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(i, j)-ω-semiopen sets and (i, j)-ω-semicontinuity in bitopological spaces

Carlos Carpintero & Ennis Rosas
Department of Mathematics,
Universidad De Oriente,
Nucleo De Sucre Cumana, Venezuela.
Facultad de Ciencias Basicas,
Universidad del Atlantico,
Barranquilla, Colombia.
carpintero.carlos@gmail.com,
ennisrafael@gmail.com

Sabir Hussain
Department of Mathematics,
College of Science,
Qassim University,
P.O.BOX 6644, Buraydah 51482, Saudi Arabia.
sabiriub@yahoo.com,
sh.hussain@qu.edu.sa

ABSTRACT

The aim of this paper is to introduce and characterize the notions of (i, j)-ω-semiopen sets as a generalization of (i, j)-semiopen sets in bitopological spaces. We also define and discuss the properties of (i, j)-ω-semicontinuous functions.

RESUMEN

El objetivo de este artículo es introducir y caracterizar las nociones de conjuntos (i, j)-ω-semiabiertos como una generalización de conjuntos (i, j)-semiabiertos en espacios bitopológicos. También definimos y discutimos las propiedades de funciones (i, j)-ω-semicontinuas.

Keywords and Phrases: Bitopological spaces, (i, j)-ω-semiopen sets, (i, j)-ω-semiclosed sets.

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1 Introduction and Preliminaries

The concept of a bitopological space was introduced by Kelly [3]. On the other hand, S. Bose [1], introduced the concept of (i,j)-semiopen sets in bitopological spaces. Recently, as generalization of closed sets, the notion of ω-closed sets was introduced and studied by Hdeib [2]. A point $x \in X$ is called a condensation point of A, if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [2], if it contains all of its condensation points. The complement of a ω-closed set is said to be ω-open. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. In this paper, we introduce the concept of (i,j)-ω-semiopen sets as a generalization of (i,j)-semiopen sets in bitopological spaces. We also define and discuss the properties of (i,j)-ω-semicontinuous functions. For a subset A of X, the closure of A and the interior of A are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A subset A of a bitopological space (X,τ_1,τ_2) is said to be (i,j)-semi open, if $A \subseteq \tau_i\text{-}cl(\tau_j\text{-}Int(A))$, where $i \neq j$, $i,j = 1,2$. The complement of a (i,j)-semiopen set is said to be a (i,j)-semiclosed. The (i,j)-semiclosure of A, denoted by (i,j)-$scl(A)$ is defined by the intersection of all (i,j)-semiclosed sets containing A. The (i,j)-semi interior of A, denoted by (i,j)-$sInt(A)$ is defined by the union of all (i,j)-semiopen sets contained in A. The family of all (i,j)-semiopen (respectively (i,j)-semiclosed) subsets of a space (X,τ_1,τ_2) is denoted by $(i,j) - SO(X)$, (respectively $(i,j) - SC(X)$). A function $f : (X,\tau_1,\tau_2) \mapsto (Y,\sigma_1,\sigma_2)$ is said to be (i,j)-semi continuous, if the inverse image of every σ_i-open set in (Y,σ_1,σ_2) is (i,j)-semi open in (X,τ_1,τ_2), where $i \neq j$, $i,j = 1,2$. A σ_i-open set U in (Y,σ_1,σ_2) means $U \in \sigma_i$.

2 (i,j)-ω-semiopen sets

A set X equipped with two topologies is called a bitopological space. Throughout this paper, spaces (X,τ_1,τ_2) (or simply X) always means a bitopological spaces on which no separation axioms are assumed unless explicitly stated.

Definition 2.1. *Let X be a bitopological space and $A \subseteq X$. Then A is said to be (i,j)-ω-semiopen, if for each $x \in A$ there exists a (i,j)-semiopen U_x containing x such that $U_x - A$ is a countable set. The complement of a (i,j)-ω-semiopen set is a (i,j)-ω-semiclosed set.*

The family of all (i,j)-ω-semiopen (respectively (i,j)-ω-semiclosed) subsets of a space (X,τ_1,τ_2) is denoted by $(i,j) - \omega - SO(X)$, (respectively $(i,j) - \omega - SC(X)$). Also the family of all $(i,j) - \omega$-semiopen sets of (X,τ_1,τ_2) containing x is denoted by $(i,j) - \omega - SO(X,x)$. Note that every (i,j)-semiopen set is a (i,j)-ω-semiopen. The following example shows that the converse is not true in general.

Example 2.2. *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a,b\}, X\}$, $\tau_2 = \{\emptyset, \{b,c\}, X\}$. Then $\{a,c\}$ is a (i,j)-ω-semiopen but not (i,j)-semiopen.*

Theorem 2.3. *Let* X *be a bitopological space and* $A \subseteq X$*. Then* A *is said to be* (i, j)-ω*-semiopen if and only if for every* $x \in A$*, there exists a* (i, j)*-semiopen set* U_x *containing* x *and a countable subset* C *such that* $U_x - C \subseteq A$*.*

Proof. Let A be a (i, j)-ω-semiopen set and $x \in A$, then there exists a (i, j)-semiopen subset U_x containing x such that $U_x - A$ is countable. Let $C = U_x - A = U_x \cap (X - A)$. Then $U_x - C \subseteq A$. Conversely, let $x \in A$. Then there exists a (i, j)-ω-semiopen subset U_x containing x and a countable subset C such that $U_x - C \subseteq A$. Thus $U_x - A \subseteq C$ and $U_x - A$ is countable. □

Theorem 2.4. *Let* X *be a bitopological space and* $C \subseteq X$*. If* C *is a* (i, j)-ω*-semiclosed set, then* $C \subseteq K \cup B$*, for some* (i, j)-ω*-semiclosed subset* K *and a countable subset* B*.*

Proof. If C is a (i, j)-ω-semiclosed set, then $X - C$ is a (i, j)-ω-semiopen set and hence by Theorem 2.3, for every $x \in X - C$, there exists a (i, j)-semiopen set U containing x and a countable set B such that $U - B \subseteq X - C$. Thus $C \subseteq X - (U - B) = X - (U \cap (X - B)) = (X - U) \cup B$, let $K = X - U$. Then K is a (i, j)-semiclosed set such that $C \subseteq K \cup B$. □

Theorem 2.5. *The union of any family of* $(i, j) - \omega$*-semiopen sets is* (i, j)-ω*-semiopen set.*

Proof. If $\{A_\alpha : \alpha \in I\}$ is a collection of (i, j)-ω-semiopen subsets of X, then for every $x \in \bigcup_{\alpha \in I} A_\alpha$, $x \in A_\alpha$, for some $\alpha \in I$. Hence, there exists a (i, j)-ω-semiopen subset U containing x, such that $U - A_\alpha$ is countable. Now as $U - (\bigcup_{\alpha \in I} A_\alpha) \subseteq U - A_\alpha$, and thus $U - (\bigcup_{\alpha \in I} A_\alpha)$ is countable. Therefore $\bigcup_{\alpha \in I} A_\alpha$ is a (i, j)-ω-semiopen set. □

Definition 2.6. *The union of all* (i, j)-ω*-semiopen sets contained in* $A \subseteq X$ *is called the* (i, j)-ω*-semi-interior of* A *and is denoted by* $(i, j) - \omega$*-SInt(A). The intersection of all* (i, j)-ω*-semiclosed sets of* X *containing* A *is called the* (i, j)-ω*-semiclosure of* A *and is denoted by* (i, j)-$\omega - SCl(A)$*.*

Remark 2.7. *The* (i, j)-ω-$SCl(A)$ *is a* (i, j)-ω*-semiclosed set and the* (i, j)-ω-$SInt(A)$ *is a* (i, j)-ω*-semiopen set.*

Theorem 2.8. *Let* X *be a bitopological space and* $A, B \subseteq X$*. Then the following properties hold:*

(1) (i, j)-ω-$SInt((i, j)$-ω-$SInt(A)) = (i, j)$-ω-$SInt(A)$.

(2) If $A \subset B$*, then* (i, j)-ω-$SInt(A) \subset (i, j)$-ω-$SInt(B)$.

(3) (i, j)-ω-$SInt(A \cap B) \subset (i, j)$-$\omega$-$SInt(A) \cap (i, j) - \omega - SInt(B)$.

(4) (i, j)-ω-$SInt(A) \cup (i, j)$-ω-$SInt(B) \subset (i, j)$-ω-$SInt(A \cup B)$.

(5) (i, j)-ω-$SInt(A)$ *is the largest* (i, j)-ω*-semiopen subset of* X *contained in* A*.*

(6) A *is* (i, j)-ω*-semiopen if and only if* $A = (i, j)$-ω-$SInt(A)$.

(7) (i, j)-ω-$SCl((i, j)$-ω-$SCl(A)) = (i, j)$-ω-$SCl(A)$.

(8) *If* $A \subset B$*, then* (i,j)-ω-$SCl(A) \subset (i,j)$-ω-$SCl(B)$.

(9) (i,j)-ω-$SCl(A) \cup (i,j)$-ω-$SCl(B) \subset (i,j)$-ω-$SCl(A \cup B)$.

(10) (i,j)-ω-$SCl(A \cap B) \subset (i,j)$-$\omega$-$SCl(A) \cap (i,j)$-$\omega$-$SCl(B)$.

Proof. (1), (2), (6), (7) and (8) follow directly from the definition of (i,j)-ω-semiopen and (i,j)-ω-semiclosed sets. (3), (4) and (5) follow from (2). (9) and (10) follow by applying (8). □

Example 2.9. *Let* X *be the real line,* $\tau_1 = \{\emptyset, \mathrm{Re}, Q^c\}$ *and* $\tau_2 = \{\emptyset, \mathrm{Re}, Q, Q^c\}$*. Take* $A = (0,1)$*,* $B = (1,2)$*, then* (i,j)-ω-$SCl(A \cap B) \subset (i,j)$-$\omega$-$SCl(A) \cap (i,j)$-$\omega$-$SCl(B)$.

Example 2.10. *Let* X *be the real line,* $\tau_1 = \{\emptyset, \mathrm{Re}, Q\}$ *and* $\tau_2 = \{\emptyset, \mathrm{Re}, Q\}$*. The collection of* $(i,j) - SO(X)$ *is* $\{\emptyset, \mathrm{Re}, Q\}$*. take* $A = Q$*,* $B = \{\pi\}$*. Then* (i,j)-ω-$SCl(A) = Q$*,* (i,j)-ω-$SCl(B) = \{\pi\}$ *and* (i,j)-ω-$SCl(A) \cup (i,j)$-ω-$SCl(B) \subset (i,j)$-ω-$SCl(A \cup B)$.

Theorem 2.11. *Let* X *be a bitopological space. Suppose* $A \subseteq X$ *and* $x \in X$*. Then* $x \in (i,j)$-ω-$SCl(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in (i,j)$-ω-$SO(X,x)$.

Proof. Suppose that $x \in (i,j)$-ω-$SCl(A)$ and we show that $U \cap A \neq \emptyset$, for all $U \in (i,j)$-ω-$SO(X,x)$. Suppose on the contrary that there exists $U \in (i,j)$-ω-$SO(X,x)$ such that $U \cap A = \emptyset$, then $A \subseteq X - U$ and $X - U$ is a (i,j)-ω-semiclosed set. This follows that (i,j)-ω-$SCl(A) \subseteq (i,j)$-ω-$SCl(X - U) = X - U$. Since $x \in (i,j)$-ω-$SCl(A)$, we have $x \in X - U$ and hence $x \notin U$. Which contradicts the fact that $x \in U$. Therefore, $U \cap A \neq \emptyset$. Conversely, suppose that $U \cap A \neq \emptyset$ for every $U \in (i,j)$-ω-$SO(X,x)$. We shall prove that $x \in (i,j)$-ω-$SCl(A)$. Suppose on the contrary that $x \notin (i,j)$-ω-$SCl(A)$. Let $U = X - (i,j)$-ω-$SCl(A)$, then $U \in (i,j)$-ω-$SO(X,x)$ and $U \cap A = (X - ((i,j)$-ω-$SCl(A))) \cap A \subseteq (X - A) \cap A = \emptyset$. This is a contradiction to the fact that $U \cap A \neq \emptyset$. Hence $x \in (i,j)$-ω-$SCl(A)$. □

Theorem 2.12. *Let* X *be a bitopological space and* $A \subset X$*. Then the following properties hold:*

(1) (i,j)-ω-$SCl(X \backslash A) = X \backslash (i,j)$-$\omega$-$SInt(A)$;

(2) (i,j)-ω-$SInt(X \backslash A) = X \backslash (i,j)$-$\omega$-$SCl(A)$.

Proof. (1). Let $x \in X \backslash (i,j)$-ω-$SCl(A)$. Then there exists $V \in (i,j)$-ω-$SO(X,x)$ such that $V \cap A = \emptyset$ and hence we obtain $x \in (i,j)$-ω-$SInt(X \backslash A)$. This shows that $X \backslash (i,j)$-ω-$SCl(A) \subset (i,j)$-ω-$SInt(X \backslash A)$. Now consider $x \in (i,j)$-ω-$SInt(X \backslash A)$. Since (i,j)-ω-$SInt(X \backslash A) \cap A = \emptyset$, we obtain $x \notin (i,j)$-ω-$SCl(A)$. Therefore, we have, (i,j)-ω-$SCl(X \backslash A) = X \backslash (i,j)$-$\omega$-$SInt(A)$.
(2). Let $x \in X \backslash (i,j)$-ω-$SInt(X - A)$. Since (i,j)-ω-$SInt(X \backslash A) \cap A = \emptyset$, we have $x \notin (i,j)$-ω-$SCl(A)$ implies $x \in X \backslash (i,j)$-ω-$SCl(A)$. Now consider $x \in X \backslash (i,j)$-ω-$SCl(A)$, then there exists $V \in (i,j)$-ω-$SO(X,x)$ such that $V \cap A = \emptyset$, hence we obtain that (i,j)-ω-$SInt(X \backslash A) = X \backslash (i,j)$-$\omega$-$SCl(A)$. □

Definition 2.13. *Let* X *be a bitopological space and* $B \subseteq X$*. Then* B *is a* (i,j)-ω*-semineighbourhood of a point* $x \in X$ *if there exists a* (i,j)-ω*-semiopen set* W *such that* $x \in W \subset B$.

Theorem 2.14. *Let* X *be a bitopological space and* $B \subseteq X$. B *is a* (i,j)-ω-*semiopen set if and only if it is a* (i,j)-ω-*semineighbourhood of each of its points.*

Proof. Let B be a (i,j)-ω-semiopen set of X. Then by definition B is a (i,j)-ω-semineighbourhood of each of its points. Conversely, suppose that B is a (i,j)-ω-semineighbourhood of each of its points. Then for each $x \in B$, there exists $S_x \in (i,j)$-ω-$SO(X,x)$ such that $S_x \subset B$. Then $B = \bigcup\{S_x : x \in B\}$. Since each S_x is a (i,j)-ω-semiopen and arbitrary union of (i,j)-ω-semiopen sets is (i,j)-ω-semiopen, B is a (i,j)-ω-semiopen in X. □

Theorem 2.15. *If each nonempty* (i,j)-ω-*semiopen set of a bitopological space* X *is uncountable, then* (i,j)-$SCl(A) = (i,j)$-ω-$SCl(A)$, *for each subset* $A \in \tau_1 \cap \tau_2$.

Proof. Clearly (i,j)-ω-$SCl(A) \subseteq (i,j)$-$SCl(A)$. On the other hand, let $x \in (i,j)$-$SCl(A)$ and B be a (i,j)-ω-semiopen set containing x. Using Theorem 2.3, there exists a (i,j)-semiopen set V containing x and a countable set C such that $V - C \subseteq B$. Follows $(V - C) \cap A \subseteq B \cap A$ and so $(V \cap A) - C \subseteq B \cap A$. Now $x \in V$, $x \in (i,j)$-$SCl(A)$ such that $V \cap A \neq \emptyset$ where $V \cap A$ is a (i,j)-ω-semiopen set, since V is a (i,j)-semiopen set and $A \in \tau_1 \cap \tau_2$. Using the hypothesis, each nonempty (i,j)-ω-semiopen set of X is uncountable and so is $(V \cap A)\backslash C$. Thus $B \cap A$ is uncountable. Therefore, $B \cap A \neq \emptyset$ implies that $x \in (i,j)$-ω-$SCl(A)$. □

Theorem 2.16. *Let* X *be a bitopological space. If every* (i,j)-ω-*semiopen subset of* X *is* τ_i-*open in* X. *Then* $(X, (i,j)$-ω-$SO(X))$ *is a topological space.*

Proof. 1. $\emptyset, X$ belong to (i,j)-ω-$SO(X)$
2. Let $U, V \in (i,j)$-ω-$SO(X)$ and $x \in U \cap V$. Then there exists (i,j)-semi open sets G, H in X containing x such that $G\backslash U$ and $H\backslash V$ are countable. Since $(G \cap H)\backslash(U \cap V) = (G \cap H) \cap ((X\backslash U) \cup (X\backslash V)) \subseteq (G \cap (X\backslash U)) \cup (H \cap (X\backslash V))$ implies that $(G \cap H)\backslash(U \cap V)$ is a countable set and by hypothesis, the intersection of two (i,j)-semi open set is (i,j)-semi open. Hence $U \cap V \in (i,j)$-ω-$SO(X))$.
3. The union follows directly.

□

3 (i,j)-ω-semicontinuous functions

Definition 3.1. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be a* (i,j)-ω-*semicontinuous, if the inverse image of every* σ_i-*open set of* Y *is* (i,j)-ω-*semiopen in* (X, τ_1, τ_2), *where* $i \neq j$, i,j=*1, 2.*

Definition 3.2. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be a* (i,j)-*semicontinuous, if the inverse image of every* σ_i-*open set of* Y *is* (i,j)-*semiopen in* (X, τ_1, τ_2), *where* $i \neq j$, i,j=*1, 2.*

Theorem 3.3. *Every* (i,j)-*semicontinuous function is* (i,j)-ω-*semicontinuous.*

Proof. The proof follows from the the fact that every (i,j)-semiopen set is (i,j)-ω-semiopen. □

However, the converse may be false.

Example 3.4. *Let* $X = \{a,b,c\}$, $\tau_1 = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$, $\tau_2 = \{\emptyset,\{a\},X\}$, $\sigma_1 = \{\emptyset,\{a,b\},X\}$, $\sigma_2 = \{\emptyset,\{a,c\},X\}$. *Then the identity function* $f:(X,\tau_1,\tau_2)\to(X,\sigma_1,\sigma_2)$ *is* (i,j)-ω-*semicontinuous but not* (i,j)-*semicontinuous.*

Theorem 3.5. *For a function* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, *the following statements are equivalent:*

(1) f *is* (i,j)-ω-*semicontinuous;*

(2) For each point $x\in X$ *and each* σ_i-*open set* F *in* Y *such that* $f(x)\in F$, *there is a* (i,j)-ω-*semiopen set* A *in* X *such that* $x\in A$, *and* $f(A)\subset F$;

(3) The inverse image of each σ_i-*closed set in* Y *is a* (i,j)-ω-*semiclosed in* X;

(4) For $A\subseteq X$, $f((i,j)\text{-}\omega\text{-}SCl(A))\subset\sigma_i\text{-}cl(f(A))$;

(5) For $B\subseteq Y$, $(i,j)\text{-}\omega\text{-}SCl(f^{-1}(B))\subset f^{-1}(\sigma_i\text{-}cl(B))$;

(6) For $C\subseteq Y$, $f^{-1}(\sigma_i\text{-}Int(C))\subset(i,j)\text{-}\omega\text{-}SInt(f^{-1}(C))$.

Proof. - (1)⇒(2): Let $x\in X$ and F be a σ_i-open set of Y containing $f(x)$. By (1), $f^{-1}(F)$ is (i,j)-ω-semiopen in X. Let $A=f^{-1}(F)$. Then $x\in A$ and $f(A)\subset F$.
(2)⇒(1): Let F be σ_i-open in Y and let $x\in f^{-1}(F)$. Then $f(x)\in F$. By (2), there is a (i,j)-ω-semiopen set U_x in X such that $x\in U_x$ and $f(U_x)\subset F$ implies $x\in U_x\subset f^{-1}(F)$. Hence $f^{-1}(F)$ is a (i,j)-ω-semiopen in X.
(1)⇔(3): This follows from the fact that for any subset B of Y, $f^{-1}(Y\backslash B)=X\backslash f^{-1}(B)$.
(3)⇒(4): Let $A\subseteq X$. Since $A\subset f^{-1}(f(A))$, we have $A\subset f^{-1}(\sigma_i\text{-Cl}(f(A)))$. By hypothesis $f^{-1}(\sigma_i\text{-Cl}(f(A)))$ is a (i,j)-ω-semiclosed set in X and hence $(i,j)\text{-}\omega\text{-}SCl(A))\subset f^{-1}(\sigma_i\text{-Cl}(f(A)))$. Follows $f((i,j)\text{-}\omega\text{-}SCl(A)))\subset f(f^{-1}(\sigma_i\text{-Cl}(f(A)))\subseteq\sigma_i\text{-Cl}(f(A))$.
(4)⇒(3): Let F be any σ_i-closed subset of Y. Then $f((i,j)\text{-}\omega\text{-}SCl(f^{-1}(F))\subset\sigma_i\text{-}cl(f(f^{-1}(F)))\subset\sigma_i\text{-}cl(F)=F$. Therefore, the $(i,j)\text{-}\omega\text{-}SCl(f^{-1}(F))\subset f^{-1}(F)$. Consequently, $f^{-1}(F)$ is a (i,j)-ω-semiclosed set in X.

(4)⇒(5): Let $B\subseteq Y$. Now, $f((i,j)\text{-}\omega\text{-}SCl(f^{-1}(B)))\subset\sigma_i\text{-Cl}(f(f^{-1}(B)))\subset\sigma_i\text{-Cl}(B)$. Consequently, $(i,j)\text{-}\omega\text{-}SCl(f^{-1}(B))\subset f^{-1}(\sigma_i\text{-Cl}(B))$.
(5)⇒(4): Let $B=f(A)$ where $A\subseteq X$. Then, $(i,j)\text{-}\omega\text{-}SCl(A)\subset(i,j)\text{-}\omega\text{-}SCl(f^{-1}(B))\subset f^{-1}(\sigma_i\text{-Cl}(B))=f^{-1}(\sigma_i\text{-Cl}(f(A)))$, and hence $f((i,j)\text{-}\omega\text{-}SCl(A))\subset\sigma_i\text{-Cl}(f(A))$.
(1)⇒(6): Let $B\subseteq Y$. Clearly, $f^{-1}(\sigma_i\text{-Int}(B))$ is a (i,j)-ω-semiopen and we have $f^{-1}(\sigma_i\text{-Int}(B))\subset(i,j)\text{-}\omega\text{-}SInt(f^{-1}\sigma_i\text{-Int}(B))\subset(i,j)\text{-}\omega\text{-}SInt(f^{-1}B)$.
(6)⇒(1): Let B be a σ_i-open set in Y. Then $\sigma_i\text{-Int}(B)=B$ and $f^{-1}(B)\subset f^{-1}(\sigma_i\text{-Int}(B))\subset(i,j)\text{-}\omega\text{-}SInt(f^{-1}(B))$. Hence, we have $f^{-1}(B)=(i,j)\text{-}\omega\text{-}SInt(f^{-1}(B))$. This implies that $f^{-1}(B)$ is a (i,j)-ω-semiopen in X. □

Definition 3.6. *The graph* $G(f)$ *of* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is said to be* (i,j)-ω-*semiclosed in* $X\times Y$*, if for each* $(x,y)\in(X\times Y)\setminus G(f)$*, there exists* $U\in(i,j)$-ω-$SO(X,x)$*,* $i,j=\{1,2\}$ *with* $i\neq j$ *and a* σ_i*-open set* V *of* Y *containing* y *such that* $(U\times V)\cap G(f)=\emptyset$*.*

Lemma 3.7. *The graph* $G(f)$ *of* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is* (i,j)-ω-*semiclosed in* $X\times Y$ *if and only if for each* $(x,y)\in(X\times Y)\setminus G(f)$*, there exists* $U\in(i,j)$-ω-$SO(X,x)$*,* $i,j=\{1,2\}$ *with* $i\neq j$ *and a* σ_i*-open set* V *of* Y *containing* y *such that* $f(U)\cap V=\emptyset$*.*

Proof. The proof is an immediate consequence of Definition 3.6. □

Theorem 3.8. *If a function* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is a* (i,j)-ω-*semicontinuous function and* (Y,σ_i) *is* T_1 $i=\{1,2\}$*, then* $G(f)$ *is* (i,j)-ω-*semiclosed.*

Proof. Let $(x,y)\in(X\times Y)\setminus G(f)$. Then $y\neq f(x)$. Since (Y,σ_i) is T_1, there exist a σ_i-open set V and W of Y such that $f(x)\in V$ and $y\notin W$ and $V\cap W=\emptyset$. Since f is (i,j)-ω-semicontinuous, there exists $U\in(i,j)$-ω-$SO(X,x)$ such that $f(U)\subset V$. Therefore, $f(U)\cap W=\emptyset$. Therefore, by Lemma 3.7, $G(f)$ is (i,j)-ω-semiclosed. □

Definition 3.9. *A bitopological space* X *is said to be a* (i,j)-ω-*semi*-T_2 *space, if for each pair of distinct points* $x,y\in X$*, there exist* $U,V\in(i,j)$-ω-$SO(X)$ *containing* x *and* y*, respectively, such that* $U\cap V=\emptyset$*.*

Theorem 3.10. *If* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is a* (i,j)-ω-*semicontinuous injective function and* (Y,σ_i) *is a* T_2 *space, then* (X,τ_1,τ_2) *is a* ω-*semi*-T_2 *space.*

Proof. The proof follows from the definition. □

Theorem 3.11. *If* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *is an injective* (i,j)-ω-*semicontinuous function with a* (i,j)-ω-*semiclosed graph, then* X *is a* (i,j)-ω-*semi*-T_2 *space.*

Proof. Let x_1 and x_2 be any pair of distinct points of X. Then $f(x_1)\neq f(x_2)$, so $(x_1,f(x_2))\in(X\times Y)\setminus G(f)$. Since the graph $G(f)$ is (i,j)-ω-semiclosed, there exist a (i,j)-ω-semiopen set U containing x_1 and $V\in\sigma_i$ containing $f(x_2)$ such that $f(U)\cap V=\emptyset$. Since f is (i,j)-ω-semicontinuous, $f^{-1}(V)$ is a (i,j)-ω-semiopen set containing x_2 such that $U\cap f^{-1}(V)=\emptyset$. Hence X is (i,j)-ω-semi-T_2. □

Definition 3.12. *A collection* $\{U_\alpha:\alpha\in I\}$ *of* (i,j)-*semiopen sets in a bitopological space* X *is called a* (i,j)-*semiopen cover of a subset* A *of* X*, if* $A\subseteq\bigcup_{\alpha\in I}U_\alpha$*.*

Definition 3.13. *A bitopological space* X *is said to be* (i,j)-*semi Lindeloff, if every* (i,j)-*semi open cover of* X *has a countable subcover. A subset* A *of bitopological space* X *is said to be* (i,j)-*semi Lindeloff relative to* X*, if every cover of* A *by* (i,j)-*semiopen sets of* X *has a countable subcover.*

Theorem 3.14. *If* X *is a bitopological space such that every* (i,j)-*semiopen subset is* (i,j)-*semi Lindeloff relative to* X*. Then every subset is* (i,j)-*semi Lindeloff relative to* X

Theorem 3.15. *For a bitopological space* X*. The following properties are equivalent:*

(1) X *is* (i,j)*-semi Lindeloff.*

(2) Every countable cover of X *by* (i,j)*-semiopen sets has a countable subcover.*

Proof. (2)⇒(1): Since every (i,j)-semiopen set is (i,j)-ω-semiopen, the proof follows. (1)⇒(2): Let $\{U_\alpha : \alpha \in I\}$ be any cover of X by (i,j)-ω-semiopen sets of X. For each $x \in X$, there exists an $\alpha_x \in I$ such that $x \in U_{\alpha_x}$. Since U_{α_x} is a (i,j)-ω-semiopen, then there exists a (i,j)-semiopen set V_{α_x} such that $x \in V_{\alpha_x}$ and $V_{\alpha_x} - U_{\alpha_x}$ is countable. The family $\{V_\alpha : \alpha \in I\}$ is a (i,j)-semiopen cover of X and X is (i,j)-semi Lindeloff. Therefore there exists a countable subcover α_{x_i} with $i \in N$ such that $X = \bigcup_{i\in N} V_{\alpha_{x_i}}$. Since $X = \bigcup_{i\in N}[(V_{\alpha_{x_i}} - U_{\alpha_{x_i}}) \cup U_{\alpha_{x_i}}] = \bigcup_{i\in N}[(V_{\alpha_{x_i}} - U_{\alpha_{x_i}}) \bigcup_{i\in N} U_{\alpha_{x_i}}]$. Since $V_{\alpha_{x_i}} - U_{\alpha_{x_i}}$ is a countable set, for each $\alpha(x_i)$, there exists a countable subset $I_{\alpha(x_i)}$ of I such that $V_{\alpha_{x_i}} - U_{\alpha_{x_i}} \subseteq \bigcup_{I_{\alpha(x_i)}} U_\alpha$ and therefore $X \subseteq \bigcup_{i\in N}(\bigcup_{\alpha\in I_{\alpha(x_i)}} U_\alpha) \cup (\bigcup_{i\in N} U_{\alpha(x_i)})$. □

Definition 3.16. *A bitopological space* X *is called pairwise Lindeloff if each pairwise open cover of* X *has a countable subcover.*

Theorem 3.17. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a* (i,j)*-*ω*-semicontinuous function. If* X *is* (i,j)*-semi Lindeloff, then* Y *is pairwise Lindeloff.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any cover of Y by σ_i-open sets. Then $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is a (i,j)-ω-semiopen cover of X. Since X is (i,j)-semi Lindeloff, there exists a countable subset I_0 of I such that $X = \bigcup_{\alpha\in I_0} U_\alpha$. Therefore, Y is a pairwise Lindeloff. □

Definition 3.18. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be:*

1 (i,j)-ω*-semiopen if* $f(U)$ *is a* (i,j)-ω*-semiopen set in* Y *for every* τ_i*-open set* U *of* X.

2 (i,j)-ω*-semiclosed if* $f(U)$ *is a* (i,j)-ω*-semiclosed set in* Y *for every* τ_i*-closed set* U *of* X.

Theorem 3.19. *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is a* (i,j)-ω*-semiopen.*

(2) $f(\tau_i - Int(U)) \subseteq (i,j)\text{-}\omega\text{-}SCl(f(U))$*, for each subset* U *of* X.

(3) $\tau_i - Int(f^{-1}(V)) \subseteq f^{-1}((i,j)\text{-}\omega\text{-}SInt(V)$*, for each subset* V *of* Y.

Proof. (1)⇒(2): Let U be any subset of X. Then $\tau_i - Int(U)$ is a τ_i-open set of X. Then $f(\tau_i - Int(U))$ is a (i,j)-ω-semiopen set of Y. Since $f(\tau_i - Int(U)) \subseteq f(U)$, $f(\tau_i - Int(U)) = (i,j)$-$\omega$-$SInt(f(\tau_i - Int(U))) \subseteq (i,j)$-$\omega$-$SInt(f(U))$.
(2)⇒(3):Let V be any subset of Y. Then $f(\tau_i - Int(f^{-1}(V))) \subseteq (i,j)$-$\omega$-$SInt(f(f^{-1}(V)))$. Hence $\tau_i - Int(f^{-1}(V)) \subseteq f^{-1}((i,j)\text{-}\omega\text{-}SInt(V))$.
(3)⇒(1): Let U be any τ_i-open set of X. Then $\tau_i - Int(U) = U$. Now, $V = \tau_i - Int(V) \subseteq \tau_i -$

$Int(f^{-1}(f(V)) \subseteq f^{-1}((i,j)\text{-}\omega\text{-}SInt(f(V))))$. Which implies that $f(V) \subseteq f(f^{-1}((i,j)\text{-}\omega\text{-}SInt(f(V))))$ $\subseteq (i,j)\text{-}\omega\text{-}SInt(f(V))$. Hence $f(V)$ is a (i,j)-ω-semiopen set of Y. Thus f is (i,j)-ω-semiopen. □

Theorem 3.20. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function, then f is a (i,j)-ω-semiclosed function if and only if for each subset V of X, the $(i,j)\text{-}\omega\text{-}SCl(f(V)) \subseteq f(\tau_i - Cl(V)))$.*

Proof. Let f be a (i,j)-ω-semiclosed function and V be any subset of X. Then $f(V) \subseteq f(\tau_i - Cl(V))$ and $f(\tau_i - Cl(V))$ is a (i,j)-ω-semiclosed set of Y. Hence $(i,j)\text{-}\omega\text{-}SCl(f(V)) \subseteq (i,j)\text{-}\omega\text{-}SCl(f(\tau_i - Cl(V))) = f(\tau_i - Cl(V)))$. Conversely, let V be a τ_i-closed set of X. Then $f(V) \subseteq (i,j)\text{-}\omega\text{-}SCl(f(V)) \subseteq f(\tau_i - Cl(V))) = f(V)$. Hence $f(V)$ is a (i,j)-ω-semiclosed set of Y. Therefore, f is a (i,j)-ω-semiclosed function □

Definition 3.21. *A bitopological space X is said to be (i,j)-ω-semiconnected, if X cannot be expressed as the union of two nonempty disjoint (i,j)-ω-semiopen sets.*

Definition 3.22. *A bitopological space X is said to be pairwise connected [5], if it cannot be expressed as the union of two nonempty disjoint sets U and V such that U is τ_i-open and V is τ_j-open, where $i,j = \{1,2\}$ and $i \neq j$.*

Theorem 3.23. *A (i,j)-ω-semicontinuous image of a (i,j)-ω-semiconnected space is pairwise connected.*

Proof. The proof is clear. □

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