CHEMICAL ENGINEERING TRANSACTIONS Volume 24, 2011
Editor Sauro Pierucci

 ISBN 978-88-95608-15-0 ISSN 1974-9791
DOI: 10.3303/CET1124056

On Convex Relaxations in Nonconvex Optimization

Tapio Westerlund*[a], Andreas Lundell[a] , Joakim Westerlund[b]

[a]Process Design and Systems Engineering, Department of Chemical Engineering
Åbo Akademi University, Biskopsgatan 8, FIN-20500 Turku, Finland.
tapio.westerlund@abo.fi

[b]Paroc Oy, FIN-21600 Parainen, Finland

Convex relaxations play an important role in many areas, especially in optimization and particularly in global optimization. In this paper we will consider some special, but fundamental, issues related to convex relaxation techniques in constrained nonconvex optimization. We will especially consider optimization problems including nonconvex inequality constraints and their relaxations. Finally, we will illustrate the results by a problem connected to *N*-dimensional allocation.

1. Introduction and Motivation

In the area of optimization, different types of relaxation techniques are used. In this paper we will focus on convex relaxations and especially on some properties related to these in connection to global optimization. Several global optimization methods are based on the principle of relaxing a nonconvex problem into convex subproblems and solving these iteratively. By using a branch and bound framework, a subdivision of the initial domain can be automated and the global optimal solution finally obtained. However, independently of the type of procedure used, it is important that the relaxations used when solving the subproblems are made as tight as possible.

2. Problem Formulation

Let us consider the following constrained nonconvex optimization problem

$$\begin{aligned} &\text{minimize} \quad f(\mathbf{x}) \\ &\text{subject to} \quad g_m(\mathbf{x}) \le 0, \quad m=1,2,\dots,M \end{aligned} \tag{1}$$

where f is a convex objective function, g are functions defining inequality constraints, M the number of inequality constraints and $\mathbf{x}$ a vector of variables in X, a convex subset of R^n. Convex constraints can be included in g, but in this case only the nonconvex constraints need to be relaxed. An attractive convex relaxation of problem (1) is obtained by replacing the nonconvex functions with their tightest convex relaxations, *i.e.*, the convex envelopes conv g. This does not, however, necessarily result in the tightest possible convex relaxation of the feasible region of the optimization problem.

Please cite this article as: Westerlund T., Lundell A. and Westerlund J., 2011, On convex relaxations in nonconvex optimization, Chemical Engineering Transactions, 24, 331-336
DOI: 10.3303/CET1124056

3. Relaxations of Functions Defining Inequality Constraints

According to Tuy (1998): "A nonconvex inequality constraint $g(\boldsymbol{x}) \leq 0$, $\boldsymbol{x} \in X$, where X is a convex set in R^n, can often be handled by replacing it with a convex inequality constraint $c(\boldsymbol{x}) \leq 0$, where $c(\boldsymbol{x})$ is a convex minorant of $g(\boldsymbol{x})$ on X. The latter inequality is then called a convex relaxation of the former. Of course, the tightest relaxation is obtained when $c(\boldsymbol{x})$ = conv $g(\boldsymbol{x})$, the convex envelope, *i.e.*, the largest convex minorant, of $g(\boldsymbol{x})$."

It should, however, be observed that conv g is the tightest relaxation of the function g over the convex set X (Sherali and Alameddine, 1990), and not the tightest convex relaxation of the set $\{\boldsymbol{x} \in X \,/\, g(\boldsymbol{x}) \leq 0\}$ itself.

From an optimization point-of-view, the convex envelope of the feasible region is even more important than convex envelopes of the constraint functions, since the tightest convex relaxation of the feasible region is not generally obtained by replacing the functions in problem (1) by their convex envelopes. Instead, the tightest convex relaxation of the problem is given by the convex envelope of the set defining its feasible region, and the convex envelope of this set is the border of its convex hull.

4. Convex Relaxation of a Feasible Region and its Convex Envelope

Problem (1) is defined by a convex objective function, variables connected to the set X, and a number of nonconvex inequality constraints. The level sets of the nonconvex inequality constraints can be defined as

$$L_\alpha^g = \bigcap_m \{\boldsymbol{x} \in X \mid g_m(\boldsymbol{x}) \leq \alpha\}.$$

The feasible region of problem (1) can, thus, be defined as the level set $L_{\alpha=0}^g$, defined by the RHS = 0 of the inequality constraints. Now, observe that a potentially good convex relaxation of the feasible region of the problem can be obtained if the feasible region $L_{\alpha=0}^g$ of the nonconvex problem is replaced with $L_{\alpha=0}^{\mathrm{conv}\,g}$, *i.e.,* by replacing the nonconvex functions g_m by their convex envelopes conv g_m. However, there is no guarantee that $L_{\alpha=0}^{\mathrm{conv}\,g}$, will result in the tightest convex relaxation of $L_{\alpha=0}^g$ and we will later on illustrate, that this is not generally the case. Thus, replacing the nonconvex functions defining the inequality constraints with their convex envelopes does not necessarily result in the convex hull $\mathrm{conv}\,L_{\alpha=0}^g$, *i.e.*, the tightest convex relaxation, of the set $L_{\alpha=0}^g$.

As mentioned previously, the convex envelope of a set is the border of its convex hull. Therefore, if the border of $\mathrm{conv}\,L_{\alpha=0}^g$ can be defined by convex functions q, different from the convex envelopes conv g, then a tighter, or at least an equally tight, convex relaxation of the feasible region of the problem will be obtained.

If the convex functions q over X are defined as $q(\boldsymbol{x}) \leq 0$, $\forall \boldsymbol{x} \in X: g(\boldsymbol{x}) \leq 0$, then $L_{\alpha=0}^g \subseteq L_{\alpha=0}^q$ and the functions q giving the tightest convex relaxation of $L_{\alpha=0}^g$ are obtained when $L_{\alpha=0}^q$ = conv $L_{\alpha=0}^g$. This results in $L_{\alpha=0}^g \subseteq L_{\alpha=0}^q \subseteq L_{\alpha=0}^{\mathrm{conv}\,g}$.

Unfortunately, we do not have a general procedure to generate convex envelopes of the type q, defining the border of the convex hull of $\mathrm{conv}\,L_{\alpha=0}^g$ for general classes of problems. We will, however, show by an example that such functions can be obtained.

5. An Illustrative Example

This example is related to a problem in N variables connected to N-dimensional allocation. A general model for such problems was presented in Westerlund et al. (2007). In the paper, items in an N-dimensional space were considered: rectangles in 2D, boxes in 3D, and so on. The sizes of the items were defined with fixed side lengths. In two related papers, Bonäs et al. (2007) and Castillo et al. (2005), the items were defined by their total areas, volumes etc. and aspect ratios were used to define maximum and minimum side lengths. The size constraint for an item in the N-dimensional case could thus generally be defined by a scalar parameter S and a product of variable side lengths x_i, restricted by defined aspect ratios in each direction in the N-dimensional space.
Thus, the size constraint for an item in an N-dimensional allocation problem can together with overlapping protection constraints (Westerlund et al., 2007), be defined as

$$\prod_{i=1}^{N} x_i \geq S. \tag{2}$$

The constraint function g, connected to an item in an optimization problem of the type (1), would then be given by

$$g(\boldsymbol{x}) = S - \prod_{i=1}^{N} x_i. \tag{3}$$

In a 2D case the constraint would contain a negative bilinear term and in the 3D case a negative trilinear term, and so on. For such terms there are known convex envelopes. However, independently of the application, the constraint function is given by equation (3). In this particular case $\boldsymbol{x} \in X \subset R_{+}^{n}$, and we observe that the function g is in fact quasiconvex. Since quasiconvex functions have convex level sets, the border of the convex hull of $L_{\alpha=0}^{g}$ must be given by the border of the level set $L_{\alpha=0}^{g}$ itself. The inequality constraint corresponding to expression (3) is written as

$$S - \prod_{i=1}^{N} x_i \leq 0. \tag{4}$$

This inequality can, however, be reformulated into convex form at the border (RHS = 0), for example, by dividing away all variables x_i except the k-th one, *i.e.*

$$q_k(\boldsymbol{x}) = S \Big/ \prod_{\substack{i=1 \\ i \neq k}}^{N} x_i - x_k \leq 0, \quad k = 1,2,\ldots,N. \tag{5}$$

Since equation (4) and its convex reformulation in equation (5), result in identical solutions at RHS = 0 both expressions must also exactly represent the same border of the level set $L_{\alpha=0}^{g}$. Thus, we may conclude that the nonconvex function g in an optimization problem of type (1) can be replaced with the convex constraints $q_k \leq 0$. In this particular case, we in fact obtain the border of the convex hull of the level set $L_{\alpha=0}^{g}$ (as it is identical to the level set itself) simply by replacing $g(\boldsymbol{x})$ in the problem (1) with the convex functions $q_k(\boldsymbol{x})$: $L_{\alpha=0}^{g} = L_{\alpha=0}^{q} \subseteq L_{\alpha=0}^{\operatorname{conv} g}$.

When solving a problem of type (1) using g, or replacing g with conv g or q_k, we obtain the minimum objective function value for the different domains as

$$f^*(\boldsymbol{x}^{\mathrm{conv}\, g}) \leq f^*(\boldsymbol{x}^q) = f^*(\boldsymbol{x}^g).$$

Note that $q_k(\boldsymbol{x}) \leq \mathrm{conv}\ g(\boldsymbol{x}) \leq g(\boldsymbol{x})$ does not hold true in this case for all $\boldsymbol{x} \in X$. Consequently, the functions q_k are neither convex minorants nor the convex envelope of the function $g(\boldsymbol{x})$ in X. Replacing g with conv g in problem (1) will thus not result in the tightest convex relaxation of the problem. Instead, a tighter convex relaxation of the set $L^g_{\alpha=0}$, is obtained by replacing g in the problem with any of the functions q_k, resulting in the convex envelope of the set $L^g_{\alpha=0}$ for all $\boldsymbol{x} \in X$.

When considering the special case where N=2 and S=50, the function g is given by

$$g(\boldsymbol{x}) = 50 - x_1 x_2.$$

Convex envelopes of bilinear terms are given in McCormick (1976). Using the convex envelope of negative bilinear terms, the convex envelope of the function $g(\boldsymbol{x})$, where the bounds on the variables are, *e.g.*, $0.5 \leq x_1, x_2 \leq 10$, is given by

$$\mathrm{conv}\, g(\boldsymbol{x}) = 50 + \max\{-10x_1 - 0.5x_2 + 5, -0.5x_1 - 10x_2 + 5\}.$$

Furthermore, the functions q_k are given by

$$q_1(\boldsymbol{x}) = 50/x_2 - x_1 \text{ and } q_2(\boldsymbol{x}) = 50/x_1 - x_2.$$

In Figures 1 and 2, the feasible region and the functions are shown. In Figure 1, the upper left figure illustrates the level curves of the original nonconvex constraint. The feasible region $L^g_{\alpha=0}$ is illustrated by the dark gray region. The upper right figure illustrates the level curves of conv g. The gray region is the relaxed convex feasible region $L^{\mathrm{conv} g}_{\alpha=0}$ obtained in this case. Finally the lower left and right figures are the level curves of q_1 and q_2 respectively. The gray regions illustrate the convex feasible regions $L^{q_1}_{\alpha=0}$ and $L^{q_2}_{\alpha=0}$. We can observe that the convex envelope of the constraint function overestimates the feasible region while the functions q_1 and q_2 express it exactly.

In Figure 2, plots of the nonconvex constraint function g and the convex functions q_1 and q_2 are illustrated. The zero-level is indicated by a white plane. Observe that the functions q_1 and q_2 exactly represent the feasible regions but under- and overestimate the constraint function g in different parts of the infeasible region and overestimate the constraint function in the entire feasible region. However, since the requirements for these functions are only to be convex and fulfill $q(\boldsymbol{x}) \leq 0,\ \forall \boldsymbol{x} \in X: g(\boldsymbol{x}) \leq 0$, they are valid reformulations of $g(\boldsymbol{x}) \leq 0$ when used as constraints of the type $q(\boldsymbol{x}) \leq 0$.

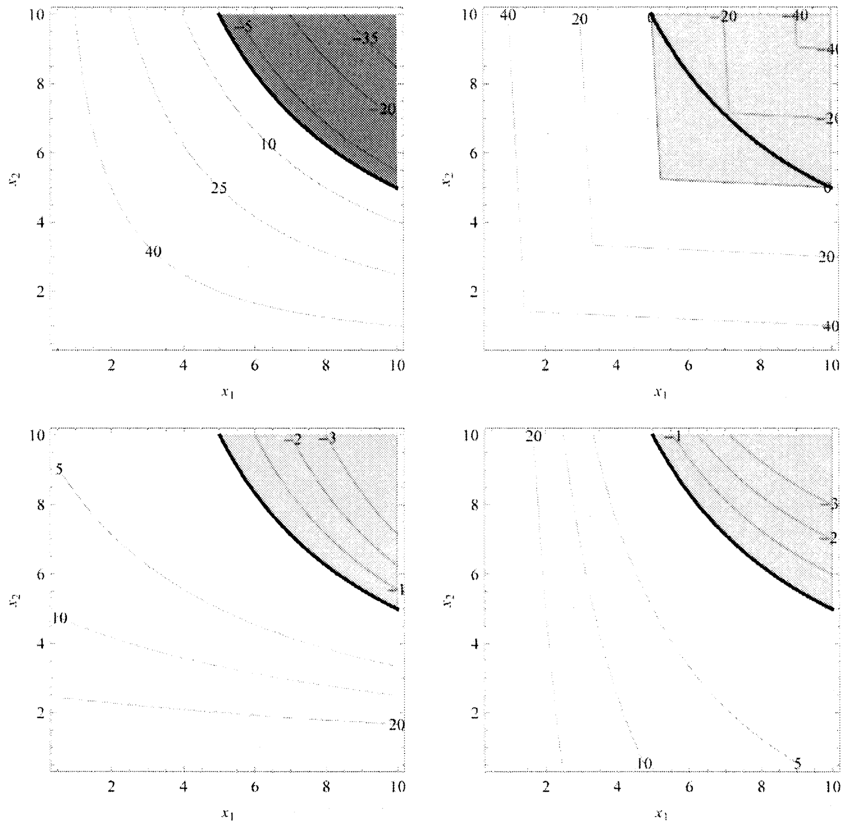

Figure 1: Level curves and feasible regions indicated for g (upper left), conv g (upper right), q_1 (lower left) and q_2 (lower right)

6. Summary

In this paper, we have considered some issues related to convex relaxation techniques in constrained nonconvex optimization. We have pointed out the importance of differentiating between convex envelopes for functions and convex envelopes for sets when creating the tightest possible convex relaxations in constrained problems. The tightest convex relaxation of $L = \{x \mid g(x) \leq 0\}$ is conv L and is generally not obtained when g is replaced by conv g. If g is replaced by a convex function q defined by $q(x) \leq 0$, $\forall x \in X : g(x) \leq 0$, a tighter convex relaxation of L can be obtained than when replacing g with conv g. This was illustrated using an example from N-dimensional allocation. When it comes to the exact convexity requirement of q it depends on the solution method used to solve the problem. If an outer approximation method like the method in Westerlund and Pörn (2002) is used, then q need only be convex in $L^{g}_{\alpha \geq 0}$. More generally, q need only be quasiconvex as quasiconvex functions have convex level sets.

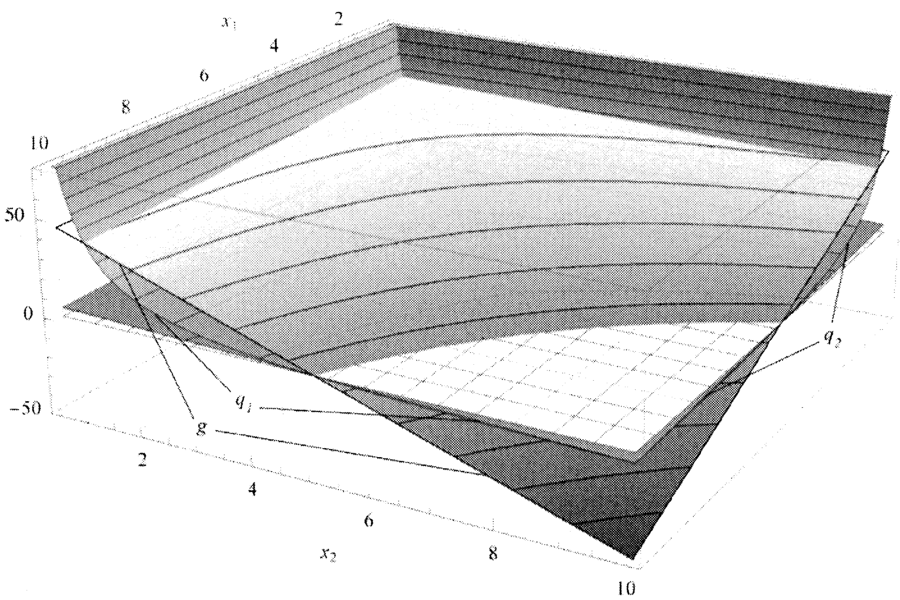

Figure 2. The functions g, q_1 and q_2, as well as, the feasible region $g(x) \leq 0$

Acknowledgements

Financial support from the Academy of Finland and the CoE in Optimization and Systems Engineering at Åbo Akademi University is gratefully acknowledged.

References

Bonäs F., Westerlund J. and Westerlund T., 2007, An interactive optimization tool for allocation problems, Chemical Engineering Transactions 11, 107–112.

Castillo I., Westerlund J., Emet S. and Westerlund T., 2005, Optimization of block layout design problems with unequal areas: A comparison of MILP and MINLP optimization methods, Computers and Chemical Engineering, 30(1), 54–69.

McCormick G.P., 1976, Mathematical programming computability of global solutions to factorable nonconvex programs: Part I – convex underestimating problems, Mathematical programming, 10, 147–175

Sherali H.D. and Alameddine A., 1990, An explicit characterization of the convex envelope of a bivariate bilinear function over special polytopes, Annals of Operations Research, 25, 197–209.

Tuy H., 1998, Convex Analysis and Global Optimization, Kluwer Academic Publishers.

Westerlund J., Papageorgiou L.G. and Westerlund T., 2007, A MILP model for *N*-dimensional allocation, Computers and Chemical Engineering, 31(12), 1702–1714.

Westerlund T. and Pörn R., 2002, Solving pseudo-convex mixed integer optimization problems by cutting plane techniques, Optimization and Engineering, 3, 253–280.