Geometric models for color perception

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ABSTRACT

In 1962 and 1974, respectively H. Yilmaz and H.L. Resnikoff published two groundbreaking articles about color perception, which were ignored by the scientific community. Yilmaz showed the striking analogy between Lorentz transformations and the modification of color perception under illuminant changes. On the other hand, Resnikoff, using mathematical techniques coming from theoretical physics, studied the possible geometrical representations of a homogeneous space of perceived colors, i.e. a space in which all the elements have “the same importance”. Both works come up to the same conclusion: the structure of the space of perceived colors can be better characterized through hyperbolic geometry, while usual color spaces have a Euclidean structure. In this work, we show how a modern revision of these important articles allows us to highlight a correlation between the colorimetric attributes and some objects of special relativity theory and quantum mechanics, opening innovative perspectives in the theoretical comprehension of perceptual phenomena related to human chromatic vision. A remarkable result of this new formalism concerns the retinal chromatic encoding expressed by the sum of an achromatic signal and two opponent chromatic signals (typically called red-green and yellow-blue). This looks as an intrinsic description of a so-called “color state”, in contrast to what happens in natural image statistics, where such an encoding is not an intrinsic result of the theory, but it is obtained through a principal component analysis.

KEYWORDS Yilmaz, Resnikoff, Jordan Algebras, relativity, quantum mechanics, mathematical models for color perception.

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1. Introduction

The scope of this paper is to give a brief and simple overview, with dissemination purposes, about a novel program of formalization of colorimetry in which both geometry and techniques typically used in mathematical physics play a fundamental role. Due to space limitations, and in order to keep the storytelling as simple as possible, we will leave many details to open access references that the interested reader may wish to consult.

2. Yilmaz's relativistic model

The value of Yilmaz's work (Yilmaz, 1962) lies in the interpretation of color perception as a relativistic phenomenon. Indeed, as Einstein showed up that space and time are relative to the single inertial observers (i.e. observers moving with constant speed with respect to each other), in the same way Yilmaz states that the colors perceived by an observer adapted to a certain broadband illuminant are relative to it. Thence it is possible to use the mathematical tools typical of Einstein's special relativity theory to model color perception. On one hand this opens new paths for a deeper comprehension of what is a color space and which are the most suitable coordinates to identify a perceived color, on the other hand it provides a mathematical formalization of the space's transformations under changes of the broadband illuminant to which the observer is adapted. This last aspect makes Yilmaz's model easy to adapt for applicative purposes, in particular for color correction of digital images.

2.1. The coordinates

As it is well-known from (Wyszecki and Stiles, 1982), there are strong physiological and psychophysical reasons behind the statement that the space of perceived colors is a 3-dimensional cone.

Thence every perceived color can be univocally identified by three coordinates. The wide range of color spaces proposed for digital and industrial applications clearly shows that the choice of these three parameters is far from being trivial.

For Yilmaz, a trichromatic observer adapted to a certain illuminant I is able to identify the colors he/she perceives by two chromatic coordinates and an achromatic one.

Let us fix three orthogonal axes, depicted in Fig. 1. The origin of the three axes corresponds to black, denoted by K, on each axis there is the value associated to a certain coordinate. Let us call these three coordinates α, β and γ.

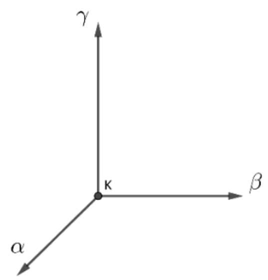

Fig. 1. Axes α, β, γ.

The achromatic coordinate γ goes from black to white, through a grayscale, while the chromatic coordinates belong to the plane α, β (we will call it chromaticity plane) shown in Fig. 2. The first chromatic coordinate is the angle ϕ, called hue, the second one is the radial coordinate ρ, called chroma.

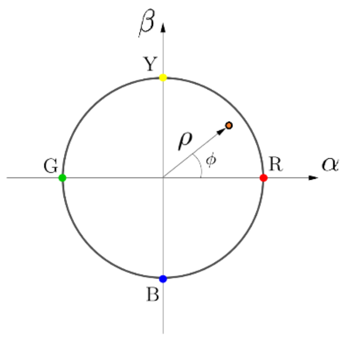

Fig. 2. Chromaticity plane.

The axes α and β represent respectively the hue oppositions red-green and blue-yellow, proposed for the first time by Hering. The existence of these two oppositions is the reason why it is impossible to perceive e.g. a reddish green or a yellowish blue. In this paper, we will follow the simplified Yilmaz framework in which well-known perceptual effects that show interdepencence between chromatic attributes are ignored. As a consequence, the space in which we will work has cylindrical shape, see Fig. 3. Using the coordinates α and β is mathematically equivalent to using the coordinates ϕ and ρ, but perceptually less immediate.

A colorimetric attribute of fundamental importance, dependent to the ones defined above, is the saturation σ. It denotes the purity level of a color and it is defined as the ratio between chroma and the achromatic coordinate $\sigma = \rho / \gamma$. The existence of a maximal perceivable saturation (i.e. a maximum attainable degree of purity that a perceived color can have) leads us to the exclusion of the points of the cylinder that do not belong to the cone of slope ∑ depicted in Fig. 4.

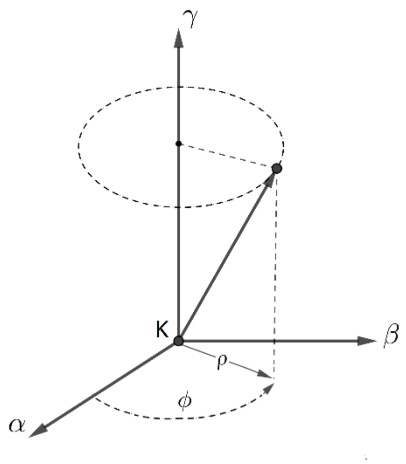

Fig. 3. Cylindrical coordinates.

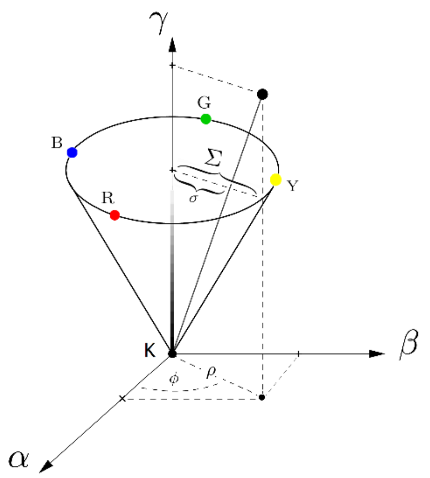

Fig. 4. Cone of perceived colors. Note that, when γ=1, we have that σ = ρ.

The perception of a stimulus constituted by a monochromatic light (e.g. a red laser) will have maximal purity, hence saturation ∑.

2.2. *The two rooms experiments*

Yilmaz motivates the introduction of relativistic concepts in a colorimetric framework through three experiments. It must be stressed that Yilmaz did not give quantitative data nor apparatus description for his experiments, thus, a doubt about the fact that they have actually been implemented still remains.

Let us consider two different broadband illuminants, we will denote them by I and I'. Let us call α, β, γ (α', β', γ', respectively) the coordinates that an observer adapted to I (to I' respectively) associates to the stimuli that he/she perceives. Yilmaz's aim is to show how the coordinates α, β, γ are transformed into the coordinates α', β', γ'.

Let us suppose we have two adjacent rooms completely painted in white. In each room, different kinds of light sources can be posed. The two rooms are separated by a wall with a tiny hole through which an observer placed in the first room is able to perceive light stimuli posed in the second room and vice-versa. Hence the presence of the hole allows the observer to perceive light stimuli belonging to an environment to which he/she is not adapted.

A piece of white paper is divided into two parts, each of them is posed in one of the two rooms. We are going to introduce just the two more emblematic experiments.

<u>Experiment 1: the perception of white is relative</u>
In this first experiment, depicted in Fig. 5, I is placed in the first room and I' in the second one. In a first phase the observer is placed in the first room and adapted to I. He/she perceives the piece of white paper placed in his/her same room as white; while the other half, placed in the second room and enlightened by I', has greenish hue and small saturation σ.

The second phase of the experiment is identical, but the roles of the two rooms are inverted: the observer is placed in the second room and adapted to I'. He/she perceives the piece of white paper placed in the room with him/her as white, while the other one, posed in the first room is perceived as having reddish hue and the same small saturation σ.

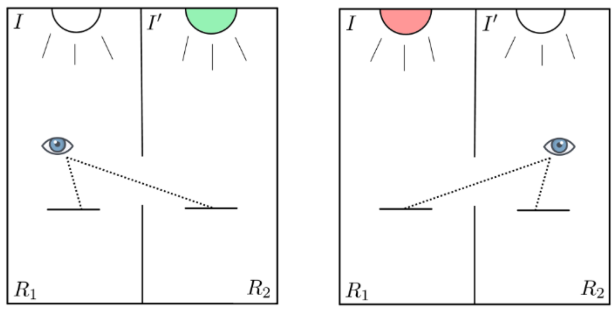

Fig. 5. Experiment 1.

<u>Experiment 2: the invariance of the spectral red</u>
In the second room is placed a monochromatic red light source, while the first room is enlightened by I in a first phase and by I' in a second phase. In both phases the observer is placed in the first room and adapted to the broadband illuminant enlightening it. He/she observes that, in both cases, the piece of white paper, posed in the second room and illuminated by the red laser, is perceived as having the same red hue and the same maximal saturation ∑, see Fig. 6.

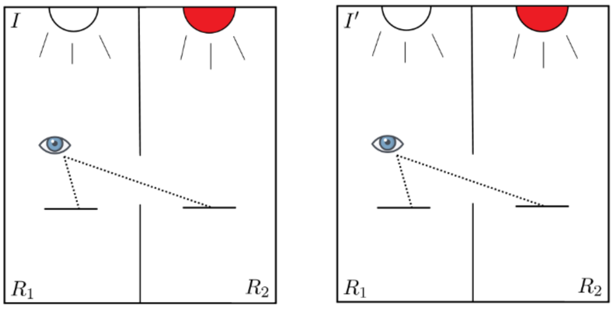

Fig. 6. Experiment 2.

Formalizing the information given by the three experiments, the coordinate transformation (linear in the variables α, β, γ) that Yilmaz obtained is the following:

$$\begin{cases} \alpha' = \frac{\alpha - \sigma\gamma}{\sqrt{1-(\sigma/\Sigma)^2}} \\ \beta' = \beta \\ \gamma' = \frac{\gamma - (\sigma/\Sigma^2)\alpha}{\sqrt{1-(\sigma/\Sigma)^2}} \end{cases}$$

The details needed to understand how to arrive to this kind of transformations are available in (Prencipe et al., 2020). In the same paper, the reader can find the description of the third Yilmaz experiment, which is more involved to discuss and not relevant for the present paper.

2.3. *Parallelisms with relativity*

The "relativistic jump" is attained recognizing in the transformation written above a so-called Lorentz boost, which allows us to find relativistic analogues for all the quantities mentioned up to now.

In the theory of special relativity, the measure of space and time is relative to the observer, hence every observer has his/her own coordinate system with respect to which he/she can measure space and time. In the simplest case of inertial observers, i.e. observers moving with constant speed with respect to each other, the transformations converting the coordinates of one observer into the coordinates of another one, are the Lorentz boosts. Thence an observer adapted to a broadband illuminant corresponds to an inertial observer in special relativity theory.

Consequently, there will be parallelisms between space-time coordinates and the coordinates of the perceptual space described in subsection 2.1. In particular, the achromatic coordinate γ corresponds to time, while the chromatic coordinates α, β (or, equivalently, ϕ and ρ) are the analogous of a two-dimensional physical space.

From the definition of saturation as σ = ρ / γ, it is easy to deduce its relativistic analogue. Indeed, it should be a ratio between a spatial and a temporal quantity, so a constant velocity. The maximum perceivable saturation ∑ corresponds to the maximum attainable velocity, i.e. the speed of light c. As a consequence of this, it is clear that Yilmaz's second experiment is nothing but a colorimetric reinterpretation of the fundamental principle of special relativity theory stating that light speed is the same for all inertial observers. For further details see (Prencipe et al., 2020).

3. Resnikoff's model and its quantum reinterpretation

As we have seen in section 1, Yilmaz's model introduces in the context of color perception analysis concepts and tools typical of special relativity theory. In this second section, we will see how Resnikoff's model allows us to see in color perception a quantum phenomenon.

Resnikoff's article (Resnikoff, 1974), gone almost unnoticed by the scientific community as Yilmaz's one, is an extraordinary (and rare) example of something that we could call "theoretical psychophysics", because he used the typical flow of thinking and mathematical techniques of theoretical physics, but applying them to the concept of perceived color, i.e. a psychophysical attribute.

More precisely, he started his analysis from the so-called Schrödinger's axioms (Schrödinger, 1920), adding a further fundamental one: the homogeneity axiom, and determining mathematically which geometric structures satisfy all the axioms. Notice that the pattern followed by Resnikoff, which characterizes the works of modern theoretical physics, is substantially different from a mere procedure of selection by interpolation, that is a work of minimization of the discrepancies with the experimental data.

Resnikoff showed that only two geometric structures are compatible with Schrödinger's axioms and his homogeneity axiom: the first one is the canonical Helmholtz-Stiles space that has many different practical expressions like LMS, RGB, XYZ, etc. all of them geometrically equivalent; the second one is a hyperbolic structure totally new in color theory. It is exactly this last one that allows a quantum interpretation, as we will detail in the following.

3.1. *The axiomatic construction of a homogeneous color space*

Erwin Schrödinger, well known for his works in quantum mechanics, dedicated many years of his scientific career to the study of color. In 1920 he wrote a series of very elegant works summarizing in a mathematically coherent framework the main results concerning color obtained by scientists like Newton, Grassmann, Maxwell and Helmholtz.

Schrödinger's axioms can be summarized stating that the space of perceived colors of trichromatic observes has the geometric structure of a convex regular cone of dimension 3.

The fact that the space of perceived colors is a cone means that a positive multiple of a perceived color (i.e. a brighter version of the color) is still a perceived color (note that this is an idealization, because the phenomenon of saturation of the photoreceptors implies that the cone is not infinite, but truncated at the glare threshold). The convexity property means that, inside the cone, the segment joining any couple of perceived colors is made up by perceived colors (this was proved by Grassmann). Regularity is a technical property that can be translated into practical terms into the statement that the cone of perceived colors has a vertex corresponding to black. Finally, the dimension of the cone is a consequence of the existence of 3 kinds of retinal cones which start the chain of neural events leading to color vision.

Resnikoff, starting from the observation that no color is "special" with respect to the others and that, thanks to the well-known phenomenon of chromatic induction, it is possible to modify the perception of any chromatic stimulus just embedding it in an appropriate background, came to the conclusion that the following postulate holds: the space of perceived colors is locally homogeneous, that is it exists an invertible transformation which maps any color to any "sufficiently similar" other color.

It can be easily proven that this postulate, justified by the induction phenomenon, together with the convexity of the cone, implies its global homogeneity, which is exactly the mathematical property characterizing the spaces where no point is special because we can pass from one point to any other one through an invertible transformation. This is the reason of the choice of adjective "homogeneous".

In summary, putting together Resnikoff's axiom and Schrödinger's ones, we can conclude that the space of perceived colors has the structure of a convex regular and homogeneous cone of dimension 3. There are only two kinds of cones of dimension 3 satisfying all these properties: the first, and the simplest one, let us denote it by P', is the set of all the nonnegative real numbers Cartesian to itself three times, which is exactly the Helmholtz-Stiles space, canonically used in colorimetry. The second one, more complex and interesting, denoted by P'', is given by the Cartesian product of the set of positive real numbers and a hyperbolic space which can be characterized in many different ways, some of them are easy to visualize, like the hyperboloid embedded in the real three-dimensional space, the upper-half plane or the Poincaré disk (i.e. the open unit disk in the real plane), others are more abstract, like, for example the space of the real symmetric positive-definite 2x2 matrices having determinant equal to 1 or the quotient space SL(2,R)/SO(2). For more details see (Provenzi, 2020).

In the following subsection, we are going to show that this second space is the most interesting one from a theoretical point of view and for the consequences related to the quantum interpretation of color vision.

3.2. *Jordan algebras and the link with quantum mechanics*

In the articles (Berthier and Provenzi 2019; Berthier, 2020) a fact of fundamental importance is stressed: the so-called classification theorem of Jordan-von Neumann-Wigner states that the two structures found by Resnikoff for the space of perceived colors coincide exactly with the only two possible forms of a symmetric cone of dimension 3, where a cone is said to be symmetric if it is convex, regular, homogeneous, open and self-dual (a technical property which is not important to explicit here).

Moreover, Koecher-Vinberg theorem states that every symmetric cone is the so-called positive cone of a (formally real) Jordan algebra.

Without going into many specialized and complicated details of the theory of Jordan algebras, which will result to be merely notional, we just say that a Jordan algebra is a vector space endowed with a commutative, but not associative product called the Jordan product and that the Jordan algebra whose positive cone is P'' is the algebra A of the real symmetric 2x2 matrices with the Jordan product between two matrices A and B of A defined as: A○B=(AB+BA)/2.

Jordan algebras have a privileged role in the modern quantum theories, where the objects are the quantum observables of a system, in duality with their quantum states. Once again, an exhaustive treatment of these concepts should deserve much more space, see e.g. (Berthier and Provenzi, 2021), hence we just underline that it is the lack of associativity of the Jordan algebra that gives a quantum character to the description of the observables and the states of the system.

To make the theoretical ideas exposed up to now more concrete, let us now talk about a feature of this quantum model for color perception that we consider particularly meaningful. It is the fact that it is possible to represent, in a very natural way, a color state through the superposition of three so-called density matrices, indicated with $\rho(r,\vartheta)$, (i.e. positive definite and with unit trace) which represent an achromatic state and two states of chromatic opponencies red-green and yellow-blue, respectively, as expressed in the following formula:

$$\rho(r,\theta) = \rho_0 + \frac{r\cos\theta}{2}[\rho(1,0) - \rho(1,\pi)] + \frac{r\sin\theta}{2}[\rho(1,0) - \rho(1,\pi)].$$

This kind of description, perfectly coherent with the human color vision, as remarked in section 1, is obtained in the quantum model just passing to a parametrization in polar coordinates of the density matrices.

The color encoding performed by the human visual system comes out in a very natural way in the framework of the quantum model and there is no need to resort to an analysis “a-posteriori”, like it is done in the context of natural image statistics, where it is shown that the principal components of a wide dataset of natural images coincide with the triplet given by the achromatic axis and two chromatic axes having opponent colors.

4. Conclusions

The power of Yilmaz’s work lies in the fact that he gave the foundations to construct a relativistic theory of color perception. Clearly here we exposed just some of the possible aspects of special relativity theory translated in the colorimetric context. The analogies between the two theories are much more, and they hide questions deserving further and deeper investigations in the colorimetric context. Furthermore, there are numerous aspects that are well suited to immediate and concrete applications. Let us imagine we have a picture taken by a digital device which is not able to automatically adapt, like a human being, to the illuminant of the scene we want to capture. We can imagine that the uncorrected image as a light stimulus posed in the second room in Yilmaz’s first experiment, with an observer posed in the first room. Indeed, it corresponds to a perception devoid of adaptation, that can be easily corrected applying a suitable Lorentz boost.

As regards Resnikoff’s work, we can say that the theoretical clarity and the lucidity of his work have been crowned, after more than 40 years after its publication, by a surprising interpretation: color perception is well suited to be naturally described by the algebraic formalism of quantum theories.

In summary, the two “forgotten” articles of Yilmaz and Resnikoff, clearly posed the bases of a quantum-relativistic color theory capable of explaining into deep and mathematically rigorous terms the phenomena of human chromatic perception. Moreover, they add a further step towards the use of hyperbolic geometry in colorimetry as also mentioned by several other authors, e.g. (Farup 2014 and Lenz et al. 2005).

5. Conflict of interest declaration

The authors declare that there is no conflict of interest concerning the content of this article.

6. Funding source declaration

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8. Short biography of the authors

Nicoletta Prencipe - Nicoletta Prencipe is a PhD student working between the Mathematics Institute of the University of Bordeaux and the Huawei Technologies Nice Research Center in Sophia Antipolis. She works on the geometry of the space of perceived colors cosupervised by Edoardo Provenzi and Joseph Meehan.

Edoardo Provenzi - Edoardo Provenzi is a mathematical physicist working as full professor in the Mathematics Institute of the University of Bordeaux. His main research interest is the geometric modeling of visual perception, color in particular, and its application to image processing and vision.

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