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Maximizing Profit Using Benders Decomposition in Two-Stage Stochastic Water Distribution Network

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Profit is the primary intention that drives any industry. It is essential to incorporate the demand uncertainty in the water distribution industry to ensure maximum profit. This paper deals with the development of a two stage stochastic approach methodology. Initially a general two stage stochastic model is developed further a model based on Benders Decomposition is developed and the results are compared. The objective of the proposed method is to calculate the maximum profit that can be achieved by satisfying the varying demands of a site using water sources. The initial model is to formulate a two-stage stochastic deterministic equivalent model. Before the reality of the uncertain data is clear, a decision must be made in the first stage. The first stage's optimal solution is fixed, and only then can the values of the uncertain parameters be determined. The second method applies a simple benders' decomposition to the two-stage stochastic framework. Benders decomposition is a technique that helps in solving huge linear programming problems. This paper develops this method to simultaneously optimize the first set of variables (amount of water to be produced and transported) and second stage variables (*Supply* and *Waste*). The model is broken into a master problem and multiple subproblems. The solutions to these subproblems define the constraints of the master problem. The master problem is solved again and again until convergence. Both methodologies are illustrated with an example case study, and the results are compared. Benders Decomposition method took 50 iterations to converge and gave a profit value of $18,037.75. The deterministic equivalent model took 26 iterations to give the same value of profit.

1. Introduction

Water is one of the major resources in process industries (Chaturvedi and Manan, 2021). Water resource consumption has been identified as one of the important global environmental issues of the 21st Century (Kumawat and Chaturvedi, 2020). Al-Redhwan et al. (2005) developed an approach based on Sensitivity analysis and Stochastic Programming to develop flexible and resilient process water networks.

Stochastic Programming is a framework for describing uncertainty in optimization issues. A stochastic program is an optimization problem in which some or all of the problem parameters are unknown but follow well-defined Probability distributions. This paradigm differs from Deterministic optimization, which assumes that all problem parameters are known precisely. Stochastic programming aims to develop a solution that optimizes some criteria set by the decision-maker while also accounting for the issue parameters' uncertainty. Many real-world decisions contain uncertainty that's why stochastic programming has utility in various fields, including banking, transportation, and energy optimization.

Among stochastic programming models, the two-stage stochastic programming model is applied in water resource allocation due to its unique advantages. The central notion behind two-stage stochastic programming is that (optimal) choice should be made using data that is accessible at the time the decision is made, rather than relying on future observations. The first stage shows the need to make quick judgments. The second stage reflects judgments that should be taken in the future, taking into account a range of probable scenarios, each of which provides a plausible manifestation of the unknown facts. The purpose is to determine the least expensive option, including the estimated costs of the first and second stages. The uncertainty about the problem parameters is modelled by a limited number of subproblems (scenarios), weighted by their occurrence

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probability, and containing a local representation of the problem uncertainties. By obtaining the optimal solutions for each scenario, one could expect to find similarities and trends among the scenarios to develop a solution that holds a good trade-off under all scenarios. Damsleth et al. (1992) proposed a two-stage stochastic model applied to a North Sea reservoir. Ling et al. (2020) adopted a novel two-stage fuzzy stochastic programming approach for the water resource allocation problem capable of addressing uncertainties with both possibility and probability distribution. Fu et al. (2018) developed an agricultural multi-water source allocation model, consisting of stochastic robust programming and two-stage random programming and introducing interval numbers and random variables to represent the uncertainties, which was proposed for the optimization of irrigation water allocation in Jiamusi City of Heilongjiang Province, China. Ling et al. (2017) established an inexact two-stage stochastic programming model was developed for supporting regional water resource allocation management under uncertainties. However, these problems give up mathematical complex formulation.

In this context, this paper uses decomposition of problem. The decomposition refers to splitting a mathematical programming problem (also known as a constrained optimization problem, of which linear programming is an exciting subset) into smaller, more manageable problems that can then be re-integrated to obtain an overall solution. Benders (1962) originally applied the idea of decomposition to mixed-integer programming problems. These problems are constrained optimization problems in which some variables can take real values, and others may only take integer values. Benders used decomposition to split the problem into a pure integer programming problem and a pure linear programming problem that could be solved iteratively, in turn, to solve the overall problem.

2. Problem definition

This paper deals with the development of a methodology for targeting profit for the water distribution network. The schematic is shown in Figure 1. The problem definition is as follows:

- There is a set S of sources where each source has a fixed maximum purification capacity.
- There is a set D of sites where each site has some uncertain demand which has to be fulfilled.
- There is a set A of individual scenarios that signify the nature of demand.
- The main aim is to maximize the profit generated by the distribution of water to the different sites by optimizing the transport, purification, and waste costs.
- Before the actual demand is observed, we suppose that pure water is transferred from the source to the site. Actual demand is known after the water arrives at each place.
- If demand isn't met, supply is depleted. The remaining water must be disposed of if the carried water exceeds the ultimate demand. The cost of wastewater removal is known.

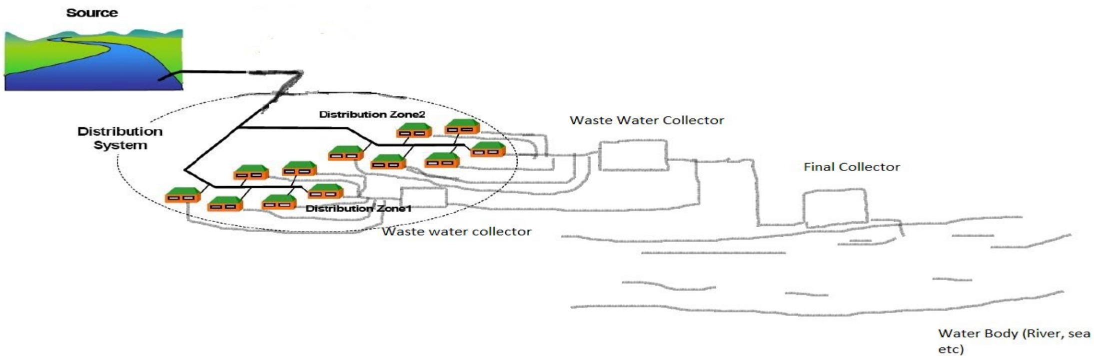

Figure 1: Schematic of the problem definition

3. Models

Different models of two-stage stochastic optimization and benders decomposition are explained in this section.

3.1. Two-Stage Stochastic Optimization-Core Model

The two-stage stochastic optimization core model is extensively analyzed and is given by Eq(1) as objective)) and subject to constraints in Eqs(2-4).

$$Min_x\ z = c^T x + E[Q(x, \omega)] \quad (1)$$

$$s.t. \quad Ax = b\,, \quad x \geq 0 \tag{2}$$

$$where, Q(x, \omega) = Min_y \; q_\omega^T y(\omega) \tag{3}$$

$$s.t., T_\omega x + W_\omega y(\omega) = h_\omega \tag{4}$$

Let *x* and *y* be two variables, and ω be the set of all possible realizations of the unknown data is given by Ω, Ω={ω_1, ,ω_s} ⊆ $\mathbb{R}^r$ where *r* is the number of random variables representing the uncertain parameters. Eq(2) define the problem in the first stage, whereas Eq(3) and Eq(4) define the problem in the second stage. In the first stage, *x* is the decision variable, c^T is the objective function's cost coefficients, and $E[Q(x, \omega)]$ is the expected value of the second stage problem's optimal solution. The coefficients are denoted by A, and the right-hand side of the first stage restrictions is denoted by b. The decision variable is *y*, the transition matrix is T, the recourse matrix is W (cost of recourse), and the right-hand side of the second stage constraints is h. It is worth noting that the second stage's parameters and decision variables are all reliant on the stochastic data's specific realization $\boldsymbol{\omega}$. The objective ***z*** is a random variable as it is a function of $\boldsymbol{\omega}$. Stochastic solvers automatically optimize the expected value of the objective variable ***z*** since a random variable cannot be maximized.

3.2. Two-Stage Stochastic Optimization-Deterministic Equivalent

Building and solving the deterministic equivalent is one of the most prevalent ways of solving a two-stage stochastic LP. Assume the uncertain parameters have a (finite) discrete distribution and that each scenario *s* occurs with probability $P(\omega_s) = p_s$for all s=1,…., S and $\sum_s p_s = 1$. $E[Q(x, \omega)] = \sum_s p_s q^T y_s$, with y_s denoting the best second-stage option for scenario *s*. After that, the deterministic equivalent is given by Eq(5) subject to the set of constraints Eq(6).

$$c^T x + p_1 q^T y_1 + p_2 q^T y_2 + \cdots \; + p_s q^T y_s \tag{5}$$

$$s.t. \; Ax \;=\; b \tag{6}$$

$$T_s x \; + W_s y_s = \;\; h_s \tag{7}$$

$$where, \;\; x \in \mathbb{R}^{\mathrm{n}}\,, y_1 \in \mathbb{R}^{\mathrm{n}}\,, y_2 \in \mathbb{R}^{\mathrm{m}}\,, y_s \in \mathbb{R}^{\mathrm{m}}, s \in S \tag{8}$$

3.3 Benders Decomposition

Original optimization problem structure;

$$\min c'\, y + Q(y), \;\; s.t. \;\; Ay \leq b, y \geq 0 \tag{9}$$

The function $Q(y)$ in Eq(9) is difficult and expensive to assess because it depends on finding an optimal solution to all sub-problems. Remember that we are seeking a rational approach to applying restrictions to some master problem, a less constrained version of the complete problem, to come up with reasonable estimates of y without solving the whole thing. The initial master problem in Benders decomposition is the first stage problem. The restrictions we apply to it correspond to the feasibility and optimality of the (second-stage) sub-problems. It is anticipated that these estimates will converge to the best solution by introducing constraints judiciously while requiring us to tackle a minor problem. Next, from the above objective function, the complicated *Q(y)* term with a variable *θ*. Bounds are placed by utilizing the information extracted from the subproblems known as optimality cuts. The assumption is that, while these bounds will eventually recreate the function *Q(y)*, they will not do so initially, making the problem significantly more straightforward. We hope to identify the best solution before the original *Q(y)* function is reproduced. Consider the sub-problem for the case when the random processes *u* takes the realization u_ω. Call this sub-problem *ω*. Eq(10) and Eq(11) state the primal sub problem and Eq(12) and Eq(13) state the dual subproblem. Dual of a minimization problem would be a maximization problem.

Primal Sub-problem:

$$Q(y, u_\omega) = \min_v q(u_\omega)' v \tag{10}$$

$$\text{s.t.} \quad W(u_\omega) v \;\leq\; h(u_\omega) - \; T(u_\omega) y \;\;, \; v \geq 0 \tag{11}$$

Dual Subproblem

$$Q(y, u_\omega) = max \;\; \pi^{'} (h(u_\omega) - T(u_\omega) y) \tag{12}$$

$$\text{s.t.} \quad \pi^{'}\, w(u_\omega) \leq \; q(u_\omega)^{'}, \pi \;\geq\; 0 \tag{13}$$

y^i is the value of the first stage decision that comes from the *i^th^* iteration of the master problem. Let π_ω^i be the optimal values of the variables in the dual subproblem (Eq(14)).

$$Q(y^i, u_\omega) = (\pi_\omega^i)'(h - Ty^i) \qquad (14)$$

By duality theory, it is known that the right-hand side of Eq(15) corresponds to the linear space that forms part of $Q(y^i, u_\omega)$ at y^i. A constraint in the primal sub-problem (Eq(10)) produces the linear piece of $Q(y^i, u_\omega)$.It is necessary to remember that when we form the dual, each constraint in the primal problem leads to a dual variable. Each primal variable leads to a dual constraint, and the constraints from the primal problem are multiplied by one of the dual variables to generate the dual problem's objective. What we are doing with Eq(15) is using constraints from the subproblem to produce a lower bound on its objective $Q(y^i, u_\omega)$.Eq(16) represents the bound which is placed on *θ*.

$$Q(y_2, u_\omega) \geq (\pi_\omega^i)'(h - Ty_2) \qquad (15)$$

$$\theta, Q(y) \geq \sum_\omega p_\omega (\pi_\omega^i)'(h - Ty) \qquad (16)$$

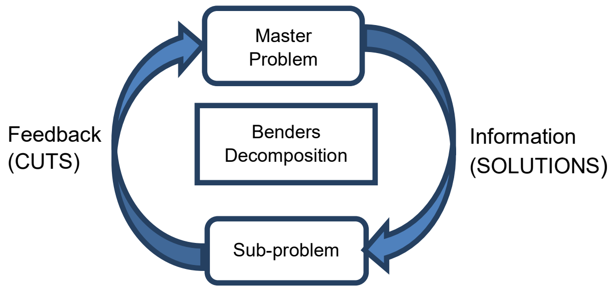

Figure 2: Schematic representation of the bender's decomposition method

4. Mathematical formulation

$$Profit = \sum_{j,\omega} p_j prob_\omega Supply_j - \sum_{i,j} c_{i,j} TransportSite_{i,j} - \sum_i c_i PurifiedWater_i - \sum_{j,w} c_j prob_\omega Waste_j \qquad (17)$$

$$PurifiedWater_i = \sum_j TransportSite_{i,j} \qquad (18)$$

$$PurifiedWater_i \leq Capacity_i \qquad (19)$$

$$Received_j = Supply_{j,\omega} + Waste_{j,\omega} \qquad (20)$$

$$Supply_{j,\omega} \leq demand_j, Supply_{j,\omega}, TransportSite_{i,j}, PurifiedWater_i, Waste_{j,\omega} \geq 0 \qquad (21)$$

The proposed methodologies are illustrated with an example consisting of multiple sources and sites which require water. The main objective of both methods was to maximize profit given by Eq(17). Eq(18) represents the first stage equation. Eq(19) is the purification capacity constraint at a particular source. Eq(20) is the second stage equation. Eq(21) is the demand constraint and also implies the non-negativity of the variables.
$Supply_j$ is the supplied water that is used at a site j, $TransportSite_{i,j}$ is the amount of water that is transported from a source *i* to site j, $PurifiedWater_i$ is the production of purified water at source *i*, $Waste_j$ is the wastewater which was extra(not used) at a site j, p_j is the supply price, $c_{i,j}$is the unit transportation cost from source *i* to site *j*, c_i is the unit purification cost, c_j is the cost of removal of wastewater, $capacity_i$ is the maximum amount of water that can be purified at factory *i*, $demand_j$ is the demand at site j.
The Primal Sub-problem is formulated as:

$$Objective = Max\{\sum_j p_j Supply_j - \sum_j c_j Waste_j\}\ OR\ \ Min\{\sum_j c_j Waste_j - \sum_j p_j Supply_j\} \qquad (22)$$

$$Supply_j + Waste_j = received_j \qquad (23)$$

$$Supply_j + SlackSupply_j = demand_j \qquad (24)$$

Supply and *waste* are the two variables of the second stage which are inter-dependent on each other and based on the uncertain demand variable. Eq(22) is the primal sub-problem objective and Eq(23) and (24) are the constraints. The Dual of the Sub-problem:

$$Max\ \sum_j received_j pi\_supplied + \sum_j demand_j pi\_supplymax \qquad (25)$$

$$pi\_supplied_j \ + pi\_supplymax_j \ \leq -p_j \tag{26}$$

$$pi\_supplied \ \leq c_j \tag{27}$$

$$pi\_supplymax_j \leq 0 \tag{28}$$

Eq(25) is the dual subproblem objective. Eq(26), Eq(27) and Eq(28) are the constraints. Here $\boldsymbol{pi\_supplied}$ and $\boldsymbol{pi\_supplymax_j}$ are the two dual variables corresponding to the primal sub-problem.

5. Case Study

Table 1 gives the demand data. Table 2 gives the data of cost of transportation. The model is solved in GAMS version 24.8.5 (CPLEX solver). Sets of sources, sites, and scenarios are declared from the start. Table 1 shows the demand outcomes and their probability. A joint probability is estimated for these 65,536 scenarios. The probability in Table 1 is considered to be independent. The Benders master problem is formed. The maximum number of iterations to be executed is provided, as well as a dynamic subset. Positive variables and equations are only specified for the first stage. Variable theta is used to represent the entire second stage function. The *cutconst* and the *cutcoeff* are two parameters that are established and used in optimality cuts. This master problem is then modeled. Next, the benders subproblem is formulated. Positive variables and equations associated with the second stage are defined. Note that the received value is constant. This is the decision passed from the first stage to the second stage. The primal subproblem and the dual subproblem are formulated using the simplex method. Apply the Benders algorithm. The first step is to solve the master problem for minimization, which is set to having no optimality cuts. The upper and lower bound for theta are defined as +inf and -inf. Now iterations are started. By duality theory, the optimal solution to the primal subproblem would be equal to the dual subproblem. When the master problem was solved without cuts, the value of the *received* variable is obtained which is used to solve the dual variables for each joint scenario. These dual variables define the optimality constraints that are passed back to the master problem. The upper bound is updated, including the dual subproblem solutions. A convergence test is introduced, i.e., If the gap between the best upper and lower bound is less than epsilon, the procedure terminates. The master problem is solved with added new constraints that update the lower bound, *received* variable, and objmaster. This process continues to the next iteration, and upper bound and lower bounds are continuously updated every iteration. The formulation has 11 single equations and 24 single variables and it takes 50 iterations to converge. The deterministic equivalent version is solved directly by specifying the equations directly in GAMS using CPLEX solver. The deterministic version has 19 single equations and 107 single variables and solution time and solution converges in 26 iterations. After solving the problem by benders, the profit value is \$18,037.75. Minus value is since it is the minimum value of (- Profit) equivalent to the maximum of (+profit). Compared with the direct implementation of the deterministic equivalent model, a similar value of \$18,037.75 was obtained.

Table 1: Possible Consequences for demand (m^3) and their probabilities

Sites	a1	a2	a3	a4
D1	120,0.15	140,0.30	160,0.15	170,0.40
D2	80,0.10	110,0.20	125,0.25	140,0.45
D3	200,0.05	230,0.10	260,0.35	275,0.50
D4	160,0.07	215,0.13	240,0.34	260,0.46
D5	180,0.08	195,0.22	205,0.30	255,0.40
D6	170,0.09	190,0.31	210,0.25	240,0.35
D7	190,0.16	220,0.14	250,0.50	280,0.20
D8	215,0.15	225,0.15	235,0.20	245,0.50

Table 2: Transport Cost from sources to sites ($/m³)

	D1	D2	D3	D4	D5	D6	D7	D8		D1	D2	D3	D4	D5	D6	D7	D8
S1	2.30	5.1	3.7	4.45	5.23	7.43	8.34	9.12	S6	9.92	9.94	5.59	4.42	2.22	2.23	3.34	3.39
S2	1.40	2.50	1.80	1.96	4.25	5.40	6.45	7.24	S7	5.56	5.51	9.91	8.82	8.83	8.84	5.56	6.69
S3	3.56	3.80	2.40	5.76	6.78	7.89	5.67	6.67	S8	8.84	8.86	8.97	9.21	9.28	9.30	9.56	9.45
S4	5.56	6.67	7.78	8.82	3.34	4.41	5.51	5.57	S9	6.71	8.81	9.95	3.45	5.51	5.78	5.69	5.90
S5	4.45	4.43	9.97	7.78	8.88	9.99	7.77	6.65	S10	12.4	13.6	14.6	16.8	17.9	14.6	3.45	3.67

Unit purification Cost=29 $/m^3$, Supply Price = 44 $/ m^3$, Cost of removal of wastewater =14 $/ m^3$, Scenarios-{a1,a2,a3,a4}, Sources- {S1,S2,S3,S4,S5,S6,S7,S8,S9,S10}, Sites- {D1,D2,D3,D4,D5,D6,D7,D8}, The total number of sources and sites are 10 and 8. Total scenarios=4^8=65,536.

6. Conclusions

In this paper, two methodologies are presented. The objective of the proposed methods is to maximize profit in the water distribution network in a dual-stage stochastic environment. The first method formulates the two-stage stochastic deterministic equivalent model. The other method applies a simple Benders' decomposition to the two-stage stochastic framework. Benders Decomposition method took 50 iterations to converge and gave a profit value of $18,037.75. The deterministic equivalent model took 26 iterations to give the same value of profit. Bender's decomposition can be more fruitful for a more complicated multi-stage stochastic problem in which there is a significant increase in the number of scenarios and variables. The results obtained via the two methods are the almost same. Current research work is directed toward the development of a multistage stochastic model using Bender's decomposition.

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