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The Heat and Mass Transfer Modeling with Time Delay

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Nonlinear hyperbolic reaction-diffusion equations with a delay in time are investigated. All equations considered here contain one arbitrary function. Exact solutions are also presented for more complex nonlinear equations in which delay arbitrarily depends on time. Exact solutions with a generalized separation of variables are found. For special cases, new exact solutions in the form of a traveling waves are obtained, some of which can be represented in terms of elementary functions. All of these solutions contain free (arbitrary) parameters, so that one can use them to solve modeling problems of heat and mass transfer with relaxation phenomena.

1. Introduction

Parabolic equation of heat- and mass-transfer has a physically paradoxical property, i.e., an infinite disturbance propagation rate, which is not observed in nature. Solving non-steady-state heat- and mass-transfer problems, it is necessary to take into account relaxation phenomena associated with the finiteness of the rate of heat and mass transfer (see, for example, Demirel, 2007). The thermal and diffusion relaxation times can vary in extremely wide limits from milliseconds (or less) to several tens of seconds and should be taken into account in solving many heat and mass transfer problems (Polyanin and Vyazmin, 2013).

The second important feature of evolutionary processes, including heat- and mass-transfer processes with chemical conversions, is that, in the general case, the rate of variations in the desired quantities in chemical, biological, physicochemical, chemical engineering, and other systems, depends not only on the state at the given time point, but also on the entire previous evolution of the process (Jou et al., 2010; Pokusaev et al., 2015). These systems are called hereditary systems. In the particular case where the state of the system is only determined by a particular time point in the past, rather than the entire evolution of the system, the system is referred to as a delayed feedback system.

Systems with delayed feedback are frequently modeled by reaction–diffusion equations, in which the kinetic function F (the rate of chemical reactions) depends on both the sought concentration function $u = u(x, t)$ and the same function, but with the delayed argument $w = u(x, t - \tau)$. The special case of $F(u, w) = f(w)$ has a simple physical interpretation, i.e., heat- and mass-transfer processes in media with local non-equilibrium have inertial properties, i.e., the system does not react to action instantaneously at the given time point t, as in the classical local equilibrium case, but it reacts by the delay time τ later.

Solving non-steady-state mass transfer problems in chemical engineering, it is necessary to take into account relaxation phenomena associated both with the finiteness of the time of transfer processes τ_1 and with the finiteness of the times τ of chemical conversions and/or the microkinetic interaction between different phases that form a single transport macromedium, exact solutions to the following non-linear hyperbolic reaction–diffusion equations are derived and analyzed in this study (see also Polyanin et al., 2015):

$$\tau_1 \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + F(u, w), \quad w = u(x, t - \tau), \tag{1}$$

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where a - diffusion coefficient, x - coordinate. It should be noted that, as a particular case, at $\tau_1 = 0$, considering equation includes parabolic equations with delay. More complex nonlinear reaction–diffusion equations with variable delay of the general form $\tau = \tau(t)$ will also be considered.
In the degenerate case, at $\tau_1 = 0$, i.e., for the parabolic equation, certain exact solutions to Eq. (1) were obtained, for examples, for travelling wave by Wu and Zou, 2001;.using complete group classification by Meleshko and Moyo, 2008; using method of generalized and functional separation by Polyanin and Zhurov, 2014a.
For kinetic function F of two general forms new exact solutions of equation (1) will be found bellow. We emphasize that for the first time exact solutions are obtained for the equation with two characteristic delay times, which have different physical meaning and which appear in different terms of equation (1). These results generalize previous solutions obtained by other authors.

2. Methods for finding solutions

The numerical solving of various nonlinear partial differential equations and systems of equations with delay and difficulties that arise in this case are described by Jackiewicza and Zubik-Kowal (2006). In any case, the general disadvantages of numerical methods include: absence universal application when changing the geometric shape of the object, the type of fluid flow, reaction kinetics and inapplicability in the presence of singular points.
Exact solutions to nonlinear differential equations promote the better understanding of the qualitative features of the processes under description (nonuniqueness, spatial localization, blowup regimes, etc.). It should be emphasized that delay substantially complicates the analysis of equations and is a factor that can lead to the instability of the systems being modeled (Jordan et al, 2008).
The term exact solutions with respect to the nonlinear delay of partial differential equations are used in the cases where a solution is expressed as follows:
- The solution can be expressed in terms of elementary functions or can be represented in the closed form (the solution is expressed in terms of indefinite or definite integrals).
- The solution can be expressed in terms of solutions to ordinary differential equations or delay ordinary differential equations (or systems of these equations).
- The solution can be expressed in terms of solutions to linear partial differential equations.
- The combinations of solutions are also allowable.

Solution methods and various applications of linear and nonlinear ordinary differential equations with delay, which are substantially simpler than nonlinear partial differential equations with delay, are described, for examples, by Bellman and Cooke, 1963; Kuang, 1993. A number of exact solutions to certain nonlinear partial differential equations with delay (as well as systems of equations with delay), which are different from reaction–diffusion equations, are given in paper Tanthanuch, 2012.
In this study, to seek exact solutions to nonlinear hyperbolic reaction–diffusion equation such as Eq. (1), we used various modifications of the methods of generalized and functional separation of variables (see, for information, handbooks by Polyanin and Manzhirov, 2007; Galaktionov and Svirshchevskii, 2007; Polyanin and Zaitsev, 2012) and the functional constraints method for parabolic delay equations (Polyanin and Zhurov, 2014b); for parabolic equations with varying transfer coefficients (Polyanin and Zhurov, 2014c). From this point on, intermediate calculations are generally omitted for the sake of brevity.
Equation (1) does not model any particular technological process. It generalizes the diffusion equation; obtained on the basis of the Fick's law in the equilibrium, for nonequilibrium processes which takes into account their own rates of perturbation propagations in the medium and its chemical transformations. The obtained results do not require any verification, since they are mathematically accurate.

3. Exact solutions to Eq. (1) with a kinetic function that depends on the ratio *w*/*u*

Let us consider Eq. (1) in the following form:

$$\tau_1 \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uF(w/u), \quad w = u(x, t-\tau(t)), \tag{2}$$

where $F(z)$ is an arbitrary function.

3.1 Equation with variable delay

1. Eq. (2) yields solution periodic with respect to x,

$$u = [C_1 \cos(\lambda x) + C_2 \sin(\lambda x)]\psi(t), \tag{3}$$

where C_1, C_2, and λ are arbitrary constants and the function $\psi(t)$ is described by the following ordinary differential equation with delay:

$$\tau_1\psi''(t)+\psi'(t)=-a\lambda^2\psi(t)+\psi(t)F(\psi(t-\tau(t))/\psi(t)).$$

2. Eq. (2) also yields solution of the form

$$u=[C_1\exp(-\lambda x)+C_2\exp(\lambda x)]\psi(t), \tag{4}$$

where the function $\psi(t)$ is described by the following ordinary functional-differential equation:

$$\tau_1\psi''(t)+\psi'(t)=a\lambda^2\psi(t)+\psi(t)F(\psi(t-\tau(t))/\psi(t)).$$

3.2 Equation with constant delay

Now we consider Eq. (2) when τ(t) = const.

1. In this case Eq. (2) yields the separable solution as the product of the functions of different arguments as Eq. (3). The function $\psi(t)$ in Eq. (3) is described by the following ordinary differential equation with delay:

$$\tau_1\psi''(t)+\psi'(t)=-a\lambda^2\psi(t)+\psi(t)F(\psi(t-\tau)/\psi(t)). \tag{5}$$

Eq. (5) yields the particular solution $\psi(t) = Ae^{\beta t}$, where A is an arbitrary constant and β is determined from the algebraic (or transcendental) equation

$$\tau_1\beta^2+\beta+a\lambda^2-F(e^{-\beta\tau})=0.$$

2. Eq. (2) also yields solution of the form Eq. (4), where the function $\psi(t)$ is described by the following delay differential equation:

$$\tau_1\psi''(t)+\psi'(t)=a\lambda^2\psi(t)+\psi(t)F(\psi(t-\tau)/\psi(t)). \tag{6}$$

Eq. (6) yields the particular solution $\psi(t) = Ae^{\beta t}$, where β is determined from the algebraic (transcendental) equation

$$\tau_1\beta^2+\beta-a\lambda^2-F(e^{-\beta\tau})=0.$$

3. Eq. (2) also yields the solution

$$u=\exp(\alpha x+\beta t)\theta(z),\quad z=\lambda x+\gamma t, \tag{7}$$

where the function $\theta(z)$ is described by the following delay ordinary differential equation:

$$(a\lambda^2-\tau_1\gamma^2)\theta''(z)+(2a\alpha\lambda-2\tau_1\beta\gamma-\gamma)\theta'(z)+(a\alpha^2-\tau_1\beta^2-\beta)\theta(z)+\theta(z)F(e^{-\beta\tau}\theta(z-\delta)/\theta(z))=0,$$
$$\delta=\gamma\tau.$$

This equation yields the particular solution $\theta(z) = Ae^{vz}$, where v is determined from the algebraic transcendental) equation

$$(a\lambda^2-\tau_1\gamma^2)v^2+(2a\alpha\lambda-2\tau_1\beta\gamma-\gamma)v+(a\alpha^2-\tau_1\beta^2-\beta)+F(e^{-\beta\tau-v\delta})=0,$$
$$\delta=\gamma\tau.$$

Solution in the form of Eq. (7) is the nonlinear superposition of two different traveling waves.

4. Let the function

$$v=V_1(x,t,b) \tag{8}$$

be any τ-periodic solution to the following linear hyperbolic equation:

$$\tau_1\frac{\partial^2 v}{\partial t^2}+\frac{\partial v}{\partial t}=a\frac{\partial^2 v}{\partial x^2}+bv,\quad v(x,t)=v(x,t-\tau) \tag{9}$$

(from this point on, for the sake of brevity, the dependence of Eqs. (8) and (13) on the parameters τ_1 and a, which appear in Eqs. (9) and (14), is not indicated explicitly). In that case, Eq. (2) yields the generalized separable solution

$$u = e^{ct} V_1(x, t, 1 + 2\tau_1 c, b), \quad b = F\left(e^{-c\tau}\right) - \tau_1 c^2 - c, \tag{10}$$

where c is an arbitrary constant.
It can be shown that the general solution to Eq. (9) subject to the aforementioned condition of τ-periodicity with respect to time has the following form:

$$\begin{aligned} V_1(x, t, b) &= \sum_{n=0}^{\infty} \exp(-\lambda_n x)\left[A_n \cos(\beta_n t - \gamma_n x) + B_n \sin(\beta_n t - \gamma_n x)\right] \\ &+ \sum_{n=1}^{\infty} \exp(\lambda_n x)\left[C_n \cos(\beta_n t + \gamma_n x) + D_n \sin(\beta_n t + \gamma_n x)\right], \end{aligned} \tag{11}$$

$$\gamma_n = \left[\frac{\sqrt{(\tau_1 \beta_n^2 + b)^2 + \beta_n^2} + \tau_1 \beta_n^2 + b}{2a}\right]^{1/2}, \quad \lambda_n = \frac{\beta_n}{2a\gamma_n}., \quad \beta_n = \frac{2\pi n}{\tau} \tag{12}$$

where A_n, B_n, C_n, and D_n are arbitrary constants at which series in Eq. (11) are convergent (the convergence can be ensured, e.g., if we set $A_n = B_n = C_n = D_n = 0$ at $n > N$, where N is any positive integer).
The following particular cases can be distinguished: τ-periodic (with respect to the time t) solutions Eq. (9) that decay at $x \to \infty$ are given by Eqs. (11) and (12) at $A_0 = B_0 = 0$, $C_n = D_n = 0$, and $n = 1, 2, \ldots$; τ-periodic (with respect to the time t) solutions bounded at $x \to \infty$ are given by Eqs. (11) and (12) at $C_n = D_n = 0$ and $n = 1, 2, \ldots$; a stationary solution is given by Eqs. (11) and (12) at $A_n = B_n = C_n = D_n = 0$ and $n = 1, 2, \ldots$.
5. Let the function

$$v = V_2(x, t, b) \tag{13}$$

be a τ-aperiodic solution to the following linear hyperbolic equation:

$$\tau_1 \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} = a \frac{\partial^2 v}{\partial x^2} + bv, \quad v(x, t) = -v(x, t - \tau) \tag{14}$$

In that case, Eq. (2) yields the generalized separable solution

$$u = e^{ct} V_2(x, t, 1 + 2\tau_1 c, b), \quad b = F\left(-e^{-c\tau}\right) - \tau_1 c^2 - c, \tag{15}$$

The general solution to Eq. (14) has the following form:

$$\begin{aligned} V_2(x, t, b) &= \sum_{n=1}^{\infty} \exp(-\lambda_n x)\left[A_n \cos(\beta_n t - \gamma_n x) + B_n \sin(\beta_n t - \gamma_n x)\right] \\ &+ \sum_{n=1}^{\infty} \exp(\lambda_n x)\left[C_n \cos(\beta_n t + \gamma_n x) + D_n \sin(\beta_n t + \gamma_n x)\right], \end{aligned} \tag{16}$$

$$\gamma_n = \left[\frac{\sqrt{(\tau_1 \beta_n^2 + b)^2 + \beta_n^2} + \tau_1 \beta_n^2 + b}{2a}\right]^{1/2}, \quad \lambda_n = \frac{\beta_n}{2a\gamma_n}., \quad \beta_n = \frac{\pi(2n-1)}{\tau} \tag{17}$$

Solutions (τ-aperiodic to the time t) that decay at $x \to \infty$ are given by Eqs. (16) and (17) at $C_n = D_n = 0$ and $n = 1, 2, \ldots$.
Eqs. (11)–(12) and (16)–(27) are very similar. However, in the first case, the first sum begins from $n = 0$ and, in the second solution, it begins from $n = 1$; the values of β_n are also different.

4. Exact solutions to Eq. (1) with a kinetic function that depends on the difference $u - w$

Let us consider Eq. (1) in the following form:

$$\tau_1 \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + bu + F(u - w), \quad w = u(x, t - \tau(t)), \tag{18}$$

where $F(z)$ is an arbitrary function.

4.1 Equation with variable delay

Eq. (18) yields solution of the form,

$$u = \varphi(x) + \psi(t), \tag{19}$$

where

$$\varphi(x) = \begin{cases} C_1 \cos(\lambda x) + C_2 \sin(\lambda x), & \lambda = \sqrt{b/a} \quad \text{at } b > 0; \\ C_1 \exp(-\lambda x) + C_2 \exp(\lambda x), & \lambda = \sqrt{-b/a} \quad \text{at } b < 0 \end{cases} \tag{20}$$

and the function $\psi(t)$ is described by the following delay differential equation:

$$\tau_1 \psi''(t) + \psi'(t) = b\psi(t) + F\big(\psi(t) - \psi(t - \tau(t))\big).$$

4.2 Equation with constant delay

Now we consider Eq. (18) when $\tau(t)$ = const.

1. In this case Eq. (18) yields the separable solution as the sum of the functions of different arguments of the form of Eqs. (19)-(20), and the function $\psi(t)$ is described by the following delay differential equation:

$$\tau_1 \psi''(t) + \psi'(t) = b\psi(t) + F\big(\psi(t) - \psi(t - \tau)\big).$$

2. At $b = 0$, Eq. (18) yields the separable solution that is quadratic with respect to x:

$$u = C_1 x^2 + C_2 x + \psi(t), \tag{21}$$

where the function $\psi(t)$ is described by the following delay differential equation:

$$\tau_1 \psi''(t) + \psi'(t) = 2C_1 a + F\big(\psi(t) - \psi(t - \tau)\big).$$

3. The solution to Eq. (18) that generalizes solution of the form Eq. (19) has the form

$$u = \varphi(x) + \theta(t), \quad z = \beta x + \gamma t, \tag{22}$$

where $\varphi(x)$ is determined by Eq. (20) and $\theta(z)$ is described by the delay ordinary differential equation:

$$(\tau_1 \gamma^2 - a\beta^2)\theta''(z) + \gamma\theta'(z) = b\theta(z) + F(\theta(z) - \theta(z - \delta)) = 0, \quad \delta = \gamma\tau.$$

At $b > 0$, Eq. (22) describes the nonlinear interaction between a periodic standing wave and a traveling wave.

4. At $b = 0$, the solution of Eq. (18) that generalizes Eq. (21) has the form

$$u = C_1 x^2 + C_2 x + \theta(t), \quad z = \beta x + \gamma t,$$

where the function $\theta(z)$ is described by the following delay ordinary differential equation:

$$(\tau_1 \gamma^2 - a\beta^2)\theta''(z) + \gamma\theta'(z) = 2C_1 a + F(\theta(z) - \theta(z - \delta)) = 0, \quad \delta = \gamma\tau.$$

5. Eq. (18) yields the degenerate generalized separable solution

$$u = t\varphi(x) + \psi(t),$$

where $\varphi(x)$ is determined by Eqs. (20) and $\psi(x)$ is described by the linear ordinary differential equation:

$$a\psi''(x) + b\psi(x) + F(\tau\varphi(x)) - \varphi(x).$$

More complex solutions to Eq. (18) can be derived using the following property. Let $u_0(x, t)$ be a solution to nonlinear Eq. (18) and $v = V_1(x, t; b)$ be any τ-periodic solution to linear Eq(9). In that case, the sum

$$u = u_0(x,t) + V_1(x,t,b) \tag{23}$$

is the solution to Eq. (18). The form of the function $V_1(x, t; b)$ is determined by Eqs. (11)–(12). For example, the traveling wave solution $u_0 = u_0(\alpha x + \beta t)$ can be used in Eq. (23) as the solution to nonlinear Eq. (18).

5. Conclusions

For hyperbolic diffusion-reaction equations with a time delay, exact solutions are obtained in an exponential form with an increment computed from a transcendental equation of a special type through delay times. New exact solutions of this equation are found in the form of a nonlinear superposition of two different traveling waves. The form of exact solutions, satisfying initial-boundary problems, is established. Certain exact solutions are described for more complex nonlinear reaction–diffusion equations such as those with variable delay of the general form $\tau = \tau(t)$.

The derived exact solutions contain free parameters (in some cases, there can be any number of these parameters) and can be used to solve certain model problems and test approximate analytical and numerical methods for solving similar or more complex nonlinear delay partial differential equations.

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