

Spline left fractional monotone approximation involving left fractional differential operators

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ABSTRACT

Let $f \in C^s([-1, 1])$, $s \in \mathbb{N}$ and L^* be a linear left fractional differential operator such that $L^*(f) \geq 0$ on $[0, 1]$. Then there exists a sequence Q_n , $n \in \mathbb{N}$ of polynomial splines with equally spaced knots of given fixed order such that $L^*(Q_n) \geq 0$ on $[0, 1]$. Furthermore f is approximated with rates fractionally and simultaneously by Q_n in the uniform norm. This constrained fractional approximation on $[-1, 1]$ is given via inequalities involving a higher modulus of smoothness of $f^{(s)}$.

RESUMEN

Sea $f \in C^s([-1, 1])$, $s \in \mathbb{N}$ y L^* un operador diferencial fraccionario lineal izquierdo tal que $L^*(f) \geq 0$ en $[0, 1]$. Entonces, existe una sucesión Q_n , $n \in \mathbb{N}$ de splines polinomiales con nodos equiespaciados de un orden fijo dado tal que $L^*(Q_n) \geq 0$ en $[0, 1]$. Además, f se aproxima con velocidades fraccionales y simultáneamente por Q_n en la norma uniforme. Esta aproximación fraccional restringida a $[-1, 1]$ se encuentra por medio de desigualdades que involucran un módulo alto de suavidad de $f^{(s)}$.

Keywords and Phrases: Monotone Approximation, Caputo fractional derivative, fractional linear differential operator, modulus of smoothness, splines.

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1 Introduction

Let $[a, b] \subset \mathbb{R}$ and for $n \geq 1$ consider the partition Δ_n with points $x_{in} = a + i \left(\frac{b-a}{n}\right)$, $i = 0, 1, \dots, n$. Hence $\bar{\Delta}_n \equiv \max_{1 \leq i \leq n} (x_{in} - x_{i-1,n}) = \frac{b-a}{n}$.

Let $S_m(\Delta_n)$ be the space of polynomial splines of order $m > 0$ with simple knots at the points x_{in} , $i = 1, \dots, n-1$. Then there exists a linear operator $Q_n : Q_n \equiv Q_n(f)$, mapping $B[a, b]$: the space of bounded real valued functions f on $[a, b]$, into $S_m(\Delta_n)$ (see [4], p. 224, Theorem 6.18).

From the same reference [4], p. 227, Corollary 6.21, we get

Corollary 1. *Let $1 \leq \sigma \leq m$, $n \geq 1$. Then for all $f \in C^{\sigma-1}[a, b]$; $r = 0, \dots, \sigma-1$,*

$$\|f^{(r)} - Q_n^{(r)}\|_{\infty} \leq C_1 \left(\frac{b-a}{n}\right)^{\sigma-r-1} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{b-a}{n}\right), \quad (1)$$

where C_1 depends only on m , $C_1 = C_1(m)$.

By denoting $C_2 = C_1 \max_{0 \leq r \leq \sigma-1} (b-a)^{\sigma-r-1}$ we obtain

Lemma 1. *([1]) Let $1 \leq \sigma \leq m$, $n \geq 1$. Then for all $f \in C^{\sigma-1}[a, b]$; $r = 0, \dots, \sigma-1$,*

$$\|f^{(r)} - Q_n^{(r)}\|_{\infty} \leq \frac{C_2}{n^{\sigma-r-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{b-a}{n}\right), \quad (2)$$

where C_2 depends only on m , σ and $b-a$. Here $\omega_{m-\sigma+1}$ is the usual modulus of smoothness of order $m-\sigma+1$.

We are motivated by

Theorem 1. *([1]) Let h, k, σ, m be integers, $0 \leq h \leq k \leq \sigma-1$, $\sigma \leq m$ and let $f \in C^{\sigma-1}[a, b]$. Let $\alpha_j(x) \in B[a, b]$, $j = h, h+1, \dots, k$ and suppose that $\alpha_h(x) \geq \alpha > 0$ or $\alpha_h(x) \leq \beta < 0$ for all $x \in [a, b]$. Take the linear differential operator*

$$L = \sum_{j=h}^k \alpha_j(x) \left[\frac{d^j}{dx^j} \right] \quad (3)$$

and assume, throughout $[a, b]$,

$$L(f) \geq 0. \quad (4)$$

Then, for every integer $n \geq 1$, there is a polynomial spline function $Q_n(x)$ of order m with simple knots at $\{a + i \left(\frac{b-a}{n}\right), i = 1, \dots, n-1\}$ such that $L(Q_n) \geq 0$ throughout $[a, b]$ and

$$\|f^{(r)} - Q_n^{(r)}\|_{\infty} \leq \frac{C}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{b-a}{n}\right), \quad 0 \leq r \leq h. \quad (5)$$

Moreover, we find

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_{\infty} \leq \frac{C}{n^{\sigma-r-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \quad h+1 \leq r \leq \sigma-1, \quad (6)$$

where C is a constant independent of f and n . It depends only on m, σ, L, a, b .

Next we specialize on the case of $a = -1, b = 1$. That is working on $[-1, 1]$.

By Lemma 1 we get

Lemma 2. Let $1 \leq \sigma \leq m, n \geq 1$. Then for all $f \in C^{\sigma-1}([-1, 1]); j = 0, 1, \dots, \sigma-1$,

$$\left\| f^{(j)} - Q_n^{(j)} \right\|_{\infty} \leq \frac{C_2}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{2}{n} \right), \quad (7)$$

where $C_2 := C_2(m, \sigma) := C_1(m) 2^{\sigma-1}$.

Since

$$\omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{2}{n} \right) \leq 2^{m-\sigma+1} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \quad (8)$$

(see [2], p. 45), we get

Lemma 3. Let $1 \leq \sigma \leq m, n \geq 1$. Then for all $f \in C^{\sigma-1}([-1, 1]); j = 0, 1, \dots, \sigma-1$,

$$\left\| f^{(j)} - Q_n^{(j)} \right\|_{\infty} \leq \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right), \quad (9)$$

where $C_2^* := C_2^*(m, \sigma) := C_1(m) 2^m$.

We use a lot in this article Lemma 3.

In this article we extend Theorem 1 over $[-1, 1]$ to the fractional level. Indeed here L is replaced by L^* , a linear left Caputo fractional differential operator. Now the monotonicity property is only true on the critical interval $[0, 1]$. Simultaneous fractional convergence remains true on all of $[-1, 1]$.

We make

Definition 1. ([3], p. 50) Let $\alpha > 0$ and $[\alpha] = m$, ($[\cdot]$ ceiling of the number). Consider $f \in C^m([-1, 1])$. We define the left Caputo fractional derivative of f of order α as follows:

$$(D_{*-1}^{\alpha} f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_{-1}^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad (10)$$

for any $x \in [-1, 1]$, where Γ is the gamma function.

We set

$$\begin{aligned} D_{*-1}^0 f(x) &= f(x), \\ D_{*-1}^m f(x) &= f^{(m)}(x), \quad \forall x \in [-1, 1]. \end{aligned} \quad (11)$$

2 Main Result

Theorem 2. Let h, k, σ, m be integers, $1 \leq \sigma \leq m$, $n \in \mathbb{N}$, with $0 \leq h \leq k \leq \sigma - 2$ and let $f \in C^{\sigma-1}([-1, 1])$, with $f^{(\sigma-1)}$ having modulus of smoothness $\omega_{m-\sigma+1}(f^{(\sigma-1)}, \delta)$ there, $\delta > 0$. Let $\alpha_j(x)$, $j = h, h+1, \dots, k$ be real functions, defined and bounded on $[-1, 1]$ and suppose $\alpha_h(x)$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ on $[0, 1]$. Let the real numbers $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \dots < \alpha_{\sigma-2} \leq \sigma - 2$. Here $D_{*-1}^{\alpha_j} f$ stands for the left Caputo fractional derivative of f of order α_j anchored at -1 . Consider the linear left fractional differential operator

$$L^* := \sum_{j=h}^k \alpha_j(x) [D_{*-1}^{\alpha_j}] \quad (12)$$

and suppose, throughout $[0, 1]$, $L^*(f) \geq 0$.

Then, for every integer $n \geq 1$, there exists a polynomial spline function $Q_n(x)$ of order $m > 0$ with simple knots at $\{-1 + i\frac{2}{n}, i = 1, \dots, n-1\}$ such that $L^*(Q_n) \geq 0$ throughout $[0, 1]$, and

$$\begin{aligned} & \sup_{-1 \leq x \leq 1} |(D_{*-1}^{\alpha_j} f)(x) - (D_{*-1}^{\alpha_j} Q_n)(x)| \leq \\ & \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right), \end{aligned} \quad (13)$$

$j = h+1, \dots, \sigma-2$.

Set

$$l_j := \sup_{x \in [-1, 1]} |\alpha_h^{-1}(x) \alpha_j(x)|, \quad h \leq j \leq k. \quad (14)$$

When $j = 1, \dots, h$ we derive

$$\begin{aligned} & \max_{-1 \leq x \leq 1} |(D_{*-1}^{\alpha_j} f)(x) - (D_{*-1}^{\alpha_j} Q_n)(x)| \leq \frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \cdot \\ & \left[\left(\sum_{\tau=h}^k l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} \right) \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right]. \end{aligned} \quad (15)$$

Finally it holds

$$\begin{aligned} & \sup_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq \\ & \frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \left[\frac{1}{h!} \sum_{\tau=h}^k l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} + 1 \right]. \end{aligned} \quad (16)$$

Proof. Set $\alpha_0 = 0$, thus $[\alpha_0] = 0$. We have $[\alpha_j] = j$, $j = 1, \dots, \sigma-2$.

Let Q_n as in Lemma 5.

We notice that $(x \in [-1, 1])$

$$\begin{aligned} & |(D_{*-1}^{\alpha_j} f)(x) - (D_{*-1}^{\alpha_j} Q_n)(x)| = \\ & \frac{1}{\Gamma(j - \alpha_j)} \left| \int_{-1}^x (x-t)^{j-\alpha_j-1} f^{(j)}(t) dt - \int_{-1}^x (x-t)^{j-\alpha_j-1} Q_n^{(j)}(t) dt \right| = \end{aligned} \tag{17}$$

$$\begin{aligned} & \frac{1}{\Gamma(j - \alpha_j)} \left| \int_{-1}^x (x-t)^{j-\alpha_j-1} (f^{(j)}(t) - Q_n^{(j)}(t)) dt \right| \leq \\ & \frac{1}{\Gamma(j - \alpha_j)} \int_{-1}^x (x-t)^{j-\alpha_j-1} |f^{(j)}(t) - Q_n^{(j)}(t)| dt \stackrel{(9)}{\leq} \end{aligned} \tag{18}$$

$$\begin{aligned} & \frac{1}{\Gamma(j - \alpha_j)} \left(\int_{-1}^x (x-t)^{j-\alpha_j-1} dt \right) \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) = \\ & \frac{1}{\Gamma(j - \alpha_j)} \frac{(x+1)^{j-\alpha_j}}{(j - \alpha_j)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) = \end{aligned} \tag{19}$$

$$\begin{aligned} & \frac{(x+1)^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \leq \\ & \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right). \end{aligned} \tag{20}$$

Hence

$$\|D_{*-1}^{\alpha_j} f - D_{*-1}^{\alpha_j} Q_n\|_{\infty, [-1, 1]} \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right), \tag{21}$$

$j = 0, 1, \dots, \sigma - 2$.

We set

$$\rho_n := C_2^* \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \left(\sum_{j=h}^k l_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1) n^{\sigma-j-1}} \right). \tag{22}$$

I. Suppose, throughout $[0, 1]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x)$, $x \in [-1, 1]$, the polynomial spline of order $m > 0$ with simple knots at the points x_{in} , $i = 1, \dots, n - 1$, on $[-1, 1]$ ($x_{in} = -1 + i \frac{2}{n}$, $i = 0, 1, \dots, n$, here $\bar{\Delta}_n = \frac{2}{n}$), so that

$$\begin{aligned} & \max_{-1 \leq x \leq 1} \left| D_{*-1}^{\alpha_j} \left(f(x) + \rho_n \frac{x^h}{h!} \right) - (D_{*-1}^{\alpha_j} Q_n)(x) \right| \leq \\ & \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right), \end{aligned} \tag{23}$$

$j = 0, 1, \dots, \sigma - 2$.

When $j = h + 1, \dots, \sigma - 2$, then

$$\max_{-1 \leq x \leq 1} |(D_{*-1}^{\alpha_j} f)(x) - (D_{*-1}^{\alpha_j} Q_n)(x)| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right), \quad (24)$$

proving (13).

For $j = 1, \dots, h$ we find that

$$D_{*-1}^{\alpha_j} \left(\frac{x^h}{h!} \right) = \sum_{\lambda=0}^{h-j} \frac{(-1)^\lambda (x+1)^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)}. \quad (25)$$

Therefore we get from (23)

$$\begin{aligned} \max_{-1 \leq x \leq 1} \left| (D_{*-1}^{\alpha_j} f)(x) + \rho_n \left(\sum_{\lambda=0}^{h-j} \frac{(-1)^\lambda (x+1)^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) - (D_{*-1}^{\alpha_j} Q_n)(x) \right| \leq \quad (26) \\ \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right), \end{aligned}$$

$j = 1, \dots, h$.

Therefore we get for $j = 1, \dots, h$, that

$$\max_{-1 \leq x \leq 1} \left| (D_{*-1}^{\alpha_j} f)(x) - (D_{*-1}^{\alpha_j} Q_n)(x) \right| \leq \quad (27)$$

$$\begin{aligned} \rho_n \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) = \\ C_2^* \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \left(\sum_{\bar{j}=h}^k l_{\bar{j}} \frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma(\bar{j}-\alpha_{\bar{j}}+1) n^{\sigma-\bar{j}-1}} \right). \\ \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) = \\ C_2^* \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \left[\left(\sum_{\bar{j}=h}^k l_{\bar{j}} \frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma(\bar{j}-\alpha_{\bar{j}}+1) n^{\sigma-\bar{j}-1}} \right) \right]. \quad (28) \end{aligned}$$

$$\begin{aligned} \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{1}{n^{\sigma-j-1}} \leq \\ C_2^* \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \frac{1}{n^{\sigma-k-1}} \left[\left(\sum_{\bar{j}=h}^k l_{\bar{j}} \frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma(\bar{j}-\alpha_{\bar{j}}+1)} \right) \right]. \quad (29) \end{aligned}$$

$$\left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \left[\right].$$

Hence for $j = 1, \dots, h$ we derived (15):

$$\begin{aligned} \max_{-1 \leq x \leq 1} |(D_{*-1}^{\alpha_j} f)(x) - (D_{*-1}^{\alpha_j} Q_n)(x)| &\leq \frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \\ &\left[\left(\sum_{\tau=h}^k l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} \right) \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right]. \end{aligned} \quad (30)$$

When $j = 0$ from (23) we obtain

$$\max_{-1 \leq x \leq 1} \left| f(x) + \rho_n \frac{x^h}{h!} - Q_n(x) \right| \leq \frac{C_2^*}{n^{\sigma-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right). \quad (31)$$

And

$$\begin{aligned} \max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| &\leq \frac{\rho_n}{h!} + \frac{C_2^*}{n^{\sigma-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) = \\ &\frac{C_2^*}{h!} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \left(\sum_{\tau=h}^k l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1) n^{\sigma-\tau-1}} \right) \\ &+ \frac{C_2^*}{n^{\sigma-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) = \\ C_2^* \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) &\left[\frac{1}{h!} \sum_{\tau=h}^k l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1) n^{\sigma-\tau-1}} + \frac{1}{n^{\sigma-1}} \right] \leq \\ &\frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \left[\frac{1}{h!} \sum_{\tau=h}^k l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} + 1 \right]. \end{aligned} \quad (32)$$

Proving

$$\begin{aligned} \max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| &\leq \\ &\frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \left[\frac{1}{h!} \sum_{\tau=h}^k l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} + 1 \right], \end{aligned} \quad (34)$$

So that (16) is established.

Also if $0 \leq x \leq 1$, then

$$\begin{aligned} \alpha_h^{-1}(x) L^*(Q_n(x)) &= \alpha_h^{-1}(x) L^*(f(x)) + \rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \\ &\sum_{j=h}^k \alpha_h^{-1}(x) \alpha_j(x) \left[D_{*-1}^{\alpha_j} Q_n(x) - D_{*-1}^{\alpha_j} f(x) - \frac{\rho_n}{h!} D_{*-1}^{\alpha_j} x^h \right] \stackrel{(23)}{\geq} \\ \rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} &- \left(\sum_{j=h}^k l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \right) = \end{aligned} \quad (35)$$

$$\begin{aligned} \rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - \rho_n &= \rho_n \left[\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - 1 \right] = \\ \rho_n \left[\frac{(x+1)^{h-\alpha_h} - \Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)} \right] &\geq \rho_n \left[\frac{1 - \Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)} \right] \geq 0. \end{aligned} \quad (36)$$

Explanation: We know that $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h - \alpha_h < 1$ and $1 \leq h - \alpha_h + 1 < 2$. Thus $\Gamma(h - \alpha_h + 1) \leq 1$ and $1 - \Gamma(h - \alpha_h + 1) \geq 0$. Hence $L^*(Q_n(x)) \geq 0$, $x \in [0, 1]$.

II. Suppose on $[0, 1]$ that $\alpha_h(x) \leq \beta < 0$. Let $Q_n(x)$, $x \in [-1, 1]$, be the polynomial spline of order $m > 0$, (as before), so that

$$\begin{aligned} \max_{-1 \leq x \leq 1} \left| D_{*-1}^{\alpha_j} \left(f(x) - \rho_n \frac{x^h}{h!} \right) - (D_{*-1}^{\alpha_j} Q_n)(x) \right| &\leq \\ \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right), & \end{aligned} \quad (37)$$

$j = 0, 1, \dots, \sigma - 2$.

Similarly as before we obtain again inequalities of convergence (13), (15) and (16).

Also if $0 \leq x \leq 1$, then

$$\alpha_h^{-1}(x) L^*(Q_n(x)) = \alpha_h^{-1}(x) L^*(f(x)) - \rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \quad (38)$$

$$\begin{aligned} \sum_{j=h}^k \alpha_h^{-1}(x) \alpha_j(x) \left[D_{*-1}^{\alpha_j} Q_n(x) - D_{*-1}^{\alpha_j} f(x) + \frac{\rho_n}{h!} (D_{*-1}^{\alpha_j} x^h) \right] &\stackrel{(37)}{\leq} \\ - \rho_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \sum_{j=h}^k l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) &= \end{aligned} \quad (39)$$

$$\rho_n \left(1 - \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) = \rho_n \left(\frac{\Gamma(h-\alpha_h+1) - (x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \leq \quad (40)$$

$$\rho_n \left(\frac{1 - (x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \leq 0,$$

and hence again $L^*(Q_n(x)) \geq 0$, $x \in [0, 1]$. □

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