CUBO A Mathematical Journal
Vol.16, Nº 01, (75–84). March 2015

Continuity via Λ_I^s-open sets

José Sanabria, Edumer Acosta, Ennis Rosas and Carlos Carpintero[1]
Departamento de Matemáticas, Núcleo de Sucre,
Universidad de Oriente,
Avenida Universidad, Cerro Colorado, Cumaná, Estado Sucre, Venezuela
jesanabri@gmail.com, edumeracostab@gmail.com, ennisrafael@gmail.com,
carpintero.carlos@gmail.com

ABSTRACT

Sanabria, Rosas and Carpintero [7] introduced the notions of Λ_I^s-sets and Λ_I^s-closed sets using ideals on topological spaces. Given an ideal I on a topological space (X, τ), a subset $A \subset X$ is said to be Λ_I^s-closed if $A = U \cap F$ where U is a Λ_I^s-set and F is a $\tau^\star$-closed set. In this work we use sets that are complements of Λ_I^s-closed sets, which are called Λ_I^s-open, to characterize new variants of continuity namely Λ_I^s-continuous, quasi-Λ_I^s-continuous y Λ_I^s-irresolute functions.

RESUMEN

Sanabria, Rosas y Carpintero [7] introdujeron las nociones de conjuntos Λ_I^s y conjuntos Λ_I^s-cerrados usando ideales sobre espacios topológicos. Dado un ideal I sobre un espacio topológico (X, τ), un subconjunto $A \subset X$ se llama Λ_I^s-cerrado si $A = U \cap F$ donde U es un Λ_I^s-conjunto y F es un conjunto $\tau^\star$-cerrado . En este trabajo usamos conjuntos que son complementos de conjuntos Λ_I^s-cerrado, los cuales son llamados Λ_I^s-abiertos, para caracterizar nuevas variantes de continuidad, denominadas, funciones Λ_I^s-continuas y funciones Λ_I^s-irresolutas.

Keywords and Phrases: Local function, Λ_I^s-open sets, Λ_I^s-irresolute functions.

2010 AMS Mathematics Subject Classification: 54C08, 54D05.

[1]Research Partially Suported by Consejo de Investigación UDO

1 Introduction

The theory of ideal on topological spaces has been the subject of many studies in recent years. It was the works of Hamlet and Jankovic [5], Abd El-Monsef, Lashien and Nasef [1] and Hatir and Noiri [2] which motivated the research in applying topological ideals to generalize the most basic properties in general topology. In 2002, Hatir and Noiri [2] introduced the notion of semi-I-open sets in topological spaces. Also, Hatir and Noiri [3] investigated semi-I-open sets and semi-I-continuous functions. Quite recently, Sanabria, Rosas and Carpintero [7] have introduced the notions of Λ_I^s-sets and Λ_I^s-closed sets to obtain characterizations of two low separation axioms, namely semi-I-T_1 and semi-I-$T_{1/2}$ spaces. In this article we introduce the notion of Λ_I^s-open sets in order to characterize new variants of continuity in ideal topological spaces.

2 Preliminaries

Throughout this paper, $P(X)$, Cl(A) and Int(A) denote the power set of X, the closure of A and the interior of A, respectively. An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following two properties:

(1) $A \in I$ and $B \subset A$ implies $B \in I$;

(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an ideal topological space and is denoted by (X, τ, I). Given an ideal topological space (X, τ, I), a set operator $(.)^\star : P(X) \to P(X)$, called a local function [6] of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I \text{for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(I, \tau)$. In general, $X^\star$ is a proper subset of X. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(I, \tau)$, called the $\star$-topology, finer than τ, is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(I, \tau)$ [5]. For any ideal topological space (X, τ, I), the collection $\beta(I, \tau) = \{V \setminus J : V \in \tau \text{ and } J \in I\}$ is a basis for $\tau^\star(I, \tau)$. According to the above, in this article, we denote by $\tau^\star$ to topology $\tau^\star(I, \tau)$ generated by $\mathrm{Cl}^\star$, that is, $\tau^\star = \{U \in P(X) : \mathrm{Cl}^\star(X - U) = X - U\}$. The elements of $\tau^\star$ are called $\tau^\star$-open and the complement of a $\tau^\star$-open is called $\tau^\star$-closed. It is well known that a subset A of an ideal topological space (X, τ, I) is $\tau^\star$-closed if and only if $A^\star \subset A$ [5].

Definition 2.1. *A subset* A *of an ideal topological space* (X, τ, I) *is said to be semi-I-open [2] if* $A \subset \mathrm{Cl}^\star(\mathrm{Int}(A))$. *The complement of a semi-I-open set is said to be semi-I-closed. The family of all semi-I-open sets of an ideal topological space* (X, τ, I) *is denoted by* $\mathrm{SIO}(X, \tau)$.

The following three notions has been introduced by Sanabria et al. [7].

Definition 2.2. *Let A be a subset of an ideal topological space* (X, τ, I). *A subset* $\Lambda_I^s(A)$ *is defined as follows:* $\Lambda_I^s(A) = \cap\{U : A \subset U,\ U \in \mathrm{SIO}(X, \tau)\}$.

Definition 2.3. *Let* (X, τ, I) *an ideal topological space. A subset* A *of* X *is said to be:*

(1) Λ_I^s*-set if* $A = \Lambda_I^s(A)$.

(2) Λ_I^s*-closed if* $A = U \cap F$*, where* U *is a* Λ_I^s*-set and* F *is an* $\tau^\star$*-closed set.*

Since each open set is semi-I-open, by Propositions 3.1(3) and 4.1 of [7], we have the following implications:

$$\text{Open} \Longrightarrow \text{semi-I-open} \Longrightarrow \Lambda_I^s\text{-set} \Longrightarrow \Lambda_I^s\text{-closed}.$$

Lemma 2.1. *(Sanabria, Rosas and Carpintero [7]) For an ideal topological space* (X, τ, I)*, we take* $\tau^{\Lambda_I^s} = \{A : A$ *is a* Λ_I^s*-set of* $(X, \tau, I)\}$*. Then the pair* $(X, \tau^{\Lambda_I^s})$ *is an Alexandroff space.*

Remark 2.1. *According to Lemma 2.1, a subset* A *of an ideal topological space* (X, τ, I) *is open in* $(X, \tau^{\Lambda_I^s})$*, if* A *is a* Λ_I^s*-set of* (X, τ, I).

Definition 2.4. *A subset* A *of an ideal topological space* (X, τ, I) *is called* Λ_I^s*-open if* $X \setminus A$ *is a* Λ_I^s*-closed set.*

In the sequel, the ideal topological space (X, τ, I) is simply denoted by X. Next we present some results related with Λ_I^s-open sets.

Lemma 2.2. *Every* $\tau^\star$*-open set is* Λ_I^s*-open.*

Proof. This follows from Proposition 4.1 of [7]. □

Lemma 2.3. *Let* $\{B_\alpha : \alpha \in \Delta\}$ *be a family of subsets of* X*. If* B_α *is* Λ_I^s*-open for each* $\alpha \in \Delta$*, then* $\bigcup\{B_\alpha : \alpha \in \Delta\}$ *is* Λ_I^s*-open.*

Proof. The proof is an immediate consequence from Proposition 4.2 of [7]. □

3 New variants of continuity

In this section we use the notions of open, Λ_I^s-open and $\tau^\star$-open sets in order to introduce new forms of continuous functions called Λ_I^s-continuous, quasi-Λ_I^s-continuous and Λ_I^s-irresolute. We study the relationships between these classes of functions and also obtain some properties and characterizations of them.

Definition 3.1. *A function* $f : (X, \tau, I) \to (Y, \sigma, J)$ *is said to be semi-*I*-irresolute [4], if* $f^{-1}(V)$ *is a semi-*I*-open set in* (X, τ, I) *for each semi-*J*-open set of* (Y, σ, J).

Theorem 3.1. *If a function* $f : (X, \tau, I) \to (Y, \sigma, J)$ *is semi-*I*-irresolute, then* $f : (X, \tau^{\Lambda_I^s}) \to (Y, \sigma^{\Lambda_J^s})$ *is continuous.*

Proof. Let V be any Λ_J^s-set of (Y, σ, J), that is $V \in \sigma^{\Lambda_J^s}$, then $V = \Lambda_J^s(V) = \cap\{W : V \subset W$ and W is semi-J-open in $(Y, \sigma, J)\}$. Since f is semi-I-irresolute, $f^{-1}(W)$ is a semi-I-open set in (X, τ, I) for each W, hence we have

$$\begin{aligned}\Lambda_I^s(f^{-1}(V)) &= \cap\{U : f^{-1}(V) \subset U \text{ and } U \in SIO(X, \tau)\}\\ &\subset \cap\{f^{-1}(W) : f^{-1}(V) \subset f^{-1}(W) \text{ and } W \in SJO(Y, \sigma)\}\\ &= f^{-1}(V).\end{aligned}$$

On the other hand, always we have $f^{-1}(V) \subset \Lambda_I^s(f^{-1}(V))$ and so $f^{-1}(V) = \Lambda_I^s(f^{-1}(V))$. Therefore, $f^{-1}(V) \in \tau^{\Lambda_I^s}$ and $f : (X, \tau^{\Lambda_I^s}) \to (Y, \sigma^{\Lambda_J^s})$ is continuous. □

Definition 3.2. *A function* $f : (X, \tau, I) \to (Y, \sigma, J)$ *is called:*

(1) Λ_I^s*-continuous, if* $f^{-1}(V)$ *is a* Λ_I^s*-open set in* (X, τ, I) *for each open set* V *of* (Y, σ, J).

(2) *Quasi-*Λ_I^s*-continuous, if* $f^{-1}(V)$ *is a* Λ_I^s*-open set in* (X, τ, I) *for each* $\sigma^\star$*-open set* V *of* (Y, σ, J).

(3) Λ_I^s*-irresolute, if* $f^{-1}(V)$ *is a* Λ_I^s*-open set in* (X, τ, I) *for each* Λ_J^s*-open set* V *of* (Y, σ, J).

Theorem 3.2. *If* $f : (X, \tau, I) \to (Y, \sigma, J)$ *is* Λ_I^s*-irresolute function, then* f *is quasi-*Λ_I^s*-continuous.*

Proof. Let V be a $\sigma^\star$-open set of (Y, σ, J), then by Lemma 2.2, we have V is a Λ_J^s-open set of (Y, σ, J) and since f is Λ_I^s-irresolute, $f^{-1}(V)$ is a Λ_I^s-open set of (X, τ, I). Therefore, f is quasi-Λ_I^s-continuous. □

The following example shows a function quasi-Λ_I^s-continuous which is not Λ_I^s-irresolute.

Example 3.1. *Let* $X = \{a, b, c\}$*,* $\tau = \{\emptyset, \{a, c\}, X\}$*,* $\sigma = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$*,* $I = \{\emptyset, \{c\}\}$ *and* $J = \{\emptyset, \{b\}\}$*. The collection of the* Λ_I^s*-open sets of* (X, τ, I) *is* $\{\emptyset, \{a, b\}, \{a, c\}, \{a\}, \{b\}, X\}$*, the collection of the* $\sigma^\star$*-open sets of* (X, σ, J) *is* $\{\emptyset, \{a\}, \{a, c\}, \{a, b\}, X\}$ *and the collection of the* Λ_J^s*-open sets of* (X, σ, J) *is* $\{\{\emptyset\}, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$*. The identity function* $f : (X, \tau, I) \to (X, \sigma, J)$ *is quasi-*Λ_I^s*-continuous, but is not* Λ_I^s*-irresolute, since* $f^{-1}(\{b, c\}) = \{b, c\}$ *and* $f^{-1}(\{c\}) = \{c\}$ *are not* Λ_I^s*-open sets.*

Theorem 3.3. *If* $f : (X, \tau, I) \to (Y, \sigma, J)$ *is quasi-*Λ_I^s*-continuous function, then* f *is* Λ_I^s*-continuous.*

Proof. Let V be an open set of (Y, σ, J), then V is $\sigma^\star$-open set of (Y, σ, J) and since f is quasi-Λ_I^s-continuous, $f^{-1}(V)$ is a Λ_I^s-open set of (X, τ, I). This shows that f is Λ_I^s-continuous. □

The following example shows a function Λ_I^s-continuous which is not quasi-Λ_I^s-continuous.

Example 3.2. *Let* $X = \{a, b, c\}$*,* $\tau = \{\emptyset, \{a, c\}, X\}$*,* $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$*,* $I = \{\emptyset, \{c\}\}$ *and* $J = \{\emptyset, \{a\}\}$*. The collection of the* Λ_I^s*-open sets of* (X, τ, I) *is* $\{\emptyset, \{a, b\}, \{a, c\}, \{a\}, \{b\}, X\}$ *and the collection of* $\sigma^\star$*-open sets of* (X, σ, J) *is* $\{\emptyset, \{a\}, \{b, c\}, \{a, b\}, \{b\}, X\}$*. The identity function* $f : (X, \tau, I) \to (X, \sigma, J)$ *is* Λ_I^s*-continuous, but is not quasi-*Λ_I^s*-continuous, because* $f^{-1}(\{b, c\}) = \{b, c\}$ *is not a* Λ_I^s*-open set.*

Corollary 3.1. *If* $f:(X,\tau,I)\to(Y,\sigma,J)$ *is a* Λ_I^s*-irresolute function, then* f *is* Λ_I^s*-continuous.*

Proof. This is an immediate consequence of Theorems 3.2 and 3.3. □

By the above results, we have the following diagram and none of these implications is reversible:

$$\Lambda_I^s\text{-irresolute} \Longrightarrow \text{quasi-}\Lambda_I^s\text{-continuous} \Longrightarrow \Lambda_I^s\text{-continuous}.$$

Proposition 3.1. *Let* $f:(X,\tau,I)\to(Y,\sigma,J)$ *and* $g:(Y,\sigma,J)\to(Z,\theta,K)$ *be two functions, where* I,J,K *are ideals on* X,Y,Z *respectively. Then:*

(1) $g\circ f$ *is* Λ_I^s*-irresolute, if* f *is* Λ_I^s*-irresolute and* g *is* Λ_J^s*-irresolute.*

(2) $g\circ f$ *is* Λ_I^s*-continuous, if* f *is* Λ_I^s*-irresolute and* g *is* Λ_J^s*-continuous.*

(3) $g\circ f$ *is* Λ_I^s*-continuous, if* f *is* Λ_I^s*-continuous and* g *is continuous.*

(4) $g\circ f$ *is quasi-*Λ_I^s*-continuous, if* f *is* Λ_I^s*-irresolute and* g *is quasi-*Λ_J^s*-continuous.*

Proof. (1) Let V be a Λ_K^s-open set in (Z,θ,K). Since g is Λ_J^s-irresolute, then $g^{-1}(V)$ is a Λ_J^s-open set in (Y,σ,J), using that f is Λ_I^s-irresolute, we obtain that $f^{-1}(g^{-1}(V))$ is a Λ_I^s-open set in (X,τ,I). But $(g\circ f)^{-1}(V)=(f^{-1}\circ g^{-1})(V)=f^{-1}(g^{-1}(V))$ and hence, $(g\circ f)^{-1}(V)$ is a Λ_I^s-open set in (X,τ,I). This shows that $g\circ f$ is Λ_I^s-irresolute.
The proofs of (2), (3) and (4) are similar to the case (1). □

In the next three theorems, we characterize Λ_I^s-continuous, quasi-Λ_I^s-continuous and Λ_I^s-irresolute functions, respectively.

Theorem 3.4. *For a function* $f:(X,\tau,I)\to(Y,\sigma)$*, the following statements are equivalent:*

(1) f *is* Λ_I^s*-continuous.*

(2) $f^{-1}(B)$ *is a* Λ_I^s*-closed set in* (X,τ,I) *for each closed set* B *in* (Y,σ)*.*

(3) *For each* $x\in X$ *and each open set* V *in* (Y,σ) *containing* $f(x)$ *there exists a* Λ_I^s*-open set* U *in* (X,τ,I) *containing* x *such that* $f(U)\subset V$*.*

Proof. (1)⇒(2) Let B be any closed set in (Y,σ), then $V=Y\backslash B$ is an open set in (Y,σ) and since f is Λ_I^s-continuous, $f^{-1}(V)$ is a Λ_I^s-open subset in (X,τ,I), but $f^{-1}(V)=f^{-1}(Y\backslash B)=f^{-1}(Y)\backslash f^{-1}(B)=X\setminus f^{-1}(B)$ and hence, $f^{-1}(B)$ is a Λ_I^s-closed set in (X,τ,I).
(2) ⇒ (1) Let V be any open set in (Y,σ), then $B=Y\backslash V$ is a closed set in (Y,σ). By hypothesis, we have $f^{-1}(B)$ is a Λ_I^s-closed set in (X,τ,I), but $f^{-1}(B)=f^{-1}(Y\setminus V)=f^{-1}(Y)\setminus f^{-1}(V)=X\setminus f^{-1}(V)$ and so, $f^{-1}(V)$ is a Λ_I^s-open set in (X,τ,I). This shows that f is Λ_I^s-continuous.
(1) ⇒ (3) Let $x\in X$ and V any open set in (Y,σ) such that $f(x)\in V$, then $x\in f^{-1}(V)$ and since f is a Λ_I^s-continuous function, $f^{-1}(V)$ is a Λ_I^s-open set in (X,τ,I). If $U=f^{-1}(V)$, then U is a Λ_I^s-open

set in (X, τ, I) containing x such that $f(U) = f(f^{-1}(V)) \subset V$.
(3) $\Rightarrow$ (1) Let V be any open set in (Y, σ) and $x \in f^{-1}(V)$, then $f(x) \in V$ and by (3) there exists a Λ_I^s-open set U_x in (X, τ, I) such that $x \in U_x$ and $f(U_x) \subset V$. Thus, $x \in U_x \subset f^{-1}(f(U_x)) \subset f^{-1}(V)$ and hence $f^{-1}(V) = \bigcup\{U_x : x \in f^{-1}(V)\}$. By Lemma 2.3, we have $f^{-1}(V)$ is a Λ_I^s-open set in (X, τ, I) and so f is Λ_I^s-continuous. □

Theorem 3.5. *For a function* $f : (X, \tau, I) \to (Y, \sigma, J)$*, the following statements are equivalent:*

(1) f *is quasi-*Λ_I^s*-continuous.*

(2) $f^{-1}(B)$ *is a* Λ_I^s*-closed set in* (X, τ, I) *for each* $\sigma^\star$*-closed set* B *in* (Y, σ, J).

(3) *For each* $x \in X$ *and each* $\sigma^\star$*-open set* V *in* (Y, σ, J) *containing* $f(x)$ *there exists a* Λ_I^s*-open set* U *in* (X, τ, I) *containing* x *such that* $f(U) \subset V$.

Proof. The proof is similar to Theorem 3.4. □

Theorem 3.6. *For a function* $f : (X, \tau, I) \to (Y, \sigma, J)$*, the following statements are equivalent:*

(1) f *is* Λ_I^s*-irresolute.*

(2) $f^{-1}(B)$ *is a* Λ_I^s*-closed set in* (X, τ, I) *for each* Λ_J^s*-closet set* B *in* (Y, σ, J).

(3) *For each* $x \in X$ *and each* Λ_J^s*-open set* V *in* (Y, σ, J) *containing* $f(x)$ *there exists a* Λ_I^s*-open set* U *in* (X, τ, I) *containing* x *such that* $f(U) \subset V$.

Proof. The proof is similar to Theorem 3.4. □

4 Λ_I^s-compactness and Λ_I^s-connectedness

In this section, new notions of compactness and connectedness are introduced in terms of Λ_I^s-open sets and semi-I-open sets, in order to study their behavior under the direct images of the new forms of continuity defined in the previous section.

Definition 4.1. *An ideal topological space* (X, τ, I) *is said to be:*

(1) Λ_I^s*-compact if every cover of* X *by* Λ_I^s*-open sets has a finite subcover.*

(2) $\tau^\star$*-compact if every cover of* X *by* $\tau^\star$*-open sets has a finite subcover.*

(3) *Semi-*I*-compact if every cover of* X *by semi-*I*-open sets has a finite subcover.*

Theorem 4.1. *Let* (X, τ, I) *be an ideal topological space, the following properties hold:*

(1) (X, τ, I) *is* Λ_I^s*-compact if and only if for every collection* $\{A_\alpha : \alpha \in \Delta\}$ *of* Λ_I^s*-closed sets in* (X, τ, I) *satisfying* $\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset$*, there is a finite subcollection* $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$ *with* $\bigcap\{A_{\alpha_k} : k = 1, \ldots, n\} = \emptyset$.

(2) (X, τ, I) *is* $\tau^\star$*-compact if and only if for every collection* $\{A_\alpha : \alpha \in \Delta\}$ *of* $\tau^\star$*-closed sets in* (X, τ, I) *satisfying* $\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset$*, there is a finite subcollection* $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$ *with* $\bigcap\{A_{\alpha_k} : k = 1, \ldots, n\} = \emptyset$.

(3) (X, τ, I) *is semi-*I*-compact if and only if for every collection* $\{A_\alpha : \alpha \in \Delta\}$ *of semi-*I*-closed sets in* (X, τ, I) *satisfying* $\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset$*, there is a finite subcollection* $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$ *with* $\bigcap\{A_{\alpha_k} : k = 1, \ldots, n\} = \emptyset$.

Proof. (1) Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of Λ_I^s-closed sets such that $\bigcap\{A_\alpha : \alpha \in \Delta\} = \emptyset$, then $\{X - A_\alpha : \alpha \in \Delta\}$ is a collection of Λ_I^s-open sets such that

$$X = X - \emptyset = X - \bigcap\{A_\alpha : \alpha \in \Delta\} = \bigcup\{X - A_\alpha : \alpha \in \Delta\},$$

that is, $\{X - A_\alpha : \alpha \in \Delta\}$ is a cover of X by Λ_I^s-open sets. Since (X, τ, I) is Λ_I^s-compact, there exists a finite subcollection $X - A_{\alpha_1}, X - A_{\alpha_2}, \ldots, X - A_{\alpha_n}$ such that

$$X = \bigcup\{X - A_{\alpha_k} : k = 1, \ldots, n\} = X - \bigcap\{A_{\alpha_k} : k = 1, \ldots, n\}.$$

This shows that $\bigcap\{A_{\alpha_k} : k = 1, \ldots, n\} = \emptyset$. Conversely, suppose that $\{U_\alpha : \alpha \in \Delta\}$ is a cover of X by Λ_I^s-open sets, then $\{X - U_\alpha : \alpha \in \Delta\}$ is a collection of Λ_I^s-closed sets such that $\bigcap\{X - U_\alpha : \alpha \in \Delta\} = X - \bigcup\{U_\alpha : \alpha \in \Delta\} = X - X = \emptyset$. By hypothesis, there exists a finite subcollection $X - U_{\alpha_1}, X - U_{\alpha_2}, \ldots, X - U_{\alpha_n}$ such that $\bigcap\{X - U_{\alpha_k} : k = 1, \ldots, n\} = \emptyset$. Follows $X = X - \emptyset = X - \bigcap\{X - U_{\alpha_k} : k = 1, \ldots, n\} = X - (X - \bigcup\{U_{\alpha_k} : k = 1, \ldots, n\}) = \bigcup\{U_{\alpha_k} : k = 1, \ldots, n\}$. This shows that (X, τ, I) is Λ_I^s-compact.
The proofs of (2) and (3) are similar to the case (1). □

Theorem 4.2. *Let* (X, τ, I) *be an ideal topological space, the following properties hold:*

(1) *If* $(X, \tau^{\Lambda_I^s})$ *is compact, then* (X, τ, I) *is semi-*I*-compact.*

(2) *If* (X, τ, I) *is* Λ_I^s*-compact, then* (X, τ, I) *is* $\tau^\star$*-compact.*

(3) *If* (X, τ, I) *is* Λ_I^s*-compact, then* (X, τ, I) *is compact.*

Proof. (1) Let $\{U_\alpha : \alpha \in \Delta\}$ any cover of X by semi-I-open sets, since every $\alpha \in \Delta$, U_α is a Λ_I^s-set and hence, $U_\alpha \in \tau^{\Lambda_I^s}$ for each $\alpha \in \Delta$. Since $(X, \tau^{\Lambda_I^s})$ is compact, there exists a finite subset Δ_0 of Δ such that $X = \bigcup\{U_\alpha : \alpha \in \Delta_0\}$. This shows that (X, τ) is semi-I-compact.
(2) Let $\{F_\alpha : \alpha \in \Delta\}$ be a collection of $\tau^\star$-closed sets of X such that $\bigcap\{F_\alpha : \alpha \in \Delta\} = \emptyset$. Since every $\tau^\star$-closed set is Λ_I^s-closed, then $\{F_\alpha : \alpha \in \Delta\}$ is a collection of Λ_I^s-closed sets and (X, τ, I) is Λ_I^s-compact. By Theorem 4.1(1), there exists a finite subset Δ_0 of Δ such that $\bigcap\{F_\alpha : \alpha \in \Delta_0\} = \emptyset$

and by Theorem 4.1(2), we conclude that (X, τ, I) is $\tau^\star$-compact.
(3) Follows from (2) and the fact that every $\tau^\star$-compact space is compact. □

Theorem 4.3. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a surjective function, the following properties hold:*

(1) *If f is Λ_I^s-irresolute and (X, τ, I) is Λ_I^s-compact, then (Y, σ, J) is Λ_J^s-compact.*

(2) *If f is semi-I-irresolute and (X, τ, I) is semi-I-compact, then (Y, σ, J) is semi-J-compact.*

(3) *If f is quasi-Λ_I^s-continuous and (X, τ, I) is Λ_I^s-compact, then (Y, σ, J) is $\sigma^\star$-compact.*

(4) *If f is Λ_I^s-continuous and (X, τ, I) is Λ_I^s-compact, then (Y, σ, J) is compact.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of Y by Λ_J^s-open sets. Since f is Λ_I^s-irresolute, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a cover of X by Λ_I^s-open sets and by the Λ_I^s-compactnes of (X, τ, I), there exists a finite subset Δ_0 of Δ such that $X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Since f is surjective, then $Y = f(X) = f(\bigcup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}) = \bigcup\{f(f^{-1}(V_\alpha)) : \alpha \in \Delta_0\} = \{V_\alpha : \alpha \in \Delta_0\}$ and this shows that (Y, θ, J) is Λ_J^s-compact.
The proofs of (2), (3) and (4) are similar to case (1). □

Definition 4.2. *An ideal topological space (X, τ, I) is said to be:*

(1) *Λ_I^s-connected if X cannot be written as a disjoint union of two nonempty Λ_I^s-open sets.*

(2) *$\tau^\star$-connected if X cannot be written as a disjoint union of two nonempty $\tau^\star$-open sets.*

(3) *semi-I-connected if X cannot be written as a disjoint union of two nonempty semi-I-open sets.*

Theorem 4.4. *Let (X, τ, I) be an ideal topological space, the following properties hold:*

(1) *If $(X, \tau^{\Lambda_I^s})$ is connected, then $(X, \tau; I)$ is semi-I-connected.*

(2) *If (X, τ, I) is Λ_I^s-connected, then (X, τ, I) is $\tau^\star$-connected.*

(3) *If (X, τ, I) is Λ_I^s-connected, then (X, τ, I) is connected.*

Proof. (1) Suppose that (X, τ, I) is not semi-I-connected, then there exist non-empty semi-I-open sets A and B such that $A \cap B = \emptyset$ and $A \cup B = X$. By Proposition 3.1(3) of [7], A and B are Λ_I^s-sets and hence, $(X, \tau^{\Lambda_I^s})$ is not connected.
(2) Suppose that (X, τ, I) is not $\tau^\star$-connected, then there exist non-empty $\tau^\star$-open sets A and B such that $A \cap B = \emptyset$ and $A \cup B = X$. By Lemma 2.2, we have A and B are Λ_I^s-open sets and so, (X, τ, I) is not Λ_I^s-connected.
(3) Follows from (2) and the fact that every $\tau^\star$-connected space is connected. □

Theorem 4.5. *For an ideal topological space (X, τ, I), the following statements are equivalent:*

(1) *(X, τ, I) is Λ_I^s-connected.*

(2) *$\emptyset$ and X are the only subsets of X which are both Λ_I^s-open and Λ_I^s-closed.*

(3) *Every Λ_I^s-continuous function of X into a discrete space Y with at least two points, is a constant function.*

Proof. (1)⇒(2) Let V be a subset of X which is both Λ_I^s-open and Λ_I^s-closed, then $X - V$ is both Λ_I^s-open and Λ_I^s-closed, so $X = V \cup (X - V)$. Since (X, τ, I) is Λ_I^s-connected, then one of those sets is $\emptyset$. Therefore, $V = \emptyset$ or $V = X$.
(2)⇒(1) Suppose that (X, τ, I) is not Λ_I^s-connected and let $X = U \cup V$, where U and V are disjoint nonempty Λ_I^s-open sets in (X, τ, I), then $U = X - V$ is both Λ_I^s-open and Λ_I^s-closed. By hypothesis, $U = \emptyset$ or $U = X$, which is a contradiction. Therefore, (X, τ, I) is Λ_I^s-connected.
(2)⇒(3) Let $f : (X, \tau, I) \to Y$ be a Λ_I^s-continuous function, where Y is a topological space with the discrete topology and contains at least two points, then X can be cover by a collection of sets which are both Λ_I^s-open and Λ_I^s-closed of the form $\{f^{-1}(y) : y \in Y\}$, from these, we conclude that there exists a $y_0 \in Y$ such that $f^{-1}(\{y_0\}) = X$ and so, f is a constant function.
(3)⇒(2) Let W be a subset of (X, τ, I) which is both Λ_I^s-open and Λ_I^s-closed. Suppose that $W \neq \emptyset$ and let $f : (X, \tau, I) \to Y$ be the function defined by $f(W) = \{y_1\}$ and $f(X - W) = \{y_2\}$ for $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Then f is Λ_I^s-continuous, since the inverse image de each open set in Y is Λ_I^s-open in X. Hence, by (3), f must be a constant function. It follows that $X = W$. □

Theorem 4.6. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a surjective function, the following properties hold:*

(1) *If f is a Λ_I^s-irresolute and (X, τ, I) is Λ_I^s-connected, then (Y, σ, J) is Λ_J^s-connected.*

(2) *If f is a semi-I-irresolute function and (X, τ, I) is semi-I-connected, then (Y, σ, J) is semi-J-connected.*

(3) *If f is a quasi-Λ_I^s-continuous function and (X, τ, I) is Λ_I^s-connected, then (Y, σ, J) is $\sigma^\star$-connected.*

(4) *If f is a Λ_I^s-continuous function and (X, τ, I) is Λ_I^s-connected, then (Y, σ) is connected.*

Proof. (1) Suppose that (Y, σ, J) is not Λ_J^s-connected, then there exist nonempty Λ_J^s-open sets H, G in (Y, σ, J) such that $G \cap H = \emptyset$ and $G \cup H = Y$. Hence, we have $f^{-1}(G) \cap f^{-1}(H) = \emptyset$, $f^{-1}(G) \cup f^{-1}(H) = X$ and moreover, $f^{-1}(G)$ and $f^{-1}(H)$ are nonempty Λ_I^s-open sets in (X, τ, I). This shows that (X, τ, I) is not Λ_I^s-connected.
The proofs of (2), (3) and (4) are similar to case (1). □

Open problems. The Theorems 4.2 and 4.4 have been proved using the fact that every semi-I-open set is Λ_I^s-open and that every $\tau^\star$-open set is Λ_I^s-open. But until today, we dont have any contra example in order to shows that the converse of such Theorems are not true. In that sense we write the following questions.

(1) Does there exists an ideal topological space (X, τ, I) which is semi-I-compact (resp. semi-I-connected) but $(X, \tau^{\Lambda_I^s})$ is not a compact (resp. connected) space.?

(2) Does there exists an ideal topological space (X, τ, I) which is $\tau^{\star}$-compact (resp. $\tau^{\star}$-connected) but (X, τ) is not Λ_I^s-compact space (resp. Λ_I^s-connected space.)?

Received: January 2014. Accepted: January 2015.

References

[1] M. E. Abd El-Monsef, E. F. Lashien, A. A. Nasef: *On* I-*open sets and* I-*continuous functions*, Kyungpook Math. J. 32 (1992), 21-30.

[2] E. Hatir, T. Noiri: *On descompositions of continuity via idealization*, Acta. Math. Hungar. 96 (4) (2002), 341-349.

[3] E. Hatir, T. Noiri: *On semi*-I-*open sets and semi*-I-*continuous functions*, Acta Math. Hungar. 107 (4) (2005), 345-353.

[4] E. Hatir, T. Noiri: *On Hausdorff spaces via ideals and semi*-I-*irresolute Functions*, Eur. J. Pure. Appl. Math. 2 (2) (2009), 172-181.

[5] D. Jankovic, T. R. Hamlett: *New topologies from old via ideals*, Amer. Math. Monthly 97 (1990), 295-310.

[6] K. Kuratowski: *Topology*, Vol. I, Academic Press, New York-London, 1966.

[7] J. Sanabria, E. Rosas, C. Carpintero: *On* Λ_I^s*-sets and the related notions in ideal topological spaces*, Math. Slovaca 63 (6) (2013), 1403-1411.