CUBO A Mathematical Journal
Vol.18, N^{o}01, (69–88). December 2016

Parametrised databases of surfaces based on Teichmüller theory

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ABSTRACT

We propose a new framework to build databases of surfaces with rich mathematical structure. Our approach is based on ideas that come from Teichmüller and moduli space of closed Riemann surfaces theory, and the problem of finding a canonical and explicit cell decomposition of these spaces. Databases built using our approach will have a graphical underlying structure, which can be built from a single graph by contraction and expansion moves.

RESUMEN

Proponemos un nuevo marco teórico para construir bases de datos de superficies con rica estructura matemática. Nuestro enfoque está basado en ideas que vienen de teoría de espacios de Teichmüller y espacios módulares de superficies de Riemann cerradas, y el problema de encontrar una descomposición celular canónica y explícita de estos espacios. Las bases de datos construidas usando nuestro enfoque tendrán una estructura gráfica subyacente, la que se puede construir a partir de un solo grafo por movimientos de expansión y contracción.

Keywords and Phrases: Database of Surfaces Design Teichmüller space Moduli Space of Riemann surfaces Canonical cell decomposition of Riemann surfaces Teichmüller surfaces descriptor.

2010 AMS Mathematics Subject Classification:

1 Introduction

The idea of surface deformation in real life is common and increasingly important for sciences and industry. For example, it is known that in the automotive industry around 90% of the time a new surface design start with the smooth modification of a prototype surface [1]. A process that mirrors the exploration of the possible paths in the Teichmüller or moduli space of Riemann surfaces. This suggests the need of encouraging the application of these theories, which already have been applied for instance to neurosciences [24, 25], but it is still basically unexploited.

A systematic study of surfaces, its parametrisation and classification are tasks of real importance in applied sciences and industrial applications. Formal studies on spaces of surfaces can be traced back to Riemann and to the work of Teichmüller [7, 11, 18]. Nowadays moduli and Teichmüller spaces theories are deep and active branches of mathematics with many connections to many other fields.

Hyperbolic geometry [16, 19] plays an important role in the understanding of Riemann surfaces because by the classical uniformization problem each Riemann surfaces S of genus $g \geq 2$ can be represented as a quotient of $\mathbb{H}/G$, where $\mathbb{H}$ is the hyperbolic upper half plane and G is a discrete group of Mobiüs transformations keeping $\mathbb{H}$ invariant.

The quest to find a canonical and explicit cell decomposition of the moduli space of closed Riemann surfaces lead to combinatorial structures and circle pattern systems. Indeed, a combinatorial analysis shows the existence of a family of graphs that contains all possible graphs corresponding to the 1-skeleton of the Voronoi cells, determined by the characteristic set W of S; and circle pattern systems of equations come from the study of realization problems of graphs. In addition, circle patterns systems lead to a polytope P_g complex which can be viewed as a parametrisation of T_g for the genus two case [2], and more generally for the hyperelliptic locus of the Teichmüller and Moduli spaces of closed Riemann surfaces for genus $g \geq 2$.

The development of applications of moduli spaces of Riemann surfaces theory can be facilitated by assigning a computable combinatorial structure to each surface S on the moduli space, M_g, or the Teichmüller space, T_g, where $g \geq 2$. Combinatorial structures can be based on W, a characteristic set of points on S, which usually is the set of Weierstrass points on S, because it carries a lot of information about a Riemann surfaces as shown on [2, 12, 10, 14, 15]. However, for specific application to choose a different set could be sensible.

Since a surface S embedded in a 3-dimensional space has a conformal structure and, if it has a negative Euler characteristic, it can be provided of a hyperbolic metric. [24, 25], We can define a new surface descriptor from the embedding of S on the Poincare disk by following the next steps:

(1) compute the uniformization metric of the surface S

(2) get a conformal model $\bar{S}$ of S defined as a quotient of the Poincaré disc $\mathbb{D}$ by a suitable Fuchsian group

(3) compute the set W of Weierstrass points of $\bar{S}$

(4) assign the Voronoi diagram, $A(S)$, determined by W to S

(5) assign a circle intersection angle θ_i to each edge i of $A(S)$

(6) define the descriptor $D_\theta(S)$ for the surface S by $D_\theta(S) = (A(S), \Theta = (\theta_i)_i)$.

For any genus $D_\theta(S)$ depends only on the Teichmüller class of S up to labelling, and circle pattern coordinates -intersection angles- can be translated to Teichmüller coordinates [2]. The technical procedures that are needed to build $D_\theta(S)$ for the first and second steps are well known [24, 25].

The location of the Weierstrass points for the genus two case are described on [10, 14, 15]. To find the locations of the Weierstrass points of a Riemann surfaces is in general challenging, for this reason we mainly study the descriptor $D_\theta(S)$ for genus 2 surfaces. Our ideas show a pathway for the implementation of the general case.

$D_\theta(S)$ for genus 2 surfaces can be simplified and associated to a linear system, whose solutions are on a polytopes complex. This will allow us to approach surfaces descriptions on a marked polytope complex- each polytope marked by a graph. We propose the polytope complex that arise as a natural structure to support database of surfaces.

In the sequel, we will describe the theory of graphs associated to Riemann surfaces based on Weierstrass points, the theory of linear systems associated to graphs, explain the construction of a polytope complex based on the previous ideas, and describe our proposal for databases designs based on Teichmüller theory.

2 Graphs associated to Riemann Surfaces

For each Riemann surface S with its hyperbolic metric, the 1-skeleton of the Voronoi cell decomposition determined by the Weierstrass points of S is a unique graph, $\hat{A}(S)$, which is a subset of S and depends only of the class of S in the Teichmüller space, $\mathcal{T}_g$, and in the moduli space $\mathcal{M}_g$, for any genus g [2].

Determining which graphs are associated to Riemann surfaces based on Weierstrass points in general is a difficult problem. We restrict ourselves to the hyperelliptic case, which allow us to find all graphs associated to Riemann surfaces computationally by considering a similar problem on the 2-dimensional sphere. Indeed, the hyperelliptic involution, τ, of S induces an action on S such that S/τ is the two dimensional hyperbolic sphere. S/τ has exactly $2g + 2$ cone points which has measure π. Then, τ projects $\hat{A}(S)$ into a graph on S/τ that we denote by $A(S)$. By a standard lifting procedure [2] we can recover S from the marked sphere S/τ. This shows the existence of a one to one correspondence between set of hyperelliptic surfaces of a given genus and a set of graphs, assigning S to $A(S)$.

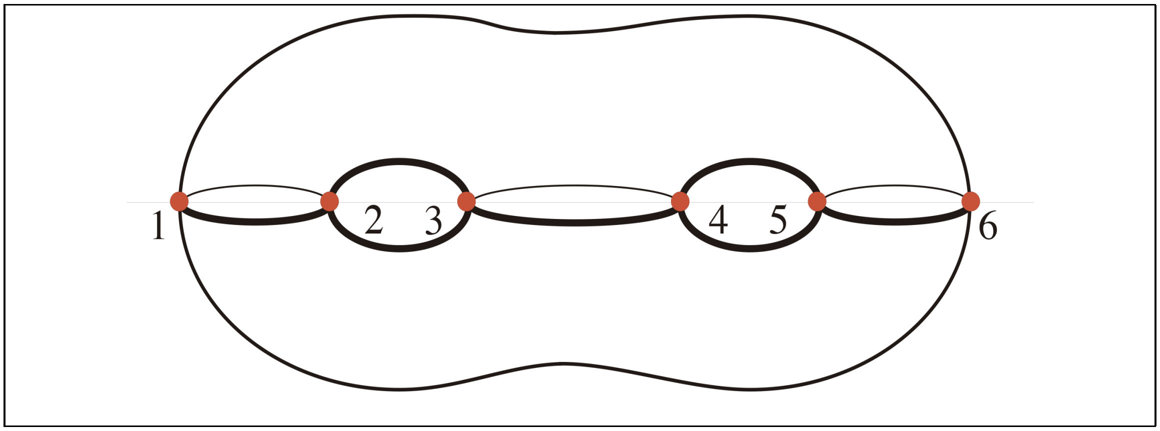

Figure 1: The Weierstrass point on a hyperelliptic surface S of genus g are the $2g+2$ fixed points of its unique hyperelliptic involution. Here, we represent the genus 2 case.

Definition 2.1. *The graph, $A(S)$, associated to a hyperelliptic Riemann surface (S, τ) of genus g is the image on S/τ under τ of the Voronoi graph determined by the Weierstrass fixed points of S.*

Proposition 2.1. *If S is a hyperelliptic hyperbolic Riemann surface of genus g, its associated graph $A(S)$ satisfies the following properties:*

(1) G *is connected*

(2) G *does not have monogons*

(3) G *divides S^2 into $2g+2$ regions*

(4) *All vertices of G have valence ≥ 3.*

The above properties of $A(S)$ motivates the following definition of the family of $CE(g)$ graphs.

Definition 2.2. *A $CE(g)$* **graph** *is a connected graph G with vertices of valence greater than two that can be embedded in the sphere S^2, determining $2g+2$ regions and having no monogons. If the valence of each vertex of G is 3, then we say that the graph is generic.*

2.1 Generic graphs: the genus two case

To find all the possible generic graphs in the genus two case, we take advantage of the fact that all generic graphs are connected by Whitehead moves and do not have any monogons. We find 20 generic graphs $G_1, G_2, \ldots G_{20}$ which are subset of the two dimensional sphere, 17 of which are non-isomorphic, see Figure 3.

Counting the number of sides on each of the faces of these graphs, we always get six numbers which are arrange in a non-increasing order. This gives a natural labelling to each generic graph,

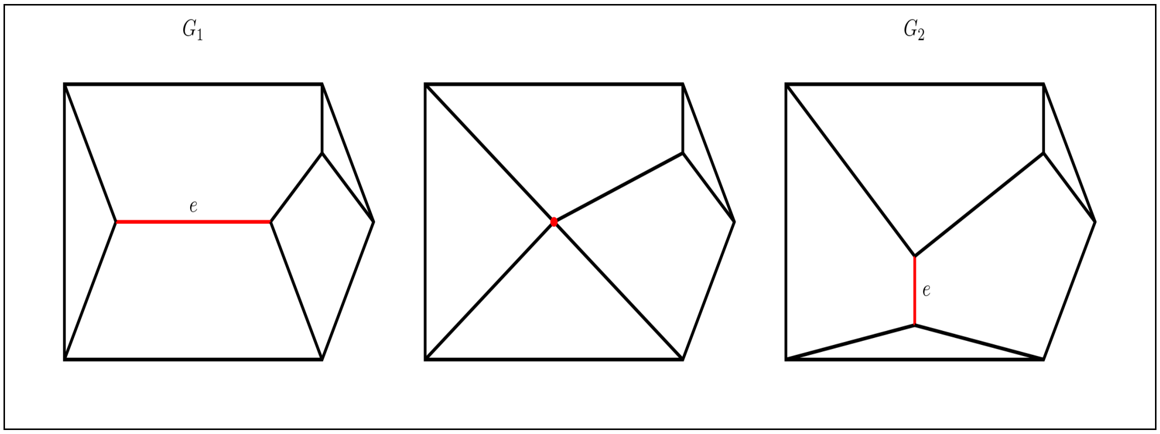

Figure 2: A Whitehead move on the red edge on graph G_1, contracts the edge to a point, as shown on the middle graph, followed by an edge expansion as shown on the right graph. The result is the new graph which is represented on the right.

called the face labelling, which is unique except for the generic graphs G_{11} and G_{12}. However, this exception is not really important because G_{11} is not associated to a Riemann surface of genus two.

On Figure 4, the combinatorics of genus two generic graphs is represented by a graph whose nodes are all possible generic graphs. We join graph G_i with G_j by an edge if there is a Whitehead move transforming G_i into G_j.

2.2 Stratification of $\mathsf{CE}(g)$

An immediate consequence of the Euler characteristic formula of the sphere is that any generic graph that belong to $\mathsf{CE}(g)$ has $4g$ vertices, $6g$ edges and by definition $2g+2$ faces. Then, the family $\mathsf{CE}(g)$ of graphs is stratified in $4g-1$ levels. Indeed, since the number of faces in all $\mathsf{CE}(g)$ graphs is $2g+2$, and each vertex is of valence not smaller than 3 then $3\nu \leq 2e$. Hence, by the Euler characteristic formula of the sphere $\nu \leq 4g$.

The maximum number of vertices that a $\mathsf{CE}(g)$ graph can have is $4g$ and the minimum number is 2, because no monogons are allowed. Thus, $\mathsf{CE}(g)$ has $4g-1$ levels, if we define the k-th stratum of $\mathsf{CE}(g)$ as the set of all graphs in $\mathsf{CE}(g)$ that have exactly $4g-(k-1)$ vertices.

The members of the first stratum of $\mathsf{CE}(g)$ are all cubic graphs and is easy to verify that graphs at this level has $2g+2$ faces and $6g$ edges. This stratum of $\mathsf{CE}(g)$ also has great importance since any graph of $\mathsf{CE}(g)$ in a higher strata can be obtained by contraction moves starting from a graph of the first strata.

The fact that cubic graphs are connected by a sequence of Whitehead moves [3], allow us to view $\mathsf{CE}(g)$ as generated by any of its cubic graphs by sequences of Whitehead moves and contraction

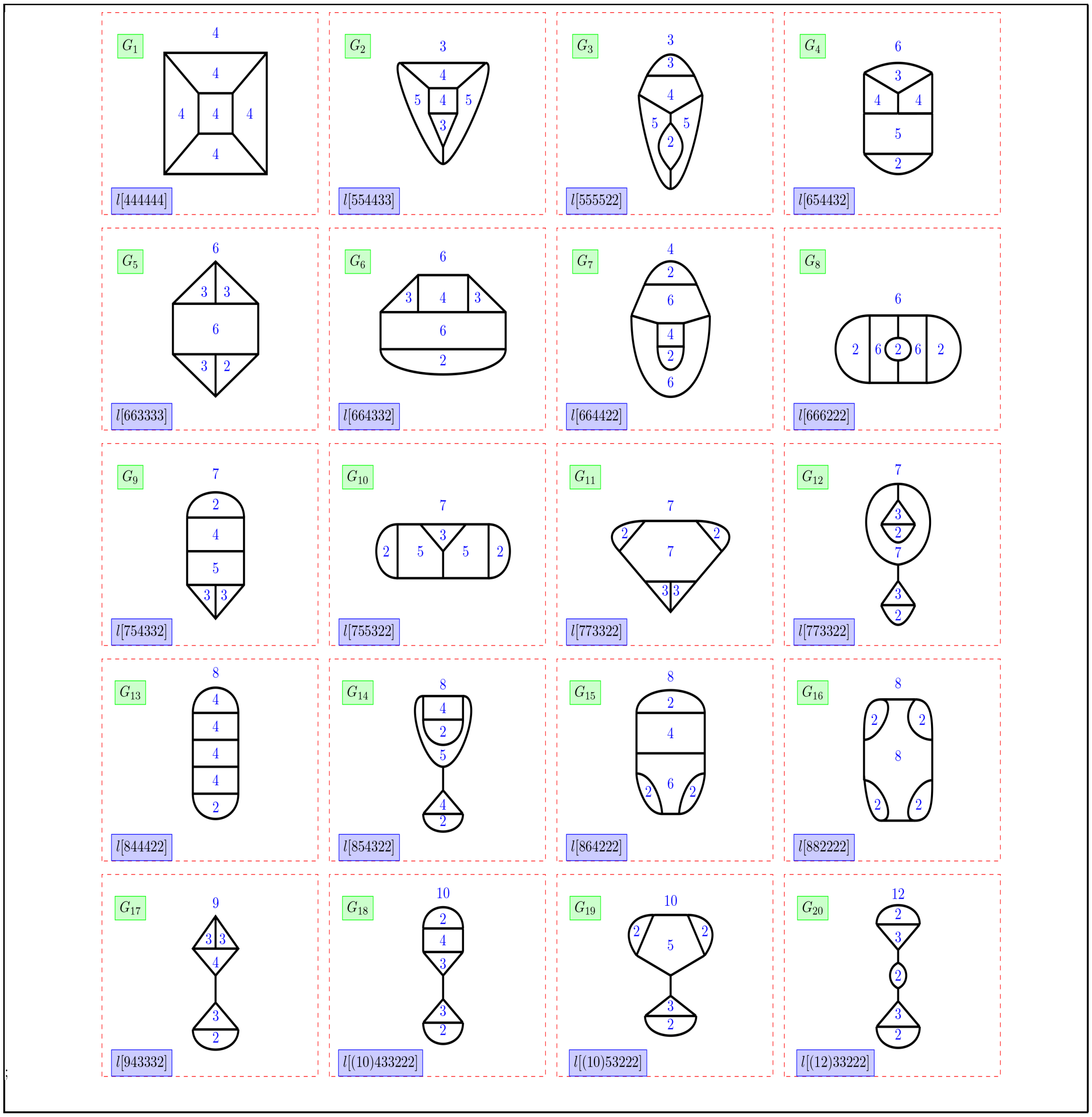

Figure 3: Genus two generic graphs and their face labelling-$\mathsf{l}[\mathsf{f}_1\mathsf{f}_2\mathsf{f}_3\mathsf{f}_4\mathsf{f}_5\mathsf{f}_6]$.

moves. In particular, any graph in $\mathsf{CE}(2)$ can be connected to the graphs which are illustrated on Figure 3.

For another equivalent point of view, any stratum of $\mathsf{CE}(\mathsf{g})$ can be obtained inductively by contracting each graph of the previous strata by a single contraction move on one of its edges in such way that no monogon is created. Furthermore, if we define a generalised Whitehead move on a graph as the graph obtained by contracting one edge and expanding a new edge in such way that no monogons are created, we have that two graphs at a fixed strata are connected by a sequence of generalised Whitehead moves.

Finally, to complete a view of how graphs are connected on $CE(g)$, its most contracted graph can generate all graphs by a sequence of contraction and expansions moves. Then, $CE(g)$ is analogous to a universe that can be generated by transforming any of its graphs by contraction or expansion moves from a single graph.

3 Generic graphs on the Teichmüller space

Informally, the Teichmülller of Riemann surfaces of genus g, T_g is a space whose elements are classes of marked surfaces, and paths on T_g can be viewed as deformation of a surface. This intuitive idea suggests that T_g or an equivalent space could be an ideal space to model the deformation of real surfaces. For a background on Teichmülller theory see [18].

Next, we list a few facts about Teichmülller theory,

(1) Let S be a fixed Riemann surface of genus g. A marked surface is pair $(R, [f])$, where R is a Riemann surface, $[f]$ is the homotopy class of a homeomorphism $f : S \to R$. Two marked surfaces $(S, [f])$ and $(S', [f'])$ are equivalent if there is a conformal map $g : R \to R'$ such that $[g \circ f] = [f']$.

(2) The Teichmüller space is the set of marked classes.

(3) The Teichmüller space $T_{g,p}$ has a natural topology, which makes it homeomorphic to an open set of $\mathbb{R}^{6g-6+2p}$, where g and p are the genus and the number of punctures of each surface in a class of $T_{g,p}$. On this space, we will only consider the case when $p = 0$.

(4) The Teichmüller space $T_{g,p}$ can be parametrised by Fenchel-Nielsen coordinates [18]

In addition to the above, if G is a cubic graph associated to S, then G can be associated to an open subset of $\mathcal{T}_g$, with its Teichmüller metric. The Fenchel-Nielsen coordinates of $\mathcal{T}_g$ allow us to see it as a $6(g-1)$ real dimensional manifold, and informally we could imagine a deformation of S by slightly changing its Fenchel-Nielsen coordinates which cannot change the graph G because all its vertices are trivalent and then G is stable under small perturbations.

The following properties which are proved in [2] establish the relationship between generic graphs and the Teichmüller space.

Proposition 3.1. *If* τ *is a hyperelliptic involution of a hyperelliptic Riemann surface* R *and* $h : R \to R'$ *is an isometry, then* R' *is hyperelliptic and its hyperelliptic involution is* $\tau' = h \circ \tau \circ h^{-1}$.

Proposition 3.2. *The associated graphs corresponding to two equivalent marked surfaces* $(R, [f])$, $(R', [f'])$ *are equal.*

Denote by $\tilde{O}_G$, the set of all points in the Teichmüller space with associated graph G, where G is a graph embedded in S^2.

Proposition 3.3. *If G is a cubic graph, then $\tilde{O}_G$ is an open set of the Teichmüller topology of $\mathcal{T}_g$.*

4 Linear systems associated to graphs

4.1 Delaunay realization problem

If a graph G on the 2-sphere is the associated graph to a hyperelliptic surface S of genus g, then G is the boundary of a cell decomposition of the 2-sphere which is S/τ, with $2g+2$ faces, each one containing an interior point with cone angles equal to π. By lifting S/τ to its two fold branched covering space S can be recovered.

The lifting of S/τ is standard. However, for a given graph G, the problem of finding a hyperbolic metric on the sphere with $2g+2$ π-cone angles whose associated graph is G is not trivial. We call this problem the *Delaunay Realization Problem (DRP)*.

A solution to the DRP defined by G can be identified with a unique hyperbolic surface S and also with a collection of circles with a set of intersection angles. This connects the realizability problem that we have described with the theory of circle patterns, which can be traced back to Koebe's work (1936), E.M. Andreev's work (1970) and Thurston [13], and has had great impact in many fields including conformal mapping, complex analysis, Teichmüller theory, brain mapping, random walks, tilings, minimal surfaces and integrable systems, numerical analysis, metric spaces and more [21].

4.2 Circle patterns and quotients

Given a closed Riemann surface S, and a cell decomposition $\{C_i\}_{i\in I}$ of S, which might have cone singularities at a vertex or at the center of a cell, we say that a circle pattern is a configuration of disks $\{D_i\}_{i\in I}$, where the boundary of each D_i contains all the vertices of C_i and no vertex of the cell decomposition is in the interior of any D_i. In a circle pattern, for each edge e of the cell decomposition, two circles have as intersection the extremes of e.

Let $\hat{S}$ be a hyperelliptic Riemann surface with metric $\tilde{d}$ and hyperelliptic involution τ. The quotient space $S = \hat{S}/\tau$ is also a metric space with the quotient metric d. We are interested in Delaunay circle patterns where the vertices of the circle pattern are the fixed points of the hyperelliptic involution. We will project a circle pattern on $\hat{S}$ to a circle pattern of the quotient S/τ.

Proposition 4.1. *A circle pattern of a cell decomposition of a hyperelliptic Riemann surface S of genus g is projected to a circle pattern of the sphere by the hyperelliptic involution of S.*

To study circle patterns, several variational approaches have been introduced and several circle packing results were proved using different functionals [23].

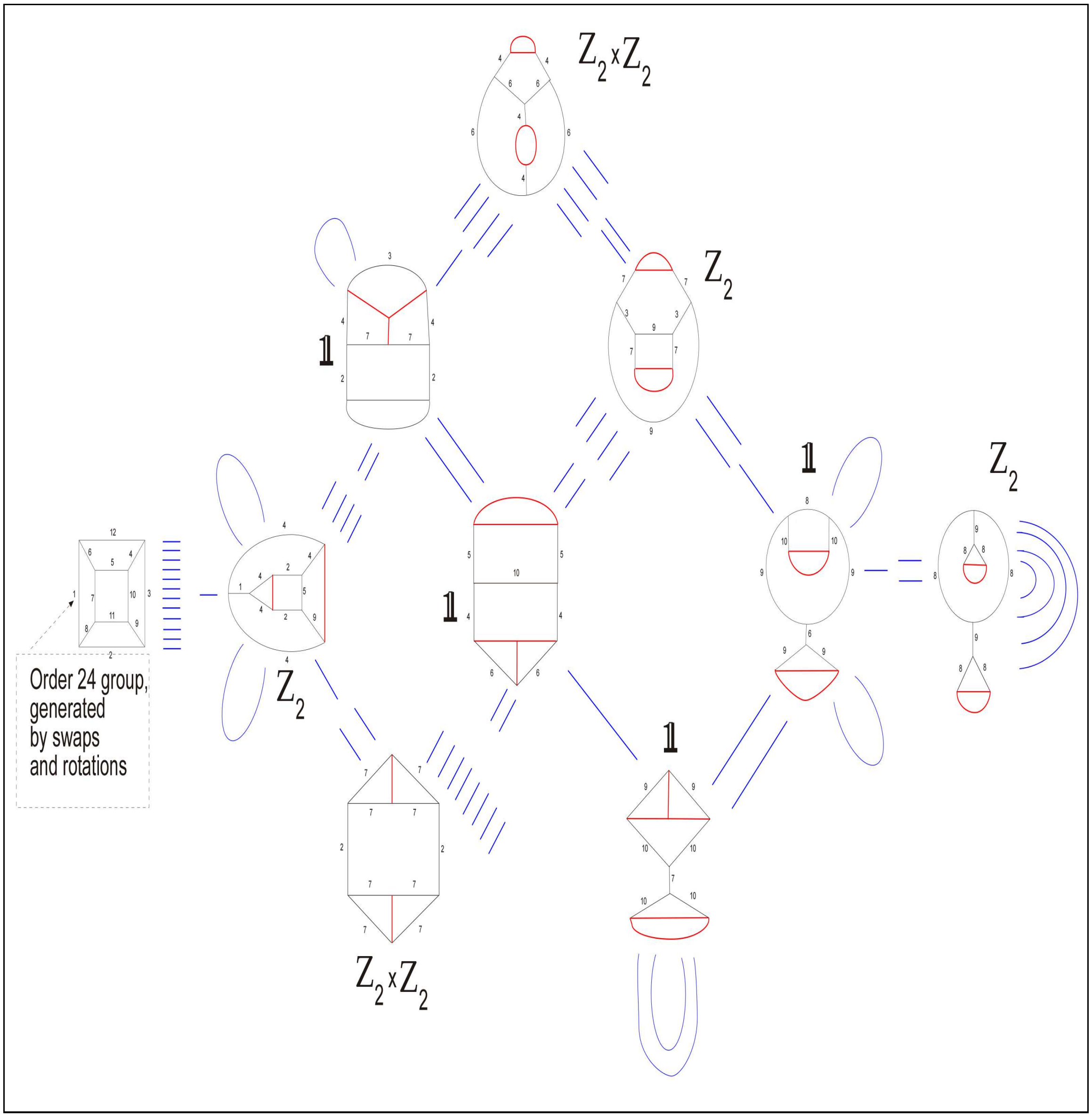

Figure 4: Above, we show the 10 realizable generic graphs with their groups of rigid symmetries. Note that Whitehead moves on red edges are prohibited.

4.3 Delaunay circle patterns, Voronoi cells and duality

A set of points $F = \{P_1, P_2, \dots, P_n\}$ on a Riemann surface S determines a Voronoi decomposition $\mathcal{V}$ of S: this is a cell decomposition determined from the points $P_i \in F$ by taking the sets of points closest to P_i, for each i. The open 2-cells are sets of form

$$V_i := \{p \in S : d(p, P_i) < d(p, P_j)\ \forall j \in I_i\}$$

where $I_i = \{1, 2, \dots, n\} - \{i\}$. Thus, $S = \cup_i \bar{V}_i$ and $G = \cup_i(\bar{V}_i - V_i)$ is a graph whose edges are geodesic segments.

The dual cell decomposition $\mathcal{V}'$ of $\mathcal{V}$ is by definition constructed by joining two vertices $P_1 \in F$ and $P_2 \in F$ by a geodesic segment, one for each common edge of the corresponding Voronoi cells V_1 and V_2. The collection $\mathcal{V}'$ is itself a cell decomposition of S with the property that each of its cells is inscribed in a unique circle: the collection of such circles is a Delaunay circle pattern for $\mathcal{V}'$. Conversely, if we start with a cell decomposition $\mathcal{V}'$ of S which has a Delaunay circle pattern, by joining the centers of adjacent circles we get a Voronoi cell decomposition of S whose centers correspond to the vertices of $\mathcal{V}'$.

A *Delaunay decomposition* of a constant curvature surface is a cellular decomposition such that the boundary of each face is a geodesic polygon which is inscribed in a circular disc, and these discs have no vertices in their interiors.

The Poincare-dual decomposition of a Delaunay decomposition with the centers of the circles as vertices and geodesics edges is a Voronoi cell decomposition. A Delaunay type circle pattern is the circle pattern formed by the circles of a Delaunay decomposition. We will allow the surface to have cone-like singularities in the hyperbolic metric at the vertices of Delauney decomposition, and centers of the circles:

A k-*cone cell* is a hyperbolic polygon with interior cone point obtained from a collection of hyperbolic polygons glued together cyclically around a common vertex, by isometric identification of edges, such that the sum of angles at the vertex is k.

A *cellular decomposition* of a surface with n-cone singularities is a collection $\{C_j\}$ of cone k_j-cone cells such that each side of each cell has been glued to a unique side of another cell (possibly the same), by hyperbolic isometry.

From a Delaunay type circle pattern, one may obtain the following data:

- A cell decomposition Σ of a 2-dimensional manifold.
- For each edge e of Σ, the exterior (respectively interior) intersection angle θ_e (respectively ${\theta_e}^* := \pi - \theta_e$). Thus, $0 < \theta_e, \theta_e^* < \pi$.
- For each face Σ, the cone angle $\Phi_f > 0$ of the cone-like singularity at the center of the circle corresponding to f. If there is no cone-like singularity at the center, then $\Phi_f = 2\pi$.

Note that the cone angle θ_ν at a vertex ν of Σ is determined by the intersection angles θ_e:

$$\theta_\nu = \Sigma\theta_e \tag{1}$$

where the sum is over all edges e around ν.

Next, we present Theorem 1.8 (ii) on [20] and [5] for the case of a closed oriented surface, the main tool that we use on this paper.

Theorem 4.1 (Springborn). *Let Σ be a cell decomposition of a closed oriented surface. Suppose the interior intersection angles are prescribed by a function $\theta^* \in (0,\pi)^{E_0}$ on the set E_0 of edges. Let $\Phi \in (0,\infty)^F$ be a function on the set F of faces, which prescribe, the cone angle corresponding to a face. A hyperbolic circle pattern corresponding to this data exists if and only if the following condition is satisfied:*

If $F' \subseteq F$ is any nonempty set of faces and $E' \subseteq E_0$ is the set of edges which are incident with any face $f \in F'$, then

$$\Sigma_{f\in F'}\Phi_f < \Sigma_{e\in E'}2\theta_e{}^*, \tag{2}$$

If it exist, the circle pattern is unique up to hyperbolic isometry.

The above observations can be integrated into a system of linear inequalities $\mathcal{L}(G,\sigma)$ to solve the Delaunay realization problem for a Delaunay graph G' embedded in S^2 with edge-labeling σ, dual to G, as a consequence of Springborn theorem.

Below $\mathcal{L}(G,\sigma)$ is defined :

$$\mathcal{L}(G,\sigma) = \begin{cases} 2\pi Q(i) < \Sigma_{j=1}^{12} 2P(i,j)\theta_j^*, & \text{for each } i \in \{1,2,\dots,255\} \\ \Sigma_{k\in J_\nu}(\pi - \theta_k{}^*) = \pi, & \text{for } \nu = 1,2,\dots,6 \\ 0 < \theta_j{}^* < \pi, & \text{for } j = 1,2,\dots,12 \end{cases} \tag{3}$$

where J_ν is the set of edges incident with vertex ν, $Q(i)$ is the number of faces in the subset i of faces, and $P(i,j)$ is the characteristic function, which is 1 if the edge j belongs to the subset i and 0 otherwise.

If $\mathcal{L}(G,\sigma)$ has a solution then, by Springborn theorem there is a hyperbolic circle pattern inducing a Delaunay cell decomposition isomorphic to G'. Hence, the two-fold covering space of the sphere that realize one of the solutions is a Delaunay triangulated Riemann surface of genus two whose dual graph projects to G', solving the Delaunay problem.

The above system can be solved using commercial computer programs that work even with thousand of constraints [4]. However, we do not need to do so for the genus two case because we can reduce the linear system that we obtain substantially, which help us to understand the structure of the solutions.

5 Solutions for the genus 2 case

To approach the problem of finding which of the 20 generic graphs that we found for the genus two case are realizable, we used the package Convex [6] which solves linear systems using symbolic algebra and allows the computation of exact solution.

Solving the reduced system, with only angle equalities and the angle constraints of the type $0 \leq \theta \leq 1$, we obtained that there are at most 10 generic graphs which are realizable. Then, we showed that the face constraint of the systems associated to generic graphs for the genus two case are consequence of the equations and angle constraints of the systems, which allow us to claim that there are exactly 10 realizable generic graphs associated to Riemann surfaces of genus two.

Proposition 5.1. *There are at most ten genus two generic Delaunay realizable graphs.*

As an additional conclusion from our computation, we can say that the face labelling of generic graphs is unique, e.g. Delaunay realizable generic graphs are uniquely determined by their labelling. Then, it is convenient to relabel the realizable generic graphs by ascending lexicographic ordering as $d_1, d_2, \ldots d_{10}$, Table 1.

Generic Delaunay realizable graphs	face labels
d1	444444
d2	554433
d3	555522
d4	654432
d5	663333
d6	664422
d7	754332
d8	773322
d9	854322
d10	943332

Table 1: Generic Delaunay realizable graphs

5.1 Independence of face constraints for generic linear systems

In this section, we will prove that all the 10 Delaunay realizable generic graphs are determined by six angle inequalities and six linear equations, corresponding to an angle equality for each vertex of a dual graph: from these all face inequalities follow, and are thus redundant. This radically improves our ability to understand these polytopes and corresponding realization solutions. Each polytope is obtained as a cube cut by hyperplanes in $\mathbb{R}^6$.

Our main result on this section is that for each Delaunay realizable generic graph the angle constraints can be deduced from angle constraints. In other words, each system $\mathcal{L}(d_i, \sigma)$ is face constraint independent.

5.2 Face inequalities for more than 3 triangles

If G is a generic graph then its dual G° determines a triangulation T of S^2 that has 255 non-empty subsets of triangles, since T has 8 triangles. Let C_i, $i = 1, 2, \ldots, 255$ be the collection of non-empty subsets of T and let E_i be the set of labels of edges which belong to C_i. The face constraint for the linear program $\mathcal{L}(G, \sigma)$ corresponding to C_i is

$$Q(i) < \sum_{j=1}^{12} P(i,j)\theta_j^* \tag{4}$$

where ${\theta_j}^*$ is the normalized exterior angle for the labeled edge j (we divided each angle by π to get $0 < {\theta_j}^* < 1$), and $Q(i) = card(C_i)$. Recall that $P(i,j) = 1$ when the edge labeled $j \in E_i$ belongs to one of the triangles in the subset C_i, where $E_i \subset \{1, \ldots, 12\}$.

From the above constraint, we can say that

$$Q(i) < \sum_{j=1}^{12} P(i,j)\theta_j^* = \sum_{j=1}^{12} \theta_j^* - \sum_{j \notin E_i} \theta_j^*$$

The following proposition reduces drastically the number of constraint that we need to consider [2].

Proposition 5.2. *Let G be a Delaunay realizable generic graph with labelling σ corresponding to a hyperelliptic Riemann surface of genus g. Then, the system $\mathcal{L}(G, \sigma)$ associated to G is independent of all face constraints corresponding to subsets of triangles with more than $2g - 1$ triangles.*

An immediate consequence of the above proposition is that in the genus two case, polytopes associated to a Delaunay realizable graph can only depend on face constraints corresponding to subsets with one, two or three triangles.

Corollary 5.1. *Let G be a Delaunay realizable generic graph with labelling σ corresponding to a hyperelliptic Riemann surface of genus 2. Then, the system L associated to G is independent of all face constraints corresponding to subsets of triangles with more than 3 triangles.*

The solutions of the angles systems associated to all generic graphs who are feasible are given on Table 2.

Proposition 5.3. *The linear system $\mathcal{L}(d_i, \sigma)$ for $i = 1, 2, \ldots, 10$ is independent of its face constraints.*

	1	2	3	4	5	6	7	8	9	10	11	12	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
P_1	1	1	1	1	1	1	1	0	1	0	1	0	+	+	+	+						
P_2	1	1	1	1	1	1	0	1	0	1	1	0	+	+	+	+	+	+	+			
P_3	1	1	0	1	1	1	1	1	1	1	0	1	+	+		+	+		+			+
P_4	0	1	1	0	1	1	1	1	1	1	0	1	+									
P_5	1	0	1	1	1	0	1	1	1	0	1	1	+	+								
P_6	0	1	1	1	0	1	1	1	0	1	1	1	+									
P_7	1	1	0	1	0	1	1	0	1	1	1	1	+	+	+	+	+	+	+		+	+
P_8	1	0	1	0	1	1	0	1	1	1	1	1	+				+					
P_9	0	1	0	1	1	1	0	1	1	1	1	1		+	+	+	+	+	+	+	+	+
P_{10}	1	1	0	1	1	1	0	1	1	0	1	1					+	+	+	+	+	+
P_{11}	$\frac{1}{2}$	1	1	1	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	+									
P_{12}	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	+	+	+	+	+	+	+	+	+	+
P_{13}	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	+									
P_{14}	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	1	+	+			+					
P_{15}	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$			+	+		+	+	+	+	+
P_{16}	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$			+			+		+	+	
P_{17}	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$			+			+		+	+	
P_{18}	1	1	0	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1								+	+	+
P_{19}	1	1	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1								+	+	
P_{20}	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	1	1	1	$\frac{1}{2}$								+		
P_{21}	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0	1	1	1	1	$\frac{1}{2}$								+		
P_{22}	1	1	0	1	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$	1	1	1								+		

Table 2: On this table, the 22 vertices $P_1, P_2, \dots P_{22}$, of the solutions of all angle systems for generic graphs of genus two is presented. Each vertex P_i is a point in $\mathbb{R}^{12}$, whose coordinates are on columns 2 to 13. The entry corresponding to the vertex P_i and the column d_j is filled with + if the vertex P_i is a vertex of the polytope of solutions corresponding to the generic graph d_j.

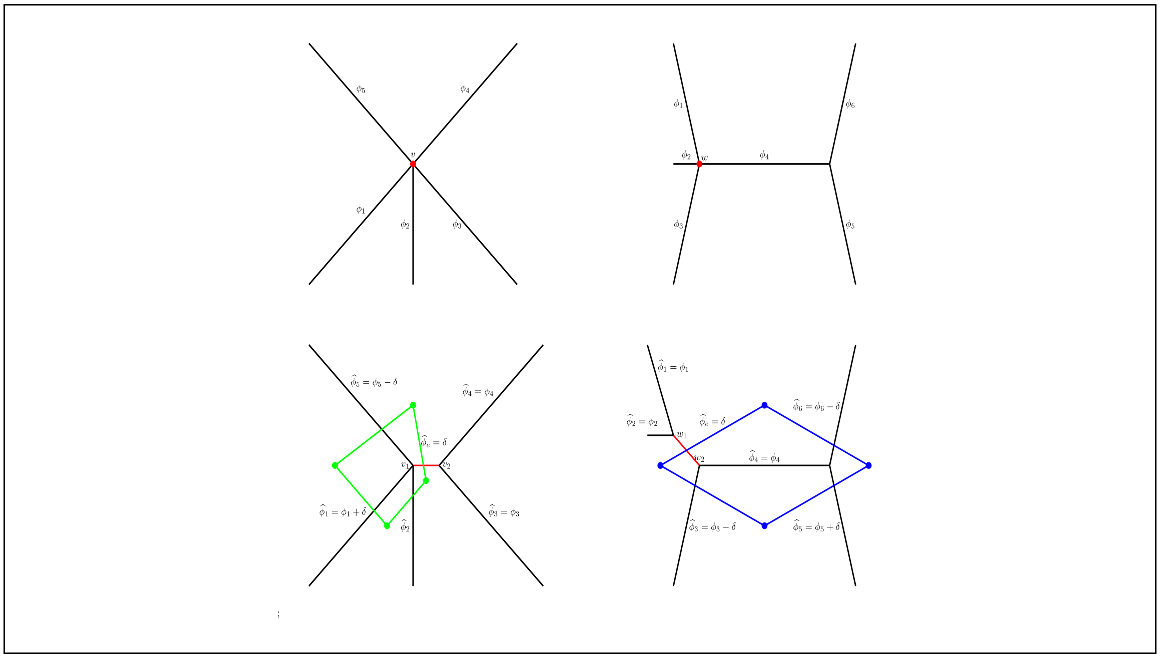

Figure 5: The solutions, ϕ, of top linear systems, $L(G)$, are modified to get solutions for the bottom graphs system by adding $\pm\delta$ to ϕ solutions of the original graph which are on the circuit and assigning δ to the new red edge.

Proof. By Corollary 1, we only need to prove the statement for all face constraints which corresponds to sets with 1, 2 or 3 triangles. In addition, to check that this linear system is independent of any given linear constraint, one can just verify that the 22 vertices given on Table 2 satisfy the constraints, which in our case can be done by simple inspection. ☐

5.3 Independence of face constraints for general genus two CE-graphs

Proposition 5.4. *The solution of the system $\mathcal{L}(G)$ is face independent for any Delaunay realizable graph G corresponding to a Riemann surface of genus two.*

The basic idea to prove the above proposition is to use induction on the number k of contractions that are needed to obtain G from a generic graph and notice that if G is a Delunay realisable graph which is obtained by $k+1$ contraction moves then there exist a solution ϕ_i of the system $\mathcal{L}(G)$ such that one of the vertices of G can be expanded to obtain a graph G' which is closer to the generic level and the solution of ϕ_i can be modified to obtain a solution of G' which is as close as we want to ϕ_i. See Figure 5 and Figure 6.

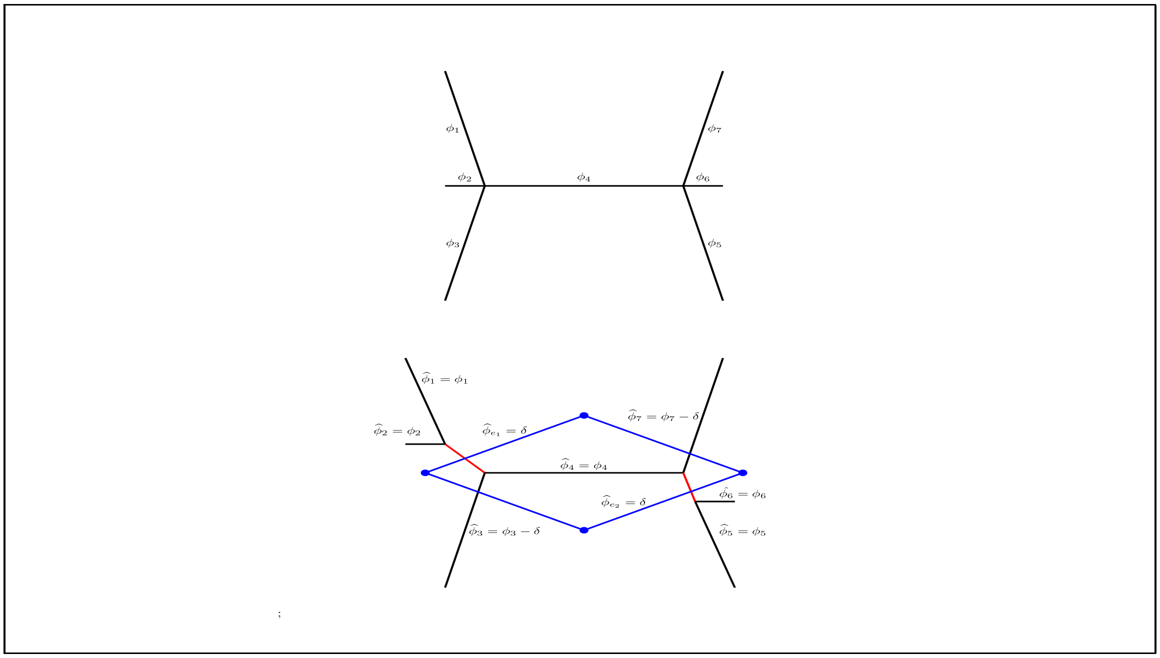

Figure 6: The solution, ϕ, of top linear system $L(G)$ is modified to get solutions for the bottom graph system by adding $\pm\delta$ to ϕ solutions of the original graph which are on the circuit and assigning δ to the new two red edges.

6 Compactification of $\mathcal{T}_2$, M_g and polytope complexes

Figure 4 not only shows how the generic Delaunay realizable graphs are connected by Whitehead moves, but the connections among the polytope complex $\mathcal{P}_2$, which have been described on Table 2. From the geometric point of view, $\mathcal{P}_2$ is a covering of Teichmüller space T_2 from which the Moduli space of Riemann surfaces M_2 can be obtained as the interior of the quotient space determined by the groups of symmetries shown on Figure 4.

Polytope complexes, as the one described for genus 2, also exist for the hyperelliptic locus of any genus $g \geq 3$. Let $\mathcal{P}_g$ be the polytope complex $\mathcal{P}_g$ obtained for $g \geq 2$. The mathematical theory relating the polytope complex $\mathcal{P}_g$, T_g and M_g is not yet completely developed. However, there is no reason why we could not take advantage of this structure for applications. For a background on polytopes see [8, 26].

7 Polytope complexes for indexing databases

It is desirable to have indexing systems for databases of surfaces which mirror the internal structure of the space where the potential members to be included in the database belongs to. This can be done for surfaces of genus two base on the polytope complex $\mathcal{P}_2$ whose vertices are given on Table

2.

The internal structure of the Teichmüller space for Riemann surfaces of genus two is given by the polytope complex $\mathcal{P}_2$ and the knowledge of its structure which is represented on Figure 4. Then, if an open database is build on top of $\mathcal{P}_2$, a person who have classified a surface $S^\star$ of genus two, using the descriptor D_θ, could include $S^\star$ in the database and annotate the database with new additional features. In addition, since the hyperbolic surface that is associated to $D_{\theta^\star}$, where $\theta^\star$ is determined by the circle pattern angles corresponding to $S^\star$, we can build and support both new surface knowledge and the applications of surfaces theory.

We think that the design of the structure of a database for surfaces of genus g could be enhanced by building on sound combinatorics and hyperbolic geometry, and propose to include the next steps in its design:

(1) Find C_g, the combinatorics of the hyperelliptic locus of T_g by generating all non-isomorphic generic graphs

(2) Construct the polytope complex $\mathcal{P}_g$ by solving the linear programs of the form $L(G)$

(3) Determine the map of the database structure, using the generic realizable graphs as vertices, and connecting these vertices by edges corresponding to Whitehead moves

(4) Finally, a new surface S entering the database would be indexed by the descriptor $D_\theta(S)$.

In addition to the above, to construct the deeper layers of C_g, we could build all non-isomorphic graphs which can be obtained by contraction moves from a graph on the generic layer of C_g. However, graphs on deeper layer of $C(g)$-non-generic- can be viewed as well as elements of the polytope C_g.

8 Conclusion

We have shown a framework to develop databases based on polytope complexes which arise by considering the combinatorics and geometry of the Teichmüller space for closed surfaces of a given genus. Databases supported on the mathematical structures that we have described are desirable because they can be used to study deformation of surfaces in real applications, and supported by rich mathematical results generations of mathematicians have developed. However, our description is not complete because we do not know a canonical cell decomposition of T_g for general g. Then, further theoretical and computational tools are needed in this area. In particular, research on the applications of canonical decomposition of the moduli space of Riemann surfaces which have punctures or boundaries [17] for the development of databases is desirable.

We hope to encourage collaboration between researchers with different backgrounds by building database of surfaces having the indexing system that we have proposed. Teichmüller and moduli space theories are in the heart of surfaces theory, modelling and its applications. Databases of surfaces which mirror the amazing structure of M_g and T_g should be one of the tools that support the increasingly complex study of surfaces.

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