CHEMICAL ENGINEERING TRANSACTIONS
VOL. 51, 2016

A publication of AIDIC
The Italian Association of Chemical Engineering
Online at www.aidic.it/cet

Guest Editors: Tichun Wang, Hongyang Zhang, Lei Tian

ISBN 978-88-95608-43-3; **ISSN** 2283-9216
DOI: 10.3303/CET1651203

Triangular C^2 Interpolants by Piecewise Rational Functions

Wenbo Yu

Jiangxi University of Science and Technology, Nanchang 330013, China
ywbo1026@163.com

In this paper a C^2 interpolation scheme on a triangle is presented. The interpolant assumes given values and derivatives of orders up to 2 at the vertices of the triangle. It is made up of partial interpolants blended with weight functions. Any partial interpolant with respect to a vertex is a piecewise quintic defined on a split of the triangle and interpolates the data at the vertex and on the two sides sharing the vertex, while the corresponding weight function is a rational polynomial in the barycentric coordinates. The resulted interpolant is a piecewise rational polynomials. By using the Bernstein-Bezier representation of polynomials, the interpolant is easy to describe and evaluate.

1. Introduction

Local interpolation over triangulations has received widespread discussion as a tool for use in free-form surface design in Computer Aided Geometric Design, finite element computation, and scattered data processing. The general setting for local interpolation of a prescribed smoothness is to setup a model on one single representative triangle. In order to attain global smoothness on the whole triangulation, a successful model must allow the interpolants on any two adjacent triangles satisfying the prescribed smoothness across the common edge. For the sake of computation, the interpolant has to be simple enough and could be a polynomial, a rational polynomial, and even their piecewise versions defined on a split of the model triangle.

In practice, the C^1 and C^2 models are of essential importance. There have been many C^1 schemes but few for C^2 ones; see Farin (1986), Farin (1990), Goodman and Said(1991), Peters (1990), and Strang and Fix (1973). Since C^2 schemes supersede C^1 ones in better approximation and visual effect (esthetic feeling), endeavor has been paid in designing C^2 schemes (Alfred, 1984; Wang, 1992).

In general, a C^r scheme of polynomial requires vertex data of order $2r$, and the degree of the polynomial can reach $4r+1$. So for a C^2 scheme, the derivatives assigned at vertices must be of orders up to 4 and the degree of the polynomial has to be 9 (Zenisek, 1970). Higher degrees mean more complexity and instability. Ideas to overcome this are to use rational polynomials or to split the *macro*triangle into *micro*triangles on which a C^2 spline is to be defined; see Alfred (1984) and Zhan (1996) for instance.

The drawback lying in the splitting trick is that the splitting causes more microtriangles. For this, a rational polynomial interpolant can be adopted. Alfeld and Barnhill (1984) developed a transfinite C^2 scheme the discrete version of which results in rational interpolant. Liu and Zhu (1995) characterized C^2 rational schemes and presented certain C^2 discrete triangular interpolants. Similar schemes can be found in Zenisek (1970).

The rational approach is able to prevent the triangle T from being split. Nonetheless, this approach exhibits a possible flaw that the degrees of the denominator and numerator could still be very high. For instance, for the scheme with quintic precision in Whelan (1986), the rational interpolant has degrees (9, 4) of the pairs of the denominator and numerator.

By combining the rational formation with the splitting technique, in the present paper, we'll develop a C^2 scheme in a quite natural way. The interpolant is a weighted sum of three partial interpolants, each of which is a C^2 piecewise quintic on a split of the triangle T with respect to a vertex. And the weight functions are rational polynomials of degrees (2, 2). Then the degrees of the resulted interplant are (7, 2).

The paper is organized as follows. In §2, as a preliminary, we formulate bivariate quintics in Bernstein-Bezier forms. Then in §3, the C^2 triangular interpolation scheme is described and the explicit formulation of the interpolant is displayed in §4. At last, a conclusion is made in §5.

Please cite this article as: Yu W.B., 2016, Triangular c2 interpolants by piecewise rational functions, Chemical Engineering Transactions, 51, 1213-1218 DOI:10.3303/CET1651203

2. A formulation of quintic polynomials

Take $T = A_0A_1A_2$ as a triangle. Let $e_i = A_{i+1}A_{i+2}$ denote the opposite edge of A_i and n_i the outer normal vector of e_i, i = 0,1,2, counting modulo 3. For $\alpha = (p, q)$ with $|\alpha| = p+q$, denote by D^α the partial differential operator $\partial^{p+q} / \partial x^p \partial y^q$. Suppose $f \in C^2(T)$. For a C^2 scheme, an interpolant $g \in C^2(T)$ of f must at least satisfies $D^\alpha(g - f)(A_i) = 0$, $0 \le |\alpha| \le 2$, $i \in Z_3$. Let $A_i \in \mathrm{R}^2$, $i \in Z_3$, form a triangle $T = A_0A_1A_2 \subset \mathrm{R}^2$. For $A = (x, y) \in \mathrm{R}^2$, denote by $\mathbf{u} = (u_0, u_1, u_2)$ its barycentric coordinates (see Farin (1986) for the detail) with respect to A_0, A_1, and A_2 (or to T), i.e.

$A = u_0A_0 + u_1A_1 + u_2A_2$, with , $u_0 + u_1 + u_2 = 1$.

An index $\lambda = (\lambda_0, \lambda_1, \lambda_2)$, $\lambda_i \in Z_+$, has length $|\lambda| = \lambda_0 + \lambda_1 + \lambda_2$. A polynomial p in P_k, the set of all the bivariate polynomials of degrees up to k, can be expressed in Bernstein-Bezier form (short for B-form)

$$p(u) := p(x, y) = \sum_{|\lambda|=k} B_{k,\lambda}(u), \quad B_{k,\lambda}(\mathrm{u}) = \frac{k!}{\lambda_0!\lambda_1!\lambda_2!} u_0^{\lambda_0} u_1^{\lambda_1} u_2^{\lambda_2}$$

where b_λ is the B-ordinates at index λ with the Bernstein basis function $B_{k,\lambda}(u)$.
Let $e \in R^2$ be a vector $e = \theta_0A_0 + \theta_1A_1 + \theta_2A_2$, with $\theta_0 + \theta_1 + \theta_2 = 0$, and $\theta = (\theta_0, \theta_1, \theta_2)$. Take $\varepsilon_0 = (1, 0, 0)$, $\varepsilon_1 = (0, 1, 0)$, and $\varepsilon_2 = (0, 0, 1)$. Define recursively

$$b_\lambda^{(0)} = b_\lambda, \ |\lambda| = k; \quad b_\lambda^{(r)} = \frac{k!}{(k-r)!} \sum_{i \in Z_3} u_i b_{\lambda+\varepsilon_i}, \ |\lambda| = k - r.$$

Then the r-th derivative of p along direction e is formulated as

$$\partial^{r|e} p(\mathrm{u}) = \frac{k!}{(k-r)!} \sum_{|\lambda|=k-r} b_\lambda^{(r)}(\theta) B_{k-r,\lambda}(\mathrm{u}) .$$

Take $\mathbf{e}_i = A_{i+1}A_{i+2}$ as the vector from A_{i+1} to A_{i+2} opposite to A_i, $i \in Z_3$. Then $n_i = \mathrm{e}_{i+2} - h_i e_i$ with $h_i = (e_i \bullet \mathrm{e}_{i+2})/(e_i \bullet \mathrm{e}_i)$ is an outer normal vector to e_i, $i \in Z_3$. Let

$$V_i = (A_{i+1} + A_{i+2})/2, \ V_{i1} = (2A_{i+1} + A_{i+2})/3, \ V_{i2} = (A_{i+1} + 2A_{i+2})/3.$$

Taking $\Lambda = \{\lambda = (\lambda_0, \lambda_1, \lambda_2); |\lambda| = 5\}$, $\Lambda_i = \{\lambda \in \Lambda; \lambda_i \ge 3\}$, $\Lambda_V = \bigcup_{i \in Z_3} \Lambda_i$ and $\Lambda_c = \{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$, and

$$E_i = p(A_i), F_{ij} = \partial_{A_iA_j} p(A_i), G_{ijk}F_i = \partial_{A_iA_j} \partial_{A_iA_k} p(A_i), \quad i, j, k \in Z_3, j, k \ne i,$$

we have (Farin, 1990)

Lemma 2.1 *Let p be a quintic of the form* $p(\mathbf{u}) = \sum_{|\lambda|=5} b_\lambda B_{5,\lambda}(\mathbf{u})$. Define $q(\mathbf{u}) = \sum_{\lambda \in \Lambda_v} b_\lambda B_{5,\lambda}(\mathbf{u})$. Then

$$q(\mathbf{u}) = \sum_{i \in Z_3} E_i \varphi_i(\mathbf{u}) + \sum_{i,j \in Z_3, i \ne j} F_{ij} \varphi_{ij}(\mathbf{u}) + \sum_{i,j,k \in Z_3, k, j \ne i} G_{ijk} \varphi_{ijk}(\mathbf{u})$$

where

$$\Phi_i(\mathrm{u}) = u_i^3(10 - 15u_i + 6u_i^2), \quad \Phi_{ij}(\mathrm{u}) = u_i^3 u_j (4 - 3u_i), \ i \ne j,$$

$$\Phi_{ijj}(\mathrm{u}) = \frac{1}{2} u_i^3 u_j^2, \ i \ne j, \quad \Phi_{ijk}(\mathrm{u}) = u_i^3 u_j u_k, \ \{i, j, k\} = \{0,1,2\}.$$

This lemma implies that the B-coordinates of p with indices Λ_v are determined by the data assigned at the vertices of T. Further, given 3 normal derivatives on *one* side of T, the B-ordinates with Λ_c can be determined. Precisely, we have, for instance.

Lemma 2.2 *On the basis of Lemma 2.1*, b_{122}, b_{212} *and* b_{221} *are determined in addition by* $\partial_{n_0} p(V_0)$, $\partial_{n_0}^2 p(V_{01})$, and $\partial_{n_0}^2 p(V_{02})$,

$$b_{122} = -\frac{8}{12} \partial_{n_0} p(V_0) + \frac{1}{6}[-(b_{140} + 4b_{131} + 4b_{113} + b_{104} + (1 + h_0)\ (b_{050} + 4b_{041} + 6b_{032} + 4b_{023} + b_{014})$$

$$-h_0(b_{041} + 4b_{032} + 6b_{023} + 4b_{014} + b_{005})] \text{ , H}$$

$$b_{221} = \frac{3}{40}[2\partial_{n_0}^2 p(V_{01}) - \partial_{n_0}^2 p(V_{02})] - \frac{5}{6} b_{030}^{(2)} + \frac{1}{3} b_{003}^{(2)} - b_{021}^{(2)}, \quad b_{212} = \frac{3}{40}[2\partial_{n_0}^2 p(V_{02}) - \partial_{n_0}^2 p(V_{01})] - \frac{5}{6} b_{003}^{(2)} + \frac{1}{3} b_{030}^{(2)} - b_{012}^{(2)} .$$

Lemma 2.3 *With the assumptions that* $\partial_{n_0} p$ *and* $\partial^2_{n_0} p$ *on edge* $\mathbf{e}_0$ *reduce to polynomials of degrees 3 and 1, respectively, the formulas in Lemma 2.2 are replaced with*

$$b_{122} = \frac{1}{6}[(-b_{140} + 4b_{131} + 4b_{113} - b_{104}) + (1+h_0)(b_{050} + 4b_{041} + 6b_{032} - 4b_{023} + b_{014}) - h_0(b_{041} - 4b_{032} + 6b_{023} - 4b_{014} + b_{005})],$$

$$b_{221} = (2b^{(2)}_{030} + b^{(2)}_{003})/3 - b^{(2)}_{021}, \quad b_{212} = (2b^{(2)}_{003} + b^{(2)}_{030})/3 - b^{(2)}_{012}.$$

In the sequel, we'll use $b^i_\lambda,\ \lambda \in \Lambda c$, to specify the B-ordinates b_λ obtained with respect to the data on the edge $\mathbf{e}_i$, $i \in Z_3$.

3. The C^2 interpolation scheme

In this section, we're presenting a C^2 interpolation scheme. Given a triangulation τ of a polygonal region $\Omega \subset R^2$, and integers k and r, $k \geq r \geq 0$, the spline space of piecewise C^r polynomials of degree k is defined as (Wang, 2001)

$$S^r_k(\tau) = S^r_k(\tau, \Omega) := \{s \in C^r(\Omega); s|_\sigma \in P_k, \sigma \text{ being any triangle in } \tau\}.$$

3.1 Partial interpolants

For the triangle $T = A_0A_1A_2$, take

$$B_{im} = (1 - t_{im})A_{i+1} + t_{im}A_{i+2},\ 0 < t_{i1} < t_{i2} < 1,\ i \in Z_3,\ m = 1, 2.$$

For any $i \in Z_3$, by connecting A_i and B_{im}, $m = 1, 2$, we form a split Δ_i of T with subtriangles $T_{i1} = A_iA_{i+1}B_{i1}$, $T_{i2} = A_iB_{i1}B_{i2}$, and $T_{i3} = A_iB_{i2}A_{i+2}$. See Figure 1. By using cofactor conforming approach developed by Wang(2001), it's easy to verify that $\dim S^2_5(\Delta_i) = 33$.

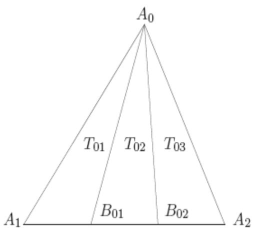

Figure 1: The partial split Δ0

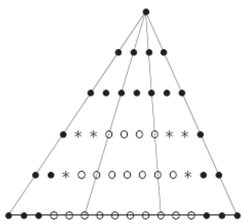

Figure 2: The stencil of {bλ ; l , l = 1, 2, 3}

Now consider the following partial interpolation problem for $i \in Z_3$.

Interpolation Problem (P) For $f \in C^2(T)$, find $s_i \in S^2_5(\Delta_i)$ such that

$$D^\alpha(s_i - f)(A_j) = 0,\ 0 \leq |\alpha| \leq 2,\ j \in Z_3; \tag{1}$$

$$\partial_{n_l}(s_i - f)(V_l) = 0,\ \partial^2_{n_l}(s_i - f)(V_{lm}) = 0,\ l \neq i,\ l \in Z_3,\ m = 1, 2, 3 \tag{2}$$

Suppose s_i is an interpolant of (P), denote $s_{il} = s_i|_{T_{il}}$, for $i \in Z_3$ and i = 1, 2, 3.

Note that the number of conditions given in problem (P) is 27, which is 6 less than the dimension of the underlying interpolation space $S^2_5(\Delta_i)$. So, for the interpolant of (P), if it exists, to be uniquely determined, additional restrictions must be imposed on the interpolation.

For the sake of simplicity, without loss of generality, the discussion in this subsection is only on Δ_0. Suppose s_0 is an interpolant of (P) and

$$s_{0l} = \sum_{|\lambda|=5} b_{\lambda;l} B_{5,\lambda}(\mathrm{u}_l) \tag{3}$$

where $u_l = (u_{0;l}, u_{1;l}, u_{2;l})$ are the barycentric coordinates of point $(x, y) \in R^2$ with respect to T_{0l}.

Note that for any $A = (x, y) \in R^2$,

$$A = u_0A_0 + u_1A_1 + u_2A_2 = u_{0;1}A_0 + u_{1;1}A_1 + u_{2;1}B_{01} = u_{0;2}A_0 + u_{1;2}B_{01} + u_{2;2}B_{02} = u_{0;3}A_0 + u_{1;3}B_{02} + u_{2;3}A_2,$$

and then $u_0 = u_{0;1} = u_{0;2} = u_{0;3}$. Base on this observation, each s_{0l} can be written as

$$s_{0l}(\mathrm{u}) = \sum_{k=0}^{5} \binom{5}{k} u_0^{5-k} g_{k;l}(u_{1;l}, u_{2;l})$$

where $g_{k;\,l}(\,u_{1;\,l},\, u_{2;\,l}\,)$ is a (univariate) polymomial of degree k, which can be seen as the k-th *layer* of s_{0l}. This allows s_0 another formulation

$$s_0(\mathrm{u}) = \sum_{k=0}^{5} \binom{5}{k} u_0^{5-k} g_k(u_1, u_2)$$

where g_k is a *univariate* piecewise polynomials of degree k, k = 0, 1, ..., 5, with successive pieces $g_{k;\,1}$, $g_{k;\,2}$, and $g_{k;\,3}$, defined on the partition $0 < t_{01} < t_{02} < 1$.
The supports of the B-ordinates are depicted in Figure 2. From Lemma 2.1, the *B*-ordinates marked with • are determined by conditions (1), and from Lemma 2.2, the conditions (2) in addition determine those marked with ***** in Figure 2.
In particular, the g_k with k = 0, 1, and 2, are all overall polynomials determined by conditions (2) and it is easy to see that, with C^2 relations, g_3 is uniquely solved by the previously determined *B*-ordinates. Refer to Schumaker (1981) for univariate splines.
For g_4 and g_5, we make the following convention that
(i) g_5 reduces to an overall quintic;
(ii) g_4 is under C^3 relations and some two successive pieces of g_4 reduce to a quartic.
Under restriction (i), g_5 is obviously determined by conditions (1). The rule (ii) means that g_4 is a C^3 cubics of two pieces with 6 parameters, just the same as the number of ***** and • on g_4. Then we conclude that

Theorem 3.1 *Under the restrictions (i) and (ii), there uniquely exists an interpolant* $s \in S_5^2(\Delta_0)$ *satisfying the conditions (1) and (2).*

3.2 The macro interpolants

Similar to s_0, the partial interpolants $s_1 \in S_5^2(\Delta_1)$ and $s_2 \in S_5^2(\Delta_2)$ of (P) are obtained uniquely under (i) and (ii). From the discussion above, we can also see that the partial interpolants $s_i, i \in Z_3$, *hold*

$s_i \,|\, \mathbf{e}_i = s_{i+1} \,|\, \mathbf{e}_i = s_{i+2} \,|\, \mathbf{e}_i$, $\partial^r_{\mathrm{n}_i} s_{i+1} \,|\, \mathbf{e}_i = \partial^r_{\mathrm{n}_i} s_{i+2} \,|\, \mathbf{e}_i$, r = 1, 2.

Define *weight* functions $w_i(\mathrm{u}) = u_i^2 / (u_0^2 + u_1^2 + u_2^2)$, $i \in Z_3$. Obviously, these functions have the properties that $w_i \in C^\infty$, $\sum_{i \in Z_3} w_i(\mathrm{u}) = 1$, $w_i(\mathrm{u})|_{e_i} = 0$, and $\partial_{\mathrm{n}_i} w_i(\mathrm{u})|_{e_i} = 0$, $i \in Z_3$.

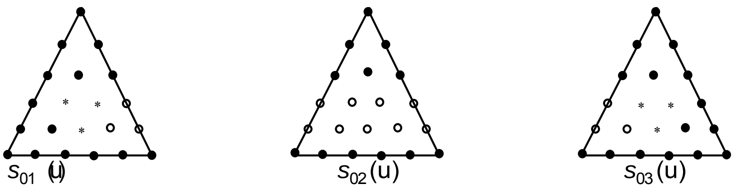

Figure 3: The stencil of s_0 in the uniform BCS u.

Now it's ready to the main result of this paper.

Theorem 3.2 *Suppose* $s_i \in S_5^2(\Delta_i)$, $i \in Z_3$, *are interpolants of (P) for f* $\in C^2(T)$ *under the restrictions of (i) and (ii). Then the blending function*

$$f_T(\mathrm{u}) = \sum_{i \in Z_3} w_i(\mathrm{u}) s_i(\mathrm{u}) \qquad (4)$$

satisfies that

$$D^\alpha (f_T - f)(A_i) = 0,\ 0 \le |\alpha| \le 2,\ i \in Z_3 \qquad (5)$$

$$\partial_{\mathrm{n}_i}(f_T - f)(V_i) = 0\ ,\ \ \partial^2_{\mathrm{n}_i}(f_T - f)(V_{im}) = 0,\ i \in Z_3,\ m = 1, 2.$$

Still, $\partial^r_{\mathrm{n}_i} f_T \,|\, \mathrm{e}_i$, $i \in Z_3$, *are polynomials of degrees* 5 − r, r = 0, 1, 2*. In addition,* f_T *reproduces polynomials of degree 5.*

Theorem 3.3 *Suppose* $s_i \in S_5^2(\Delta_i)$, $i \in Z_3$, *satisfies* $D^\alpha(s_i - f)(A_j) = 0,\ 0 \le |\alpha| \le 2,\ j \in Z_3$; $\partial_{n_l} s_i \mid e_l$ and $\partial^2_{n_l} s_i \mid e_l$ *are of degrees 3 and 1, respectively,* $l \ne i$, $l \in Z_3$. *In addition, suppose* s_i, $i \in Z_3$, *are under the restrictions of (i) and (ii), and* f_T *is defined as in (4). Then* f_T *reproduces polynomials of degree 3 and satisfies (5). On* e_i, $\partial_{n_i} f_T$ *and* $\partial^2_{n_i} f_T$, $i \in Z_3$, *are polynomials of degrees 3 and 1, respectively.*

The usage of rational functions can be dated back to the well-known BBG scheme (Barnhill et al, 1973) and Nielson's side-vertex scheme (Nielson, 1979). The interpolation schemes in Liu and Zhu (1995) and Xu, et al (2000) can also be regarded as side-vertex schemes, since a partial interpolant on a triangle in those schemes interpolates data at a vertex and on its opposite side. In our method, a partial interpolant interpolates data on two sides sharing a vertex and then our interpolation can be called a side-side scheme. The weight functions used in side-vertex schemes and in side-side schemes are different. For instance, the weight functions used in Liu and Zhu (1995) are

$$w_i(\mathrm{u}) = u_{i+1}^2 u_{i+2}^2 / (u_0^2 u_1^2 + u_1^2 u_2^2 + u_2^2 u_3^2),\quad i \in Z_3 .$$

Obviously, the macro interpolant as a whole is a piecewise rational function with degrees (7, 2) for the pairs of the denominator and numerator. By contrast, for the scheme with quintic precision in Liu and Zhu (1995), the rational interpolant has degrees (9, 4).

4. Evaluation of the interpolant

Suppose $f \in C^2(T)$. Let $q(\mathbf{u}) = \sum_{\lambda \in \Lambda_V} b_\lambda B_{5,\lambda}(\mathrm{u})$ be the polynomial in Lemma 2.1, with the symbol p replaced with f. Then q is nothing but the polynomial interpolating function f at the vertices of T such that

$$D^\alpha(q - f)(A_i) = 0,\ 0 \le |\alpha| \le 2,\ i \in Z_3 .$$

Now fix one $i \in Z_3$. By assigning $\partial_{n_i}(p-f)(V_i) = 0,\ \partial^2_{n_i} f(p-f)(V_{im}) = 0, m = 1, 2$, the B-ordinates $b^i_\lambda,\ \lambda \in \Lambda_C$, can be determined via Lemma 2.2.

Continuing the preceding, we will give an expression of the partial interpolant $s_i \in S_5^2(\Delta_i)$, $i \in Z_3$, and hence the macro interpolant f_T. In order to express s_i, owing to the symmetry, we only illustrate the formulation of s_0, and the others can be obtained by rotation with the index i.

Note that a B-form polynomial in a barycentric coordinates system (BCS) can be expressed in any other BCS with an affine transform between the two systems. For the interpolant $s_0 \in S_5^2(\Delta_0)$, we prefer to write

$$s_{0l}(\mathrm{u}) = s_0 \mid T_{0l} = \sum_{|\lambda|=5} b_{\lambda;l} B_{5,\lambda}(\mathrm{u}) = \sum_{k=0}^{5} \binom{5}{k} u_0^{5-k} g_{k;l}(u_1, u_2),\quad l = 1, 2, 3 \tag{6}$$

in the uniform BCS u with T rather than that with T_{0l} as in (3). This form has the advantages that

$$b_{122;1} = b^2_{212},\ b_{221;1} = b^2_{122},\ b_{212;1} = b^2_{221},\ b_{122;3} = b^1_{221},\ b_{221;3} = b^1_{212},\ b_{212;3} = b1^2_{122};$$

$$b_{\lambda;l} = b_\lambda,\quad \text{for}\ \ \lambda_0 \ne 1, 2,\ \ l = 1, 2, 3;\quad b_{\lambda;1} = b_\lambda,\quad \text{for}\ \ \lambda \in \Lambda_1;\quad b_{\lambda;3} = b_\lambda,\quad \text{for}\ \ \lambda \in \Lambda_2.$$

Since the segments A_0B_{01} and A_0B_{02} are on the lines

$$\Gamma_1:\quad \gamma_1 := t_{01}u_1 - (1 - t_{01})u_2,\quad \Gamma_2:\quad \gamma_1 := t_{02}u_1 - (1 - t_{02})u_2,$$

respectively, we can write

$$s_0(\mathrm{u}) = \sum_{|\lambda|=5,\ \lambda_0 \ne 1,2} b_\lambda B_{5,\lambda}(\mathrm{u}) + 10u_0^2 g_3(u_1, u_2) + 5u_0 g_4(u_1, u_2) \tag{7}$$

where g_3 and g_4 are piecewise C^2 cubics and quartics in u_1 and u_2, and

$$g_3 = \begin{cases} g_{31}, & \gamma_1 \ge 0, \\ g_{32}, & \gamma_1 < 0 < \gamma_2, \\ g_{33}, & \gamma_2 \le 0, \end{cases} \qquad g_4 = \begin{cases} g_{41}, & \gamma_1 \ge 0, \\ g_{42}, & \gamma_1 < 0 < \gamma_2, \\ g_{43}, & \gamma_2 \le 0, \end{cases}$$

4.1 The representation of g_3

The unknown B-ordinates for g_3 include $b_{203;1}$, $b_{230;2}$, $b_{221;2}$, $b_{212;2}$, b_{203};2, and $b_{230;3}$. We only provide the formula of $b_{203;1}$ and the other can be derived similarly.

The C^2 smoothness of g_3 implies that there are two numbers c_1 and c_2, such that

$$g_{31}(u_1,u_2)+c_1[\gamma_1(u_1,u_2)]^3 = g_{32}(u_1,u_2), \quad g_{32}(u_1,u_2)+c_2[\gamma_2(u_1,u_2)]^3 = g_{33}(u_1,u_2).$$

Identifying the coefficients of u_1 and u_2 in these equations, we get

$$c_1 = \frac{(b_{221;3}-b_{221;1})(t_{02}-1)-(b_{212;3}-b_{212;1})t_{02}}{t_{01}(t_{01}-1)(t_{02}-t_{01})}, \quad c_2 = \frac{(b_{221;3}-b_{221;1})(t_{01}-1)-(b_{212;3}-b_{212;1})t_{01}}{t_{02}(t_{02}-1)(t_{01}-t_{02})}$$

$$b_{203;1} = b_{203;3} - c_1(t_{01}-1)^3 - c_2(t_{02}-1)^3.$$

4.2 The representation of g_4

Under (ii), g_4 is of C^3, and two pieces of g_4, e.g. g_{41} and g_{42} are identical, i.e.

$$g_{41}(u_1,u_2) = g_{42}(u_1,u_2), \quad g_{42}(u_1,u_2)+d_2[\gamma_2(u_1,u_2)]^4 = g_{43}(u_1,u_2),$$

where d_2 is the C^3 cofactor of g_4 on Γ_2. Then

$$d_2 = \frac{b_{122;3}-b_{122;1}}{t_{02}^2(1-t_{02})^2}, \quad b_{113;1} = b_{113;3}+d_2 t_{02}(1-t_{02})^3, \quad b_{104;1} = b_{104;3}+d_2(1-t_{02})^4.$$

The other B-ordinates $b_{140;2}$, $b_{131;2}$, $b_{122;2}$, $b_{113;2}$, $b_{104;2}$, $b_{140;3}$, and $b_{131;3}$ are formulated similarly and omitted here.

5. Conclusion

We have shown in this paper a C^2 local interpolation method and the detailed representation of the interpolant. In literature, Alfeld (1984) developed a C^2 scheme on the so called twice HCT split for only C^2 data. Whelan (1986) displayed a C^2 interpolant of piecewise nonics with C^4 data, which indeed a special case of the schemes in Zenisek(1970). Both are comparatively complicated because of the large number of split pieces of the macrotriangle of the former and the high degree of polynomials and high order of derivatives of the latter. We can see that the scheme in this paper overcomes these drawbacks to a large extent. With the explicit formulation of the interpolant with Bernstern-Bezier technique, it’s also easy to implement the evaluation of this interpolation scheme.

Acknowledgments

This paper was supported by Science and Technology Research Project of Jiangxi Provincial Education Department (No. GJJ150696).

References

Alfeld P., 1984, A bivariate C^2 Clough-Tocher scheme, Computer Aided Geometric Design 1, 257-267.
Alfeld P., Barnhill R. E., 1984, A transfinite C^2 interpolant over triangles, Rocky Mountain J. Math. 14, 17-39.
Barnhill R. E., Birkhoff G., Gordon W. J., 1973, Smooth interpolation in triangles, J. Approx. Theory 8, 114-128.
Farin G., 1986, Triangular Berstein-Bezier patches, Computer Aided Geometric Design 3, 83-127.
Farin G., 1990, Curves and surfaces for computer aided geometric design, Academic Press.
Goodman, T.N.T., Said H. B., 1991, A C^1 triangular interpolant suitable for scattered data interpolation, Communcation in Applied Numerical Methods 7, 479-485.
Liu X., Zhu Y., 1995, A characterization of certain C^2 discrete triangular interpolants, CAGD 12, 329-348.
Nielson G. M., 1979, The side-vertex method for interpolation in triangles, J. Approx. Theory, 25,318–336.
Peters J., 1990, Local smooth surface interpolation: a classification, CAGD 7,191-195.
Schumaker L., 1981, Spline Functions: Basic Theory, Wiley, New York.
Strang G., Fix G., 1973, An Analysis of the Finite Element Method, Prentice-Hall, New York.
Wang R., 2001, Multivariate Spline Functions and Their Applications, Kluwer Academic Publishers.
Wang T., 1992, A C2-quintic spline interpolation scheme on triangulation, CAGD 9, 379-386.
Whelan T., 1986, A representation of a C^2 interpolant over triangles, CAGD 3, 53–66.
Xu G., Chu C., Xue W., 2000, Triangular C^m interpolation by rational functions, Approximation Theory and its Applications 16, no. 4, 35–54.
Zenisek A., 1970, Interpolation polynomials on the triangle, Numer, Math. 15, 383-296.
Zhan Y., 1996, A C^2 interpolation scheme over triangulations, SPIE 2644, 232–237.