Heisenberg-Pauli-Weyl inequality)*
Let $f \in L^2(d\mu_\alpha)$*, then*

$$||rf||_{2,\mu_\alpha} ||\lambda \mathscr{H}_\alpha(f)||_{2,\mu_\alpha} \geqslant (\alpha + 1)||f||^2_{2,\mu_\alpha}. \tag{4.2}$$

Proof. Let $h \in L^2(d\mu_\alpha)$; $||h||_{2,\mu_\alpha} = 1$. From Lemma 4.1, we get

$$\int_0^{+\infty} \big[|h(r)|^2 \ln(|h(r)|^2) - |h(r)|^2 \ln(|g_t(r)|)\big] d\mu_\alpha(r) \geqslant 0. \tag{4.3}$$

Then,

$$E_{\mu_\alpha}(|h|^2) \leqslant \ln\big(t^{2\alpha+2}\big) + \frac{1}{2t^2}\int_0^{+\infty} |h(r)|^2 r^2 d\mu_\alpha(r). \tag{4.4}$$

Since $||\mathscr{H}_\alpha(h)||_{2,\mu_\alpha} = ||h||_{2,\mu_\alpha} = 1$, we get also

$$E_{\mu_\alpha}(|\mathscr{H}_\alpha(h)|^2) \leqslant \ln\big(t^{2\alpha+2}\big) + \frac{1}{2t^2}\int_0^{+\infty} |\mathscr{H}_\alpha(h)(\lambda)|^2 \lambda^2 d\mu_\alpha(\lambda), \tag{4.5}$$

adding (4.4) and (4.6), it follows that

$$||rh||^2_{2,\mu_\alpha} + ||\lambda \mathscr{H}_\alpha(h)||^2_{2,\mu_\alpha} \geqslant 2t^2\big[E_{\mu_\alpha}(|h|^2) + E_{\mu_\alpha}(|\mathscr{H}_\alpha(h)|^2) - 2\ln(t^{2\alpha+2})\big].$$

Applying Theorem 3.5, we obtain

$$\begin{aligned}||rh||^2_{2,\mu_\alpha} + ||\lambda\mathscr{H}_\alpha(h)||^2_{2,\mu_\alpha} &\geqslant 2t^2\left[(2\alpha+2)(1-2\ln 2) - 2(2\alpha+2)\ln t\right] \\ &= 2t^2(2\alpha+2)(1-\ln 2t^2).\end{aligned}$$

In particular, for $t = \frac{1}{\sqrt{2}}$; we deduce that for every $h \in L^2(d\mu_\alpha); ||h||_{2,\mu_\alpha} = 1$,

$$||rh||^2_{2,\mu_\alpha} + ||\lambda\mathscr{H}_\alpha(h)||^2_{2,\mu_\alpha} \geqslant 2\alpha+2. \tag{4.6}$$

Let $f \in L^2(d\mu_\alpha)$, replacing h by $\dfrac{f}{||f||^2_{2,\mu_\alpha}}$ in relation (4.6), we claim that for every $f \in L^2(d\mu_\alpha)$,

$$||rf||^2_{2,\mu_\alpha} + ||\lambda\mathscr{H}_\alpha(f)||^2_{2,\mu_\alpha} \geqslant (2\alpha+2)||f||^2_{2,\mu_\alpha}. \tag{4.7}$$

On the other hand, for $f \in L^2(d\mu_\alpha)$ and $t > 0$, we denote by f_t the dilated of f defined by

$$f_t(r) = f(tr),$$

then, f_t belongs to $L^2(d\mu_\alpha)$ and we have

$$\begin{aligned}||f_t||^2_{2,\mu_\alpha} &= \int_o^{+\infty} |f_t(r)|^2 d\mu_\alpha(r) \\ &= \frac{1}{t^{2\alpha+2}}\int_o^{+\infty} |f(r)|^2 d\mu_\alpha(r) \\ &= \frac{1}{t^{2\alpha+2}}||f||^2_{2,\mu_\alpha}. \end{aligned}\tag{4.8}$$

Moreover

$$\begin{aligned}||rf_t||^2_{2,\mu_\alpha} &= \int_o^{+\infty} r^2|f_t(r)|^2 d\mu_\alpha(r) \\ &= \frac{1}{t^{2\alpha+4}}||rf||^2_{2,\mu_\alpha}, \end{aligned}\tag{4.9}$$

and

$$||\lambda\mathscr{H}_\alpha(f_t)||^2_{2,\mu_\alpha} = \int_o^{+\infty} \lambda^2|\mathscr{H}_\alpha(f_t)(\lambda)|^2 d\mu_\alpha(\lambda) \tag{4.10}$$

$$\begin{aligned}\mathscr{H}_\alpha(f_t)(\lambda) &= \int_o^{+\infty} f_t(x)j_\alpha(\lambda x)d\mu_\alpha(x) \\ &= \frac{1}{t^{2\alpha+2}}\mathscr{H}_\alpha(f)(\frac{\lambda}{t}). \end{aligned}\tag{4.11}$$

Then

$$||\lambda\mathscr{H}_\alpha(f_t)||^2_{2,\mu_\alpha} = \frac{1}{t^{2\alpha}}||\lambda\mathscr{H}_\alpha(f)||^2_{2,\mu_\alpha}. \tag{4.12}$$

Now, we assume that $\|rf\|_{2,\mu_\alpha}$ and $\|\lambda\mathscr{H}_\alpha(f)\|_{2,\mu_\alpha}$ are both non zero and finite, hence the same holds for f_t , $t > 0$ and from relation (4.7), we have

$$\|rf_t\|^2_{2,\mu_\alpha} + \|\lambda\mathscr{H}_\alpha(f_t)\|^2_{2,\mu_\alpha} \geqslant (2\alpha+2)\|f_t\|^2_{2,\mu_\alpha}. \tag{4.13}$$

Then, by relations (4.8), (4.9) and (4.12), we get for every $t > 0$

$$\frac{1}{t^{2\alpha+4}}\|rf\|^2_{2,\mu_\alpha} + \frac{1}{t^{2\alpha}}\|\lambda\mathscr{H}_\alpha(f)\|^2_{2,\mu_\alpha} \geqslant (2\alpha+2)\frac{1}{t^{2\alpha+2}}\|f\|^2_{2,\mu_\alpha},$$

or

$$\frac{1}{t^2}\|rf\|^2_{2,\mu_\alpha} + t^2\|\lambda\mathscr{H}_\alpha(f)\|^2_{2,\mu_\alpha} \geqslant (2\alpha+2)\|f\|^2_{2,\mu_\alpha}.$$

In particular for

$$t = t_0 = \sqrt{\frac{\|rf\|_{2,\mu_\alpha}}{\|\lambda\mathscr{H}_\alpha(f)\|_{2,\mu_\alpha}}}.$$

We obtain

$$\|\lambda\mathscr{H}_\alpha(f)\|_{2,\mu_\alpha}\|rf\|_{2,\mu_\alpha} \geqslant (\alpha+1)\|f\|^2_{2,\mu_\alpha}.$$

□

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