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A Parameter Uniform Almost First Order Convergent Numerical Method for a Semi-Linear System of Singularly Perturbed Delay Differential Equations

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***Abstract*—In this paper an initial value problem for a semi-linear system of two singularly perturbed first order delay differential equations is considered on the interval (0,2]. The components of the solution of this system exhibit initial layers at 0 and interior layers at 1. A numerical method composed of a classical finite difference scheme on a piecewise uniform Shishkin mesh is suggested. This method is proved to be almost first order convergent in the maximum norm uniformly in the perturbation parameters.**

***Keywords*-Singular Perturbation problems, boundary layers, semi-linear delay differential equations, finite difference schemes, Shishkin mesh, parameter uniform convergence.**

I. Introduction

Singularly perturbed delay differential equations play an important role in the modelling of several physical and biological phenomena like first exit time problems in modelling of activation of neuronal variability [3], bistable devices [8] and evolutionary biology [6] and in a variety of models for physiological processes or diseases [9],[10] and [11]. These systems also find applications in Belousov- Zhabotinskii reaction (BZ reaction) models and the modelling of biological oscillators [6].

A model of tumor growth that includes the immune system response and a cycle-phase-specific drug presented in [13] is cited here. The model considers three populations: immune system, population of tumor cells during interphase and population of tumor during mitosis.

The governing equations of the system are

$$\frac{dT_I}{dt} = 2a_4T_M - (c_1I + d_2)T_I - a_1T_I(t-\tau)$$
$$\frac{dT_M}{dt} = a_1T_I(t-\tau) - d_3T_M - a_4T_M - c_3T_MI - k_1(1-e^{-k_2u})T_M$$
$$\frac{dI}{dt} = k + \frac{\rho I(T_I+T_M)^n}{\alpha+(T_I+T_M)^n} - c_2IT_I - c_4T_MI - d_1I - k_3(1-e^{k_4u})I$$
$$\frac{du}{dt} = -\gamma u$$

with

$$T_I(t) = \phi_1(t) \text{ for } t \in [-\tau, 0]$$
$$T_M(t) = \phi_2(t) \text{ for } t \in [-\tau, 0]$$
$$I(t) = \phi_3(t) \text{ for } t \in [-\tau, 0]$$
$$u(0) = u_0.$$

Here,
$T_I(t)$ - population of tumor cells during interphase at time t
$T_M(t)$ -population to tumor cells during mitosis at time t
$I(t)$ -population of immune system at time t
$u(t)$ -amount of drug present at time t
τ -the resident time of cells in interphase
d_2T_I, d_3T_M, d_1I - proportions of natural cell death or apoptosis
a_1, a_4 - the rate at which cells cycle are reproduce
c_i -losses from encounters of tumor cells with immune cells
$\frac{\rho I(T_I+T_M)^n}{\alpha+(T_I+T_M)^n}$ - non-linear growth of the immune population due to stimulus by tumor cells
k -constant rate at which the immune cells grow, in the absence of tumor cells
ρ, α, n -parameters depending on the type of tumor being considered and the health of the immune system.

Thus, an initial value problem for a system of semilinear delay differential equations is used to model tumor growth. Here, the parameters may take large values, for instance the value of k is 1.3×10^4 in the paper cited. In these cases, the system becomes singularly perturbed.

Motivated by this, in this paper, the following semilinear system of singularly perturbed delay differential equations is considered:

$$\vec{T}\vec{u} = E\vec{u}'(x) + \vec{f}(x, u_1, u_2) + B(x)\vec{u}(x-1) = \vec{0} \quad \text{on } (0,2], \quad \vec{u} = \vec{\phi} \text{ on } [-1, 0]. \tag{1}$$

For all $x \in [0,2]$, $\vec{u}(x) = (u_1(x), u_2(x))^T$ and $\vec{f}(x, u_1, u_2) = (f_1(x, u_1, u_2), f_2(x, u_1, u_2))^T$. E, $B(x)$ are 2×2 matrices. $E = \text{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ with $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$, $B(x) = \text{diag}(\vec{b})$, $\vec{b} = (b_1(x), b_2(x))$.
It is assumed that the nonlinear terms satisfy

$$\frac{\partial f_k(x)}{\partial u_k} \geq \beta > 0, \ \frac{\partial f_k(x)}{\partial u_j} \leq 0, \quad k, j = 1, 2, \ k \neq j \tag{2}$$

$$\min_{1 \leq i \leq 2} \left(\sum_{j=1}^{2} \frac{\partial f_i(x)}{\partial u_j} + b_i(x) \right) \geq \alpha > 0, \tag{3}$$

$$b_i(x) \leq 0, \ i = 1, 2 \tag{4}$$

for x in $[0,2] \times \mathbb{C}^2$ where $\mathbb{C} = C^0([-1,2]) \cap C^1((0,2]) \cap C^2((0,1) \cup (1,2))$.
These conditions and the implicit function theorem ensure that a unique solution $\vec{u} \in \mathbb{C}^2$ exists for the problem (1).
The solution $\vec{u}(x)$ has initial layers at $x = 0$ and interior layers at $x = 1$. Both the components u_1 and u_2 have layers of width $O(\varepsilon_2)$ and the component u_1 has an additional sublayer of width $O(\varepsilon_1)$.
For any vector-valued function $\vec{y}$ on $[0,2]$ the following norms are introduced:
$\| \vec{y}(x) \| = \max_i |y_i(x)|, i = 1, 2$ and
$\| \vec{y} \| = \sup\{\| \vec{y}(x) \| : x \in [0,2]\}$.
A mesh $\bar{\Omega}^N = \{x_i\}_{i=0}^N$ is a set of points satisfying $0 = x_0 < x_1 < ... < x_N = 2$.
A mesh function $V = \{V(x_i)\}_{i=0}^N$ is a real valued function defined on $\bar{\Omega}^N$. The discrete maximum norm for the above function is defined by $\| V \|_{\bar{\Omega}^N} = \max_{i=0,1,...,N} |V(x_i)|$ and
$\| \vec{V} \|_{\bar{\Omega}^N} = max\{\| V_1 \|_{\bar{\Omega}^N}, \| V_2 \|_{\bar{\Omega}^N}\}$ where the vector mesh functions $\vec{V} = (V_1, V_2)^T = \{V_1(x_i), V_2(x_i)\}$, $i = 0, 1, .., N$.

Throughout the paper C denotes a generic positive constant, which is independent of x and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.

II. Analytical Results

The problem (1) can be rewritten in the form

$$\begin{aligned} &\varepsilon_1 u_1'(x) + f_1(x, u_1, u_2) + b_1(x)\phi_1(x-1) = 0\\ &\varepsilon_2 u_2'(x) + f_2(x, u_1, u_2) + b_2(x)\phi_2(x-1) = 0,\\ &\qquad x \in (0,1]\\ &\vec{u}(0) = \vec{\phi}(0) \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\varepsilon_1 u_1'(x) + f_1(x, u_1, u_2) + b_1(x)u_1(x-1) = 0\\ &\varepsilon_2 u_2'(x) + f_2(x, u_1, u_2) + b_2(x)u_2(x-1) = 0,\\ &\qquad x \in (1,2]\\ &\vec{u}(1) \text{ known from (5).} \end{aligned} \tag{6}$$

$$\begin{aligned} &\vec{T_1}\vec{u} := E\vec{u}'(x) + \vec{g}(x, u_1, u_2) = \vec{0}, \quad x \in (0,1]\\ &\vec{T_2}\vec{u} := E\vec{u}'(x) + \vec{f}(x, u_1, u_2)\\ &\qquad + B(x)\vec{u}(x-1) = \vec{0}, \quad x \in (1,2]\\ &\text{where}\\ &\vec{g}(x, u_1, u_2) = \vec{f}(x, u_1, u_2) + B(x)\vec{\phi}(x-1). \end{aligned} \tag{7}$$

The reduced problem corresponding to (7) is given by

$$\vec{g}(x, r_1, r_2) = \vec{0}, \quad x \in (0,1] \tag{8}$$

$$\vec{f}(x, r_1, r_2) + B(x)\vec{r}(x-1) = \vec{0}, \quad x \in (1,2]. \tag{9}$$

The implicit function theorem and conditions (2),(3) and (4) ensure the existence of a unique solution for (8) and (9).

This solution $\vec{r}$ has derivatives which are bounded independently of ε_1 and ε_2.

Hence,

$|r_1^{(k)}(x)| \leq C; \quad |r_2^{(k)}(x)| \leq C; \; k = 0, 1, 2, 3; \; x \in [0, 2]$.

The following Shishkin decomposition [1], [2] of the solution $\vec{u}$ is considered:

$\vec{u} = \vec{v} + \vec{w}$, where the smooth component $\vec{v}(x)$ is the solution of the problem

$$\begin{aligned} &E\vec{v}'(x) + \vec{g}(x, v_1, v_2) = \vec{0}, \quad x \in (0,1]\\ &E\vec{v}'(x) + \vec{f}(x, v_1, v_2) + B(x)\vec{v}(x-1) = \vec{0},\\ &\qquad x \in (1,2]\\ &\vec{v}(0) = \vec{r}(0) \end{aligned} \tag{10}$$

and the singular component $\vec{w}(x)$ satisfies

$$\begin{aligned} &E\vec{w}'(x) + \vec{g}(x, v_1 + w_1, v_2 + w_2)\\ &\qquad - \vec{g}(x, v_1, v_2) = \vec{0}, x \in (0,1]\\ &E\vec{w}'(x) + \vec{f}(x, v_1 + w_1, v_2 + w_2) - \vec{f}(x, v_1, v_2)\\ &\qquad + B(x)\vec{w}(x-1) = \vec{0}, \quad x \in (1,2]\\ &\vec{w}(0) = \vec{u}(0) - \vec{v}(0). \end{aligned} \tag{11}$$

The bounds of the derivatives of the smooth component are contained in

Lemma 1: The smooth component $\vec{v}(x)$ satisfies $|v_k^{(i)}(x)| \leq C, \quad k = 1, 2; \quad i = 0, 1$ and $|v_k''(x)| \leq C\varepsilon_k^{-1}, \; k = 1, 2$.

Proof:

The smooth component $\vec{v}$ is further decomposed as follows:

$\vec{v} = \vec{\hat{q}} + \vec{\tilde{q}}$ where $\vec{\hat{q}}$ is the solution of

$$g_1(x, \hat{q}_1, \hat{q}_2) = 0 \tag{12}$$

$$\varepsilon_2 \frac{d\hat{q}_2}{dx} + g_2(x, \hat{q}_1, \hat{q}_2) = 0, \; x \in (0,1] \tag{13}$$

$$\hat{q}_2(0) = v_2(0); \;\; \hat{q}_1(0) = v_1(0) \tag{14}$$

and

$$f_1(x, \hat{q}_1, \hat{q}_2) + b_1(x)\hat{q}_1(x-1) = 0 \tag{15}$$

$$\varepsilon_2 \tfrac{d\hat{q}_2}{dx} + f_2(x, \hat{q}_1, \hat{q}_2) + b_2(x)\hat{q}_2(x-1) = 0, \quad x \in (1,2] \tag{16}$$

$\hat{q}_2(1)$ and $\hat{q}_1(1)$ are known from (12) and (13).

$\vec{\tilde{q}}$ is the solution of

$$\begin{aligned} &\varepsilon_1 \frac{d\tilde{q}_1}{dx} + g_1(x, \tilde{q}_1 + \hat{q}_1, \tilde{q}_2 + \hat{q}_2)\\ &\qquad - g_1(x, \hat{q}_1, \hat{q}_2) = -\varepsilon_1 \frac{d\hat{q}_1}{dx} \end{aligned} \tag{17}$$

$$\varepsilon_2\frac{d\tilde{q}_2}{dx}+g_2(x,\tilde{q}_1+\hat{q}_1,\tilde{q}_2+\hat{q}_2) -g_2(x,\hat{q}_1,\hat{q}_2)=0,\ x\in(0,1] \quad \tilde{q}_1(0)=\tilde{q}_2(0)=0 \tag{18}$$

and

$$\varepsilon_1\frac{d\tilde{q}_1}{dx}+f_1(x,\tilde{q}_1+\hat{q}_1,\tilde{q}_2+\hat{q}_2)-f_1(x,\hat{q}_1,\hat{q}_2) +b_1(x)\tilde{q}_1(x-1)=-\varepsilon_1\frac{d\hat{q}_1}{dx} \tag{19}$$

$$\varepsilon_2\frac{d\tilde{q}_2}{dx}+f_2(x,\tilde{q}_1+\hat{q}_1,\tilde{q}_2+\hat{q}_2)-f_2(x,\hat{q}_1,\hat{q}_2) +b_2(x)\tilde{q}_2(x-1)=0,\ x\in(1,2] \tag{20}$$

$\tilde{q}_1(1)$ and $\tilde{q}_2(1)$ are known from (17) and (18). Let $x\in[0,1]$.
Using (8), (12) and (13),

$$a_{11}(x)(\hat{q}_1-r_1)+a_{12}(x)(\hat{q}_2-r_2)=0 \tag{21}$$

$$\varepsilon_2\frac{d}{dx}(\hat{q}_2-r_2)+a_{21}(x)(\hat{q}_1-r_1) +a_{22}(x)(\hat{q}_2-r_2)=-\varepsilon_2\frac{dr_2}{dx}, \tag{22}$$

where,

$$a_{ij}(x)=\frac{\partial g_i}{\partial u_j}(x,\xi_i(x),\eta_i(x)),\quad i,j=1,2;$$
$\xi_i(x),\eta_i(x)$ are intermediate values.

Using (21) in (22),

$$\varepsilon_2\frac{d}{dx}(\hat{q}_2-r_2)+\left(a_{22}(x)-\frac{a_{12}(x)a_{21}(x)}{a_{11}(x)}\right) \times(\hat{q}_2-r_2)=-\varepsilon_2\frac{dr_2}{dx}$$

Consider the linear operator,

$$l_1(z):=\varepsilon_2 z'+\left(a_{22}(x)-\frac{a_{12}(x)a_{21}(x)}{a_{11}(x)}\right)z= -\varepsilon_2\frac{dr_2}{dx}, \tag{23}$$

where, $z=\hat{q}_2-r_2$.

This operator satisfies the maximum principle [1].

Thus, $\parallel \hat{q}_2-r_2\parallel\leq C\varepsilon_2$ and $\parallel \frac{d(\hat{q}_2-r_2)}{dx}\parallel\leq C$.

Using this in (21), $\parallel \hat{q}_1-r_1\parallel\leq C\varepsilon_2$.

Hence, $\parallel \hat{q}_2\parallel\leq C$, $\parallel \frac{d\hat{q}_2}{dx}\parallel\leq C$ and $\parallel \hat{q}_1\parallel\leq C$.
Differentiating (22),

$$\varepsilon_2\frac{d^2}{dx^2}(\hat{q}_2-r_2)+a'_{21}(x)(\hat{q}_2-r_2) +a_{21}(x)\frac{d}{dx}(\hat{q}_2-r_2)+a'_{22}(x)(\hat{q}_1-r_1) +a_{22}(x)\frac{d}{dx}(\hat{q}_1-r_1)=-\varepsilon_2\frac{d^2r_2}{dx^2}. \tag{24}$$

Hence,$\parallel \frac{d^2\hat{q}_2}{dx^2}\parallel\leq C\varepsilon_2^{-1}$.
Differentiating (21) twice and using the above estimates of $\frac{d^2\hat{q}_2}{dx^2}$,

$$\parallel \frac{d^2\hat{q}_1}{dx^2}\parallel\leq C\varepsilon_2^{-1}. \tag{25}$$

From (17) and (18),

$$\varepsilon_1\frac{d\tilde{q}_1}{dx}+a^*_{11}(x)\tilde{q}_1+a^*_{12}(x)\tilde{q}_2=-\varepsilon_1\frac{d\hat{q}_1}{dx} \tag{26}$$
$$\varepsilon_2\frac{d\tilde{q}_2}{dx}+a^*_{21}(x)\tilde{q}_1+a^*_{22}(x)\tilde{q}_2=0 \tag{27}$$
$$\tilde{q}_1(0)=\tilde{q}_2(0)=0 \tag{28}$$

where,

$$a^*_{ij}(x)=\frac{\partial g_i}{\partial u_j}(x,\zeta_i(x),\chi_i(x)),\quad i,j=1,2;$$
$\zeta_i(x),\chi_i(x)$ are intermediate values.

From equations (26) and (27),

$$\parallel \tilde{q}_i\parallel\leq C,\quad i=1,2 \tag{29}$$
$$\parallel \frac{d\tilde{q}_i}{dx}\parallel\leq C,\quad i=1,2 \tag{30}$$
$$\parallel \frac{d^2\tilde{q}_i}{dx^2}\parallel\leq C\varepsilon_i^{-1},\quad i=1,2. \tag{31}$$

Hence from the bounds for $\vec{\tilde{q}}$ and $\vec{\hat{q}}$, the required bounds of $\vec{v}$ follow.

Let $x \in [1,2]$.

Using (9), (15) and (16),

$$\begin{aligned} p_{11}(x)(\hat{q}_1 - r_1) + p_{12}(x)(\hat{q}_2 - r_2) + \\ b_1(x)(\hat{q}_1(x-1) + r_1(x-1)) = 0 \end{aligned} \tag{32}$$

$$\begin{aligned} \varepsilon_2 \frac{d}{dx}(\hat{q}_2 - r_2) + p_{21}(x)(\hat{q}_1 - r_1) \\ + p_{22}(x)(\hat{q}_2 - r_2) + b_2(x)(\hat{q}_2(x-1) - r_2(x-1)) \\ = -\varepsilon_2 \frac{dr_2}{dx} \end{aligned} \tag{33}$$

where,

$$p_{ij}(x) = \frac{\partial f_i}{\partial u_j}(x, \kappa_i(x), \lambda_i(x)), \quad i,j = 1,2;$$
$$\kappa_i(x), \lambda_i(x) \text{ are intermediate values.}$$

Using (32) in (33),

$$\begin{aligned} \varepsilon_2 \frac{d}{dx}(\hat{q}_2 - r_2) + \left(p_{22}(x) - \frac{p_{12}(x)p_{21}(x)}{p_{11}(x)}\right) \\ \times(\hat{q}_2 - r_2) - \tfrac{p_{21}}{p_{11}}(x) b_1(x)(\hat{q}_1(x-1) \\ - r_1(x-1)) + b_2(x)(\hat{q}_2(x-1) - r_2(x-1)) \\ = -\varepsilon_2 \frac{dr_2}{dx} \end{aligned}$$

Consider the linear operator,

$$\begin{aligned} l_2(z) := \varepsilon_2 z' + \left(p_{22}(x) - \frac{p_{12}(x)p_{21}(x)}{p_{11}(x)}\right) z \\ + b_2(x) z(x-1) \\ = -\varepsilon_2 \frac{dr_2}{dx} - \tfrac{p_{21}}{p_{11}}(x) b_1(x)(\hat{q}_1(x-1) \\ - r_1(x-1)), \end{aligned} \tag{34}$$

where, $z = \hat{q}_2 - r_2$.

This operator satisfies the maximum principle [12].

Hence using similar arguments as in the interval $[0,1]$ and the bounds of $\vec{q}$ and $\vec{\hat{q}}$ in the interval $[0,1]$, the required bounds in the interval $[1,2]$ are derived.

Lemma 2: The singular component $\vec{w}(x)$ satisfies, for any $x \in [0,1]$,

$$\begin{aligned} |w_i(x)| &\leq Ce^{\frac{-\alpha x}{\varepsilon_2}}; \quad i = 1,2 \\ |w_1'(x)| &\leq C(\varepsilon_1^{-1} e^{\frac{-\alpha x}{\varepsilon_1}} + \varepsilon_2^{-1} e^{\frac{-\alpha x}{\varepsilon_2}}) \\ |w_2'(x)| &\leq C\varepsilon_2^{-1} e^{\frac{-\alpha x}{\varepsilon_2}} \\ |w_i''(x)| &\leq C\varepsilon_i^{-1}(\varepsilon_1^{-1} e^{\frac{-\alpha x}{\varepsilon_1}} + \varepsilon_2^{-1} e^{\frac{-\alpha x}{\varepsilon_2}}), \\ & \qquad i = 1,2 \end{aligned}$$

For $x \in [1,2]$,

$$\begin{aligned} |w_i(x)| &\leq Ce^{\frac{-\alpha(x-1)}{\varepsilon_2}}; \quad i = 1,2 \\ |w_1'(x)| &\leq C(\varepsilon_1^{-1} e^{\frac{-\alpha(x-1)}{\varepsilon_1}} + \varepsilon_2^{-1} e^{\frac{-\alpha(x-1)}{\varepsilon_2}}) \\ |w_2'(x)| &\leq C\varepsilon_2^{-1} e^{\frac{-\alpha(x-1)}{\varepsilon_2}} \\ |w_i''(x)| &\leq C\varepsilon_i^{-1}(\varepsilon_1^{-1} e^{\frac{-\alpha(x-1)}{\varepsilon_1}} \\ & \qquad + \varepsilon_2^{-1} e^{\frac{-\alpha(x-1)}{\varepsilon_2}}), \quad i = 1,2 \end{aligned}$$

Proof:
From equations (11),

$$\varepsilon_1 w_1'(x) + s_{11}(x) w_1(x) + s_{12}(x) w_2(x) = 0 \tag{35}$$

$$\begin{aligned} \varepsilon_2 w_2'(x) + s_{21}(x) w_1(x) + s_{22}(x) w_2(x) = 0, \\ x \in (0,1] \end{aligned} \tag{36}$$

$w_1(0) = u_1(0) - v_1(0); \quad w_2(0) = u_2(0) - v_2(0)$
and

$$\begin{aligned} \varepsilon_1 w_1'(x) + s_{11}^*(x) w_1(x) + s_{12}^*(x) w_2(x) \\ + b_1(x) w_1(x-1) = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} \varepsilon_2 w_2'(x) + s_{21}^*(x) w_1(x) + s_{22}^*(x) w_2(x) + \\ b_2(x) w_2(x-1) = 0, \quad x \in (1,2] \end{aligned} \tag{38}$$

$w_1(1) = u_1(1) - v_1(1); \quad w_2(1) = u_2(1) - v_2(1)$
Here, $s_{ij}(x) = \dfrac{\partial g_i}{\partial u_j}(x, \nu_i(x), \upsilon_i(x))$ and $s_{ij}^*(x) = \dfrac{\partial f_i}{\partial u_j}(x, \phi_i(x), \phi_i^*(x)); \nu_i(x), \upsilon_i(x), \phi_i(x)$, $\phi_i^*(x)$ are intermediate values.

From equations (35),(36),(37) and (38), the bounds of the singular component $\vec{w}$ can be derived as in [5] in the domains $[0,1]$ and $[1,2]$.

III. Shishkin Mesh

A piecewise uniform Shishkin mesh $\overline{\Omega}^N = \overline{\Omega^-}^N \cup \Omega^{+N}$ where $\overline{\Omega^-}^N = \{x_j\}_0^{\frac{N}{2}}$ and $\Omega^{+N} = \{x_j\}_{\frac{N}{2}+1}^{N}$ with N mesh-intervals is now constructed on $\overline{\Omega} = [0,2]$,as follows, for the case $\varepsilon_1 < \varepsilon_2$. In the case $\varepsilon_1 = \varepsilon_2$ a simpler construction requiring just one parameter τ suffices. The interval $[0,1]$ is subdivided into 3 sub-intervals $[0,\tau_1] \cup (\tau_1,\tau_2] \cup (\tau_2,1]$. The parameters $\tau_r,\ r = 1,2$, which determine the points separating the uniform meshes, are defined by $\tau_0 = 0$, $\tau_3 = \frac{1}{2}$,

$$\begin{aligned} \tau_2 &= \min\left\{\frac{1}{2}, \frac{\varepsilon_2}{\alpha}\ln N\right\} \quad \text{and} \\ \tau_1 &= \min\left\{\frac{\tau_2}{2}, \frac{\varepsilon_1}{\alpha}\ln N\right\}. \end{aligned} \tag{39}$$

Clearly $0 < \tau_1 < \tau_2 \le \frac{1}{2}$. Then, on the sub-interval $(\tau_2,1]$ a uniform mesh with $\frac{N}{4}$ mesh points is placed and on each of the sub-intervals $(0,\tau_1]$ and $(\tau_1,\tau_2]$, a uniform mesh of $\frac{N}{8}$ mesh points is placed. Similarly, the interval $[1,2]$ is also divided into 3 sub-intervals $[1,1+\tau_1],(1+\tau_1,1+\tau_2],(1+\tau_2,2]$ having the same number of mesh intervals as in $[0,1]$.

Note that, when both the parameters $\tau_r,\ r = 1,2$, take on their lefthand value, the Shishkin mesh becomes a classical uniform mesh on $[0,2]$.

IV. Discrete Problem

The initial value problems (5) and (6) are discretised using the backward Euler scheme on the piecewise uniform fitted mesh $\bar{\Omega}^N$. The discrete problem is

$$\begin{aligned} T_N\vec{U}(x_j) &:= ED^-\vec{U}(x_j) \\ &+ \vec{g}(x_j,U_1(x_j),U_2(x_j)) = 0, \quad j = 1(1)\frac{N}{2} \end{aligned} \tag{40}$$

$$\begin{aligned} \tilde{T}_N\vec{U}(x_j) &:= ED^-\vec{U}(x_j) + \vec{f}(x_j,U_1(x_j),U_2(x_j)) \\ &= -B(x_j)\vec{U}(x_j-1), \quad j = \frac{N}{2}+1(1)N \end{aligned} \tag{41}$$

$$\vec{U}(0) = \vec{u}(0) \text{ and}$$

$$D^-\vec{U}(x_j) = \frac{\vec{U}(x_j) - \vec{U}(x_{j-1})}{x_j - x_{j-1}},\ j = 1(1)N.$$

Lemma 3: For any mesh functions $\vec{Y}$ and $\vec{Z}$ with $\vec{Y}(0) = \vec{Z}(0)$,

$$\| \vec{Y} - \vec{Z} \| \le C \| T_N\vec{Y} - T_N\vec{Z} \|$$

Proof:

$$\begin{aligned} T_N\vec{Y} - T_N\vec{Z} =& \\ & ED^-\vec{Y}(x_j) + \vec{g}(x_j,Y_1(x_j),Y_2(x_j)) \\ & -ED^-\vec{Z}(x_j) - \vec{g}(x_j,Z_1(x_j),Z_2(x_j)) \\ =& ED^-(\vec{Y} - \vec{Z})(x_j) \\ & +\frac{\partial\vec{g}}{\partial u_1}(x_j,\vec{\xi}(x_j),\vec{\eta}(x_j))(Y_1 - Z_1) \\ & +\frac{\partial\vec{g}}{\partial u_2}(x_j,\vec{\xi}(x_j),\vec{\eta}(x_j))(Y_2 - Z_2) \\ & = (T_N')(\vec{Y} - \vec{Z}) \end{aligned}$$

where T_N' is the Frechet derivative of T_N and the notation $\frac{\partial\vec{g}}{\partial u_i}(x_j,\vec{\xi}(x_j),\vec{\eta}(x_j)),\ i = 1,2$ is used to express the difference between the mid-values for the components g_1 and g_2. Since T_N' is linear, it satisfies the discrete maximum principle and discrete stability result [5].Hence

$$\| \vec{Y} - \vec{Z} \| \le C \| T_N'(\vec{Y} - \vec{Z}) \| = C \| T_N\vec{Y} - T_N\vec{Z} \|$$

and the lemma is proved.

Parameter - uniform bounds for the error are given in the following theorem, which is the main result of this paper.

Theorem 1: Let $\vec{u}$ be the solution of the problem (1) and $\vec{U}$ be the solution of the discrete problem (40),(41). Then

$$\| \vec{U} - \vec{u} \| \le CN^{-1}\ln N \tag{42}$$

TABLE I

Values of D_{ε}^{N}, D^{N}, p^{N}, p^{*} *and* $C_{p^*}^{N}$ *for* $\varepsilon_1 = \dfrac{\eta}{16}$, $\varepsilon_2 = \dfrac{\eta}{4}$ and $\alpha = 0.9$.

η	Number of mesh points N				
	128	256	$\cdots$	8192	16384
2^{0}	0.150E-01	0.806E-02	$\cdots$	0.271E-03	0.136E-03
2^{-3}	0.211E-01	0.121E-01	$\cdots$	0.619E-03	0.336E-03
2^{-6}	0.218E-01	0.125E-01	$\cdots$	0.619E-03	0.336E-03
2^{-9}	0.218E-01	0.125E-01	$\cdots$	0.619E-03	0.336E-03
2^{-12}	0.218E-01	0.125E-01	$\cdots$	0.619E-03	0.336E-03
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
2^{-27}	0.218E-01	0.125E-01	$\cdots$	0.619E-03	0.336E-03
D^{N}	0.218E-01	0.125E-01	$\cdots$	0.619E-03	0.336E-03
p^{N}	0.800E+00	0.854E+00	$\cdots$	0.880E+00	
C_p^{N}	0.249E+01	0.249E+01	$\cdots$	0.196E+01	0.186E+01
Computed order of $\vec{\varepsilon}$-uniform convergence, $p^* = 0.8$					
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^{N} = 2.48$					

Proof:
Let $x \in [0,1]$.
From the above lemma,

$$\parallel \vec{U} - \vec{u} \parallel \leq C \parallel T_N\vec{U} - T_N\vec{u} \parallel$$

Consider $\parallel T_N\vec{u} \parallel = \parallel T_N\vec{u} - T_N\vec{U} \parallel$
Hence,

$$\begin{aligned}\parallel T_N\vec{u} - T_N\vec{U} \parallel &= \parallel T_N\vec{u} \parallel \\ &= \parallel T_N\vec{u} - \vec{T_1}\vec{u} \parallel \\ &= E|(D^-\vec{u} - \vec{u}')(x)| \\ &\leq E|(D^-\vec{v} - \vec{v}')(x)| \\ &+ E|(D^-\vec{w} - \vec{w}')(x)|\end{aligned}$$

Since the bounds for $\vec{v}$ and $\vec{w}$ are the same as in [5] , the required result follows.
Let $x \in [1,2]$.
From the above lemma,

$$\begin{aligned}\parallel \vec{U} - \vec{u} \parallel &\leq C \parallel \tilde{T}_N\vec{U} - \tilde{T}_N\vec{u} \parallel \\ &\leq C \parallel B(x_j)(\vec{U} - \vec{u})(x_j - 1) \parallel \\ &\leq C \parallel \vec{U} - \vec{u} \parallel \\ &\leq CN^{-1}\ln N\end{aligned}$$

V. Numerical Results

The numerical method proposed in this paper is illustrated through an example presented in this section.

Example Consider the initial value problem

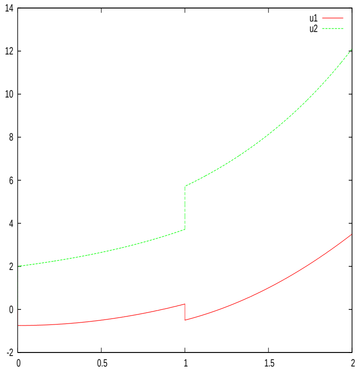
$$\begin{aligned}\varepsilon_1 u_1'(x) + 3u_1(x) - \frac{1}{4}exp(-u_1^2)(x) - u_2(x) \\ -x^2 + 1 - u_1(x-1) = 0 \\ \varepsilon_2 u_2'(x) + 4u_2(x) - cos(u_2(x)) - u_1(x) - \\ e^x - u_2(x-1) = 0; \quad x \in (0,1]\end{aligned}$$

$\vec{u}(x) = \vec{0}; \quad x \in [-1, 0]$.
The above quasi linear problem is solved using the numerical method suggested in this paper utilising the continuation method found in [2].
The maximum pointwise errors and the rate of convergence for this IVP are calculated using the two - mesh algorithm in [2] and are presented in Table 1.

The notations $D^N, p_N, C_p^N, C_{p^*}^N$ and p^* bear the same meaning as in [2] but the methods to arrive at them are modified for the vector solution.

A graph of the numerical solution is presented in Figure 1 for $N = 2048$ and $\eta = 2^{-15}$. The sharper initial layers at $x = 0$ and interior layers at $x = 1$ are evident.

Fig. 1. Numerical solution



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