Extended Supersymmetries in One Dimension

F. Toppan

This work covers part of the material presented at the Advanced Summer School in Prague. It is mostly devoted to the structural properties of Extended Supersymmetries in One Dimension. Several results are presented on the classification of linear, irreducible representations realized on a finite number of time-dependent fields. The connections between supersymmetry transformations, Clifford algebras and division algebras are discussed. A manifestly supersymmetric framework for constructing invariants without using the notion of superfields is presented. A few examples of one-dimensional, N-extended, off-shell invariant sigma models are computed. The relation between supersymmetry transformations and graph theory is outlined. The notion of the fusion algebra of irreps tensor products is presented. The relevance of one-dimensional Supersymmetric Quantum Mechanics as a way to extract information on higher dimensional supersymmetric field theories is discussed.

Keywords: Supersymmetric Quantum Mechanics, M-theory.

1 Introduction

Supersymmetric Quantum Mechanics [1] is under intensive development and remarkable new features have been discovered in recent years. This attention is due both to the wide range of applicability of one-dimensional supersymmetric theories and especially superconformal quantum mechanics [2] for extremal black holes [3], in the AdS-CFT correspondence [4] (when setting AdS_2), in investigating partial breaking of extended supersymmetries [5, 6], as well as for its underlying mathematical structures. It is well known that large N (up to $N = 32$, starting from the maximal, eleven-dimensional supergravity) one-dimensional supersymmetric quantum mechanical models are automatically derived [7] from dimensional reduction of higher-dimensional supersymmetric field theories. Large N one-dimensional supersymmetry on the other hand (possibly in the $N \to \infty$ limit) even emerges in condensed matter phenomena. Controlling one-dimensional N-extended supersymmetry for arbitrary values of N (that is, the nature of its representation theory, how to construct manifestly supersymmetric invariants, etc.) is a technical, but challenging program with important consequences in many areas of physics, see e.g. the discussion in [8] concerning the nature of *on-shell versus off-shell* representations, for its implications in the context of the supersymmetric unification of interactions.

Over the years, progress has come from two lines of attack. In the pivotal work of [9] irreducible representations were investigated to analyze supersymmetric quantum mechanics. The special role played by Clifford algebra was pointed out [10]. Clifford algebras were also used in [11] to construct representations of the extended one-dimensional supersymmetry algebra for arbitrarily large values of N. Another line of attack involved using superspace, so that manifest invariants could be constructed through superfields. For low values of N this is indeed the most convenient approach. However, with increasing N, the associated superfields become highly reducible and require the introduction of constraints to extract irreducible representations. This approach soon becomes impractical for large N. Indeed, only very recently a manifestly $N = 8$ superfield formalism for one-dimensional theory has been introduced, see [12] and references therein. A manifest superfield formalism is however lacking for larger values of N.

In this work we discuss our results [13], [14], [15] concerning the classification of linear irreducible representations realized on a finite number of time-dependent, bosonic and fermionic, fields. The connection with Clifford algebras and division algebras is discussed, as well as the construction of off-shell invariant actions and some associations with graph theory. Several important topics that have appeared recently in the literature, like the nature of the non-linear representations will not be discussed here. There are reviews ([16]) that cover this and other aspects. Similarly, the quite important connection with supersymmetric integrable systems in (1+1) dimensions (such as the supersymmetric extension of the KdV equation, will not be discussed since they have been covered elsewhere [17]).

The scheme of the paper is as follows. The next Section deals with the relevance of one-dimensional Supersymmetric Quantum Mechanics for understanding higher-dimensional supersymmetric field theory. Some selected examples of dimensional reductions are pointed out. The relation between irreducible representations of one-dimensional N Extended Supersymmetry Algebra and Clifford algebras is explained in Section **3**. Section **4** reviews the classification of Clifford algebras and their relation with division algebras, following [18]. In Section **5** the results of [14] concerning the classification of irreducible representations with length-4 field content are reported. Section **6** computes off-shell invariant actions of one-dimensional sigma models within a manifestly supersymmetric formalism which does not require the introduction of superfields. In Section **7** an $N = 8$ invariant action constructed in terms of octonionic structure constants is presented. The classification in [15] of nonequivalent $N = 5, 6$ supersymmetry transformations with the same field content is given in Section **8** and **9**. A graphical presentation of supersymmetry transformations in terms of N-colored oriented graphs is discussed. Section **10** introduces the fusion algebra produced by tensoring irreducible representations and presents it in graphical form.

2 N Extended supersymmetries in $D = 1$ and dimensional reduction of supersymmetric theories in higher dimensions

One important motivation for investigating N Extended Supersymmetries in one dimension is the fact that their rich algebraic setting can furnish useful information concerning the construction of supersymmetric theories in a higher dimension (such as super-Yang-Mills, supergravity, etc.) Supersymmetric quantum mechanics with large number N encodes large information of these theories. The simplest way to see this is through dimensional reduction, where all space-dimensions are frozen and the only remaining dependence is in terms of a time-like coordinate. The usefulness of this procedure is due to the fact that in such a framework we can make use of powerful mathematical tools (essentially based on the available classification of Clifford algebras) which are not available in higher dimensions.

It should be remembered that a four-dimensional field theory with N extended supersymmetries corresponds, once it is dimensionally reduced to one-dimension, to a supersymmetric quantum mechanics with four times ($4N$) the number of the original extended supersymmetries [7]. The most interesting case, in the context of the unification program, corresponds to eleven-dimensional supergravity (the low-energy limit of M-theory), which is reduced to an $N = 8$ four-dimensional theory and later to an $N = 32$ one-dimensional supersymmetric quantum mechanical system.

In this section we will discuss the dimensional reduction of supersymmetric theories from $D = 4$ to $D = 1$ in some specific examples. We will prove how certain $D = 4$ problems can be reformulated in a $D = 1$ language.

It is convenient to start with a dimensional analysis of the following theories:

i) the free particle in one (time) dimension ($D = 1$) and, for the ordinary Minkowski space-time ($D = 4$),

iia) the scalar boson theory (with quartic potential $\frac{\lambda}{4!}\phi^4$),

iib) the Yang-Mills theory and, finally,

iic) the gravity theory (expressed in the vierbein formalism).

We further make a dimensional analysis of the above three theories when dimensionally reduced (*à la Scherk*) to a one (time) dimensional $D = 1$ quantum mechanical system.

In the following we will repeat the dimensional analysis for the supersymmetric version of these theories.

Case i) – the $D = 1$ free particle

It is described by a dimensionless action S given by

$$S = \frac{1}{m}\int dt\dot{\varphi}^2. \tag{2.1}$$

The dot denotes, as usual, the time derivative. The dimensionality of the time t is the inverse of the mass; we can therefore set ($[t] = -1$). By assuming φ being dimensionless ($[\varphi] = 0$), an overall constant (written as $\frac{1}{m}$) of mass dimension -1 has to be inserted to make S non-dimensional. Summarizing, we have, for the above $D = 1$ model,

$$\begin{aligned}[t]_{D=1} &= -1,\\ [\frac{\partial}{\partial t}]_{D=1} &= 1,\\ [\varphi]_{D=1} &= 0,\\ [m]_{D=1} &= 1,\\ [S]_{D=1} &= 0.\end{aligned} \tag{2.2}$$

The suffix $D = 1$ has been added for later convenience, since the theory corresponds to a one-dimensional model.

Case iia) – the $D = 4$ scalar boson theory

The action can be presented as

$$S = \int d^4x\left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 - \frac{1}{4!}\lambda\Phi^4\right) \tag{2.3}$$

A non-dimensional action S is obtained by setting, in mass dimension,

$$\begin{aligned}[\Phi]_{D=4} &= 1,\\ [\partial_\mu]_{D=4} &= 1,\\ [M]_{D=4} &= 1,\\ [\lambda]_{D=4} &= 0.\end{aligned} \tag{2.4}$$

Case iib) – the $D = 4$ pure QED or Yang-Mills theories

The gauge-invariant action is given by

$$S = \frac{1}{e^2}\int d^4x\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{2.5}$$

where the antisymmetric stress-energy tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = [D_\mu, D_\nu], \tag{2.6}$$

with D_μ the covariant derivative, expressed in terms of the gauge connection A_μ

$$D_\mu = \partial_\mu - eA_\mu. \tag{2.7}$$

e is the charge (the electric charge for QED). The action is non-dimensional, provided that

$$\begin{aligned}[A_\mu]_{D=4} &= 1,\\ [F_{\mu\nu}]_{D=4} &= 2,\\ [e]_{D=4} &= 0.\end{aligned} \tag{2.8}$$

Case iic) – The pure gravity case

The action is constructed, see [19] for details, in terms of the determinant E of the vierbein e_μ^a and the curvature scalar R. It is given by

$$S = -\frac{6}{8\pi G_N}\int d^4x\,ER. \tag{2.9}$$

The overall constant (essentially the inverse of the gravitational constant G_N) is now dimensional ($[G_N]_{D=4} = -2$). The non-dimensional action is recovered by setting

$$\begin{aligned}[e_\mu^a]_{D=4} &= 0,\\ [R]_{D=4} &= 2.\end{aligned} \tag{2.10}$$

Let us now discuss the dimensional reduction from

$$D = 4 \Rightarrow D = 1.$$

Let us suppose that the three space dimensions belong to some compact manifold M (e.g. the three-sphere S^3) and let us freeze the dependence of the fields on the space-dimensions (application of the time derivative ∂_0 leads to non-vanishing results, while application of the space-derivatives ∂_i, for $i = 1, 2, 3$, gives zero). Our space-time is now given by $\mathbf{R} \times M$. We get that the integration over the three space variables contributes just to an overall factor, the volume of the three-dimensional manifold M. Therefore

$$\int d^4x \equiv Vol_M \int dt. \tag{2.11}$$

Since

$$[Vol_M]_{D=4} = -3 \tag{2.12}$$

we can express $Vol \equiv \frac{1}{m^3}$, where m is a mass-term. A factor $\frac{1}{m}$ contributes as an overall factor in one-dimensional theory, while the remaining part $\frac{1}{m^2}$ can be used to rescale the fields. We have, e.g., for dimensional reduction of the scalar boson theory that

$$\varphi_{D=1} \equiv \frac{1}{m}\phi_{D=4}. \tag{2.13}$$

The dimensional reduction of the scalar boson theory *ii a)* is therefore given by

$$S = \frac{1}{m}\int dt\left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}M^2\varphi^2 + \lambda_{D=1}\frac{1}{4!}\varphi^4\right) \tag{2.14}$$

where we have

$$\begin{aligned} [\varphi]_{D=1} &= 0, \\ [M]_{D=1} &= 1, \\ [\lambda_1]_{D=1} &= 2. \end{aligned} \tag{2.15}$$

The $D = 1$ coupling constant λ_1 is related to the $D = 4$ non-dimensional coupling constant λ by the relation

$$\lambda_1 = \lambda\, m^2. \tag{2.16}$$

We proceed in a similar way in the case of Yang-Mills theory. We can rescale the $D = 4$ Yang-Mills fields A_μ to the $D = 1$ fields $B_\mu = \frac{1}{m}A_\mu$. The $D = 1$ charge e is rescaled to $e_1 = e\,m$. We have, symbolically, for the dimensionally reduced action, a sum of terms of the type

$$S = \frac{1}{m}\int dt\left(\dot{B}^2 + e_1\dot{B}B^2 + e_1^2B^4\right), \tag{2.17}$$

where

$$\begin{aligned} [B]_{D=1} &= 0, \\ [e_1]_{D=1} &= 1. \end{aligned} \tag{2.18}$$

The situation is different as far as gravity theory is concerned. In that case the overall factor Vol_M/G_N produces the dimensionally correct $\frac{1}{m}$ overall factor of the one-dimensional theory. This implies that we do not need to rescale the dimensionality of the vierbein e_μ^a and of the curvature. Summarizing, we have the following results

$$\begin{array}{lll} \text{scalar boson} & \Phi: & [\Phi]_{D=4} = 1 \Rightarrow [\Phi]_{D=1} = 0 \\ \text{gauge connection} & A_\mu: & [A_\mu]_{D=4} = 1 \Rightarrow [A_\mu]_{D=1} = 0 \\ \text{vierbein} & e_\mu^\alpha: & [e_\mu^\alpha]_{D=4} = 1 \Rightarrow [e_\mu^\alpha]_{D=1} = 0 \\ \text{electric charge} & e: & [e]_{D=4} = 0 \Rightarrow [e]_{D=1} = 1 \end{array} \tag{2.19}$$

Let us now discuss the $N = 1$ supersymmetric version of the three $D = 4$ theories above. First, we have the chiral multiplet, described in [19], in terms of the chiral superfields Φ, $\overline{\Phi}$. Next the vector multiplet V, the vector-multiplet in the Wess-Zumino gauge, the supergravity multiplet in terms of vierbein and gravitinos and, finally, the gauged supergravity multiplet presenting an extra set of auxiliary fields. The total content of fields is given by the following table, which presents also the $D = 4$ and respectively the $D = 1$ dimensionality of the fields (in the latter case, after dimensional reduction). We have

(2.20)

chiral multiplet	:	$\Phi, \overline{\Phi}$
fields content	:	(2, 4, 2)
$D = 4$ dimensionality	:	$[1, \frac{3}{2}, 2]_{D=4}$
$D = 1$ dimensionality	:	$[0, \frac{1}{2}, 1]_{D=1}$
vector multiplet	:	$V = V^\dagger$
fields content	:	(1, 4, 6, 4, 1)
$D = 4$ dimensionality	:	$[0, \frac{1}{2}, 1, \frac{3}{2}, 2]_{D=4}$
$D = 1$ dimensionality	:	$[-1, -\frac{1}{2}, 0, \frac{1}{2}, 1]_{D=1}$
vector multiplet	:	V in the WZ gauge
fields content	:	(3, 4, 1)
$D = 4$ dimensionality	:	$[1, \frac{3}{2}, 2]_{D=4}$
$D = 1$ dimensionality	:	$[0, \frac{1}{2}, 1]_{D=1}$
supergravity multiplet	:	e_μ^a, ψ_μ^α
fields content	:	(16, 16)
$D = 4$ dimensionality	:	$[0, \frac{1}{2}]_{D=4}$
$D = 1$ dimensionality	:	$[0, \frac{1}{2}]_{D=1}$
gauged sugra multiplet	:	$e_\mu^a, \psi_\mu^\alpha, b^i$
fields content	:	(6, 12, 6)
$D = 4$ dimensionality	:	$[0, \frac{1}{2}, 1]_{D=4}$
$D = 1$ dimensionality	:	$[0, \frac{1}{2}, 1]_{D=1}$

Some comments are in order: the vector multiplet corresponds, in $D = 1$ language, to the $N = 4$ "enveloping representation" [14] (1, 4, 6, 4, 1). The latter is a reducible, but non-decomposable representation of the $N = 4$ supersymmetry. Its irreducible multiplets are split into (1, 4, 3, 0, 0) and (0, 0, 3, 4, 1). The Wess-Zumino gauge, in $D = 1$ language, corresponds to selecting the latter $N = 4$ irreducible multiplet, whose fields present only non-negative dimensions.

The $N = 2$ four-dimensional super-QED involves coupling a set of chiral superfields together with the vector multiplet. Due to the dimensional analysis, the corresponding one-dimensional multiplet is the (5, 8, 3) irrep of $N = 8$ given by (2, 4, 2)+(3, 4, 1).

As far as supergravity theories are concerned, the original supergravity multiplet corresponds to four irreducible $N = 4$ one-dimensional multiplets, while the gauged supergravity multiplet is obtained, in the $D = 1$ viewpoint, in terms of three irreducible $N = 4$ multiplets whose total number of fields is (6, 12, 6).

The multiplet of the physical degrees of freedom of eleven-dimensional supergravity (44 components for the graviton, i.e. the components of a $SO(9)$ traceless symmetric tensor, the 128 fermionic components of the gravitinos and the 84 components of the three form) can be accommodated into the (44, 128, 84) multiplet of an N-extended one-dimensional supersymmetry. As will be shown later, 128 bosons and 128 fermions can accommodate at most 16 off-shell supersymmetries that are linearly realized. It is under question whether an off-shell formulation of eleven-dimensional supergravity indeed exists. In any case it would require at least 32768 bosonic (and an equal number of fermionic) degrees of freedom to produce an $N = 32$ supersymmetry representation in $D = 1$.

3 Supersymmetric quantum mechanics and Clifford algebras

In this section we discuss several results, based on ref. [13], concerning the classification of irreducible representations (from now on "irreps") of the N-extended one-dimensional supersymmetry algebra and their connection with Clifford algebras.

The N extended $D = 1$ supersymmetry algebra is given by

$$\{Q_i, Q_j\} = \eta_{ij} H, \tag{3.21}$$

where the Q_i's are the supersymmetry generators (for $i, j = 1, \ldots, N$) and $H \equiv -i\frac{\partial}{\partial t}$ is a Hamiltonian operator (t is the time coordinate). If the diagonal matrix η_{ij} is pseudo--Euclidean (with signature (p, q), $N = p + q$) we can speak of generalized supersymmetries. For convenience we limit the discussion here (despite the fact that our results can be straightforwardly generalized to pseudo-Euclidean supersymmetries, having applicability, e.g., to supersymmetric spinning particles moving in pseudo-Euclidean manifolds) to ordinary N-extended supersymmetries. Therefore for our purposes $\eta_{ij} \equiv \delta_{ij}$.

The (D-modules) representations of the (3.21) supersymmetry algebra realized in terms of *linear* transformations acting on *finite* multiplets of fields satisfy the following properties. The total number of bosonic fields equal the total number of fermionic fields. For irreps of the N-extended supersymmetry the number of bosonic (fermionic) fields is given by d, with N and d linked through

$$\begin{aligned} N &= 8l + n, \\ d &= 2^{4l} G(n), \end{aligned} \tag{3.22}$$

where $l = 0, 1, 2, \ldots$ and $n = 1, 2, 3, 4, 5, 6, 7, 8$. $G(n)$ appearing in (3.22) is the Radon-Hurwitz function [13]

n	1	2	3	4	5	6	7	8
$G(n)$	1	2	4	4	8	8	8	8

(3.23)

The modulo 8 property of the irreps of the N-extended supersymmetry is a consequence of the famous modulo 8 property of Clifford algebras. The connection between supersymmetry irreps and Clifford algebras is specified later.

The $D = 1$ dimensional reduction of the maximal $N = 8$ supergravity produces a supersymmetric quantum mechanical system with $N = 32$ extended number of supersymmetries. It is therefore convenient to explicitly report the number of bosonic/fermionic component fields in any given irrep of (3.21) for any N up to $N = 32$. We get the table

$N=1$	1	$N=9$	16	$N=17$	256	$N=25$	4096
$N=2$	2	$N=10$	32	$N=18$	512	$N=26$	8192
$N=3$	4	$N=11$	64	$N=19$	1024	$N=27$	16384
$N=4$	4	$N=12$	64	$N=20$	1024	$N=28$	16384
$N=5$	8	$N=13$	128	$N=21$	2048	$N=29$	32768
$N=6$	8	$N=14$	128	$N=22$	2048	$N=30$	32768
$N=7$	8	$N=15$	128	$N=23$	2048	$N=31$	32768
$N=8$	8	$N=16$	128	$N=24$	2048	$N=32$	32768

(3.24)

The bosonic (fermionic) fields entering an irreducible multiplet can be grouped together according to their dimensionality. Sometimes instead of "dimension", the word "spin" is used to refer to the dimensionality of the component fields. This choice of word finds some justification when discussing the $D = 1$ dimensional reduction of higher-dimensional supersymmetric theories. The number (equal to l) of different dimensions (i.e. the number of different spin states) of a given irrep, will be referred to as the *length* l of the irrep. Since there are at least two different spin states (one for bosons, the other for fermions), obtained when all bosons (fermions) are grouped together within the same spin, the minimal length of an irrep is $l = 2$.

A general property of (linear) supersymmetry in any dimension is the fact that the states of highest spin in a given multiplet are auxiliary fields, whose supersymmetry transformations are given by total derivatives. Just for $D = 1$ total derivatives coincide with the (unique) time derivative. Using this specific property of the one-dimensional supersymmetry it was proven in [13] that all finite linear irreps of the (3.21) supersymmetry algebra fall into classes of equivalence, each class of equivalence being singled out by an associated minimal length ($l = 2$) irreducible multiplet. It was further proven that the minimal length irreducible multiplets are in 1-to-1 correspondence with a subclass of Clifford algebras (those which satisfy a Weyl property). The connection goes as follows. The supersymmetry generators acting on a length-2 irreducible multiplet can be expressed as

$$Qi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i \cdot H & 0 \end{pmatrix}, \tag{3.25}$$

where σ_i and $\widetilde{\sigma}_i$ are matrices entering a Weyl type (i.e. block antidiagonal) irreducible representation of the Clifford algebra relation

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i & 0 \end{pmatrix}, \quad \{\Gamma_i, \Gamma_j\} = 2\eta_{ij}. \tag{3.26}$$

The Q_i's in (3.25) are supermatrices with vanishing bosonic and non-vanishing fermionic blocks, acting on an irreducible multiplet m (thought of as a column vector) which can be either bosonic or fermionic. We conventionally consider a length-2 irreducible multiplet as bosonic if its upper half part of component fields is bosonic and its lower half is fermionic. It is fermionic in the converse case. The connection between Clifford algebra irreps of the Weyl type and minimal length irreps of the N-extended one-dimensional supersymmetry is such that D, the dimensionality of the (Euclidean, in the present case) space-time of the Clifford algebra (3.26) coincides with the number N of the extended supersymmetries, according to

♯ of space-time dim. (Weyl-Clifford)	⇔	♯ of extended su.sies (in 1-dim.)
D	=	N

(3.27)

The matrix size of the associated Clifford algebra (equal to $2d$, with d given in (3.22)) corresponds to the number of (bosonic plus fermionic) fields entering the one-dimensional N-extended supersymmetry irrep.

The classification of Weyl-type Clifford irreps, furnished in [13], can be easily recovered from the well-known classification of Clifford irreps, given in [20] (see also [21] and [22]).

The (3.25) Q_i's matrices realizing the N-extended supersymmetry algebra (3.21) on length-2 irreps have entries which are either c-numbers or are proportional to the Hamiltonian H. Irreducible representations of higher length ($l \geq 3$) are systematically produced [13] through repeated applications of the dressing transformations

$$Q_i \mapsto \hat{Q}_i^{(k)} = S^{(k)} Q_i {S^{(k)}}^{-1} \tag{3.28}$$

realized by diagonal matrices $S^{(k)}$'s ($k = 1, \ldots, 2d$) with entries $s_{ij}^{(k)}$ given by

$$s_{ij}^{(k)} = \delta_{ij}(1 - \delta_{jk} + \delta_{jk} H). \tag{3.29}$$

Some remarks are in order [13]:

i) The dressed supersymmetry operators Q_i' (for a given set of dressing transformations) have entries which are integral powers of H. A subclass of the Q_i' s dressed operators is given by the local dressed operators, whose entries are *non-negative* integral powers of H (their entries have no $\frac{1}{H}$ poles). A local representation (irreps fall into this class) of an extended supersymmetry is realized by local dressed operators. The number of the extension, given by $N'(N' \leq N)$, corresponds to the number of local dressed operators.

ii) The local dressed representation is not necessarily an irrep. Since the total number of fields (d bosons and d fermions) is unchanged under dressing, the local dressed representation is an irrep iff d and N' satisfy the (3.22) requirement (with N' in place of N).

iii) The dressing changes the dimension (spin) of the fields of the original multiplet m. Under the $S^{(k)}$ dressing transformation (3.28), $m \mapsto S^{(k)} m$, all fields entering m are unchanged apart from the k-th one (denoted, e.g., as φ_k and mapped to $\dot{\varphi}_k$). Its dimension is changed from $[k] \mapsto [k] + 1$. This is why the dressing changes the length of a multiplet. As an example, if the original length-2 multiplet m is a bosonic multiplet with d spin-0 bosonic fields and d spin-$\frac{1}{2}$ fermionic fields (in the following such a multiplet will be denoted as $(x_i; \psi_j) \equiv (d, d)_{s=0}$, for $i, j = 1, \ldots, d$), then $S^{(k)} m$, for $k \leq d$, corresponds to a length-3 multiplet with $d - 1$ bosonic spin-0 fields, d spin-$\frac{1}{2}$ fermionic fields and a single spin-1 bosonic field (in the following we employ the notation $(d - 1, d, 1)_{s=0}$ for such a multiplet).

Let us now fix the overall conventions. The most general multiplet is of the form $(d_1, d_2, \ldots, d_l)$, where d_i for $i = 1, 2, \ldots, l$ specify the number of fields of a given spin $s + \frac{i-1}{2}$. The spin s, i.e. the spin of the lowest component fields in the multiplet, will also be referred to as the "spin of the multiplet". When looking purely at the representation properties of a given multiplet the assignment of an overall spin s is arbitrary, since the supersymmetry transformations of the fields are not affected by s. Introducing a spin is useful for tensoring multiplets and becomes essential for physical applications, e.g. in the construction of supersymmetric invariant terms entering an action.

In the above multiplet l denotes its length, d_l the number of auxiliary fields of highest spins transforming as time-derivatives. The total number of odd-indexed equal the total number of even-indexed fields, i.e. $d_1 + d_3 + \ldots = d_2 + d_4 + \ldots = d$. The multiplet is bosonic if the odd-indexed fields are bosonic and the even-indexed fields are fermionic (the multiplet is fermionic in the converse case). For a bosonic multiplet the auxiliary fields are bosonic (fermionic) if the length l is an odd (even) number.

Just like the overall spin assignment, the assignment of a bosonic (fermionic) character to a multiplet is arbitrary since the mutual transformation properties of the fields inside a multiplet are not affected by its statistics. Therefore, multiplets always appear in dually related pairs so that to any bosonic multiplet there exists its fermionic counterpart with the same transformation properties (see also [23]).

Throughout this paper we assign integer valued spins to bosonic multiplets and half-integer valued spins to fermionic multiplets.

As pointed out before, the most general $(d_1, d_2, \ldots, d_l)$ multiplet is recovered as a dressing of its corresponding N-extended length-2 (d, d) multiplet. In [13] it was shown that all dressed supersymmetry operators producing any length-3 multiplet (of the form $(d-p, d, p)$ for $p = 1, \ldots, d-1$) are of local type. Therefore, for length-3 multiplets, we have $N' = N$. This implies, in particular, that the $(d-p, d, p)$ multiplets are non-equivalent irreps of the N-extended one-dimensional supersymmetry. As concerns length $l \geq 4$ multiplets, the general problem of finding irreps was not addressed in [13]. It was shown, as a specific example, that the dressing of the length-2 (4, 4) irrep of $N = 4$, realized through the series of mappings $(4, 4) \mapsto (1, 4, 3) \mapsto (1, 3, 3, 1)$, produces at the end a length-4 multiplet $(1, 3, 3, 1)$ carrying only three local supersymmetries ($N' = 3$). Since the relation (3.22) is satisfied when setting the number of extended supersymmetries acting on a multiplet equal to 3 and the total number of bosonic (fermionic) fields entering a multiplet equal to 4, as a consequence, the $(1, 3, 3, 1)$ multiplet corresponds to an irreducible representation of the $N = 3$ extended supersymmetry.

Based on an algorithmic construction of representatives of Clifford irreps, we present an iterative method for classifying all irreducible representations of higher length for arbitrary N values of the extended supersymmetry.

4 Clifford algebras and division algebras

Due to the relation between Supersymmetric Quantum Mechanics and Clifford algebras, we present here a classification of the irreducible representations of Clifford algebras in terms of an algorithm which allows us to single out, in arbitrary signature space-times, a representative in each irreducible class of representations of Clifford's gamma matrices. The class of irreducible representations is unique apart from special signatures, where two non-equivalent irreducible representations are linked by sign flipping ($\Gamma^\mu \leftrightarrow -\Gamma^\mu$). The construction goes as follows. First, we prove that starting from a given spacetime-dimensional representation of Clifford's Gamma matrices, we can recursively construct $D+2$ spacetime dimensional Clifford Gamma matrices with the help of two recursive algorithms. Indeed, it is a simple exercise to verify that if γ_i's denotes the d-dimensional Gamma matrices of a $D = p + q$ spacetime with (p, q) signature (namely, providing a representation for the $C(p, q)$ Clifford algebra) then $2d$-dimensional $D+2$ Gamma matrices (denoted as Γ_j) of a $D+2$ spacetime are produced according to either

Table 1: Table with the maximal Clifford algebras (up to $d = 256$)

(4.33)

1	*	2	*	4	*	8	*	16	*	32	*	64	*	128	*	256	*
(1, 0)	⇒	(2, 1)	⇒	(3, 2)	⇒	(4, 3)	⇒	(5, 4)	⇒	(6, 5)	⇒	(7, 6)	⇒	(8, 7)	⇒	(9, 8)	⇒
						(1, 4)	→	(2, 5)	→	(3, 6)	→	(4, 7)	→	(5, 8)	→	(6, 9)	→
					↗												
				(0,3)													
					↘												
						(5,0)	→	(6, 1)	→	(7, 2)	→	(8, 3)	→	(9, 4)	→	(10, 5)	→
								(1, 8)	→	(2, 9)	→	(3, 10)	→	(4, 11)	→	(5, 12)	→
							↗										
						(0,7)											
							↘										
								(9, 0)	→	(10, 1)	→	(11, 2)	→	(12, 3)	→	(13, 1)	→
														(1, 12)	→	(2, 13)	→
													↗				
												(0, 11)					
													↘				
														(13, 0)	→	(14, 1)	→
																(1, 16)	→
															↗		
														(0, 15)			
															↘		
																(17, 0)	→

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}, \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix} \tag{4.30}$$
$$(p,q) \mapsto (p+1, q+1).$$

or

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}, \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix} \tag{4.31}$$
$$(p,q) \mapsto (q+2, p).$$

It is immediately clear, e.g., that the two-dimensional real-valued Pauli matrices τ_A, τ_1, τ_2 which realize the Clifford algebra $C(2,\ 1)$ are obtained by applying either (4.30) or (4.31) to the number 1, i.e. the one-dimensional realization of $C(1,\ 0)$. We have indeed

$$\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.32}$$

All Clifford algebras are obtained by recursively applying algorithms (4.30) and (4.31) to the Clifford algebra $C(1,\ 0)$ ($\equiv 1$) and the Clifford algebras of the series $C(0, 3+4m)$ (with m non-negative integer), which must be previously known. This is in accordance with the scheme illustrated in the Table1.

Concerning the previous table, some remarks are in order. The columns are labeled by the matrix size d of the maximal Clifford algebras. Their signature is denoted by the $(p,\ q)$ pairs. Furthermore, the underlined Clifford algebras in the table can be named as "primitive maximal Clifford algebras". The remaining maximal Clifford algebras appearing in the table are "maximal descendant Clifford algebras". They are obtained from the primitive maximal Clifford algebras by iteratively applying the two recursive algorithms (4.30) and (4.31). Moreover, any non-maximal Clifford algebra is obtained from a given maximal Clifford algebra by deleting a certain number of Gamma matrices (as an example, Clifford algebras in even-dimensional spacetimes are always non-maximal).

It is immediately clear from the above construction that the maximal Clifford algebras are encountered if and only if the condition

$$p - q = 1{,}5 \bmod 8 \tag{4.34}$$

is matched.

The notion of a Clifford algebra of the generalized Weyl type, namely satisfying the (3.26) condition, has already been introduced. All maximal Clifford algebras, both primitive and descendant, are *not* of the generalized Weyl type. As already pointed out, the notion of generalized Weyl spinors is based on *real*-valued representations of Clifford algebras which, for classification purposes, are more convenient to use w.r.t. the *complex* Clifford algebras that one in general deals with. For this reason generalized Weyl spinors exist also in odd-dimensional space-time, see formula (3.26), while standard Weyl spinors only exist in even-dimensional spacetimes. This can be understood by analyzing a single example. The *real* irrep $C(0,\ 7)$, with all negative signs, is 8-dimensional, see table (4.33), while the *real* irrep $C(7,\ 0)$ is 16-dimensional, but of generalized Weyl type (3.26). Accordingly, Euclidean 8-dimensional fundamental spinors can be understood either as the 8-dimensional "Non-Weyl" spinors of $C(0,\ 7)$, or as 8-dimensional "Weyl-projected" $C(7,\ 0)$ spinors. In the *complex* case, the sign flipping $C(0,7) \mapsto C(7,0)$ can be realized by multiplying all Gamma matrices by the imaginary unit "i". No doubling of the matrix size of the Γ's is found and the notion of Weyl spinors cannot be applied. One faces a similar situation in one-dimensional spacetime. In the *complex* case we can realize $C(1,\ 0)$ with 1 and $C(0,\ 1)$ with i (both one-dimensional). On the other hand, in the *real* case, $C(0,\ 1)$ can only be realized through the 2-dimensional irrep

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which is block-antidiagonal. Throughout the text Weyl (Non--Weyl) spinors are always referred to the (3.26) property with respect to *real*-valued Clifford algebras. Non-maximal Clifford algebras are of the Weyl type if and only if they are produced from a maximal Clifford algebra by deleting at least one spatial Gamma matrix which, without loss of generality, can always be chosen as the one with diagonal entries.

Let us now illustrate how non-maximal Clifford algebras are produced from the corresponding maximal Clifford algebras. The construction goes as follows. We illustrate at first the example of the non-maximal Clifford algebras obtained from the 2-dimensional maximal Clifford irrep $C(2,\ 1)$ furnished by the three matrices τ_1, τ_2, τ_A given in (4.32). If we restrict the Clifford algebra to τ_1, τ_A, i.e. if we delete τ_2 from the previous set, we get the 2-dimensional irrep $C(1,\ 1)$. If we further delete τ_1 we are left with τ_A only, which provides the 2-dimensional irrep $C(0,\ 1)$ discussed above. On the other hand, deleting τ_A from $C(2,\ 1)$ leaves us with τ_1, τ_2, the 2-dimensional irrep $C(2,\ 0)$.

To summarize, from the 2-dimensional irrep of the "maximal Clifford algebra" $C(2,\ 1)$ we obtain the 2-dimensional irreps of the non-maximal Clifford algebras $C(1,\ 1)$, $C(0,\ 1)$ and $C(0,\ 2)$ through a "Γ-matrices deleting procedure". Please note that, through deleting, we cannot obtain from C(1, 2) the irrep , since the latter is one-dimensional.

In full generality, non-maximal Clifford algebras are produced from the corresponding maximal Clifford algebras according to the following table, which specifies the number of time-like or space-like Gamma matrices that should be deleted, as well as the generalized Weyl (W) character or not (NW) of the given non-maximal Clifford algebra. We get

$$
\begin{array}{|c|c|}
\hline
W & NW \\
\hline
\begin{array}{c}(0 \bmod 8) \subset (1 \bmod 8)\\ (p,q) \Leftarrow (p+1,q)\end{array} & \begin{array}{c}(2 \bmod 8) \subset (1 \bmod 8)\\ (p,q) \Leftarrow (p,q+1)\end{array} \\
\hline
\begin{array}{c}(4 \bmod 8) \subset (5 \bmod 8)\\ (p,q) \Leftarrow (p+1,q)\end{array} & \begin{array}{c}(3 \bmod 8) \subset (1 \bmod 8)\\ (p,q) \Leftarrow (p,q+2)\end{array} \\
\hline
\begin{array}{c}(6 \bmod 8) \subset (1 \bmod 8)\\ (p,q) \Leftarrow (p+3,q)\end{array} & \\
\hline
\begin{array}{c}(7 \bmod 8) \subset (1 \bmod 8)\\ (p,q) \Leftarrow (p+2,q)\end{array} & \\
\hline
\end{array}
\quad (4.35)
$$

In the above entries $x \bmod 8$ specifies the mod 8 residue of $t-s$ for any given (t,s) spacetime. Non-maximal Clifford algebras are denoted by $p \equiv t$, $q \equiv s$, while maximal Clifford algebras are denoted by (p', q'), with $p' \geq p$, $q' \geq q$. The differences $p'-p$, $q'-q$ denote how many Clifford gamma matrices (of time-like or respectively space-like type) have to be deleted from a given maximal Clifford algebra to produce the irrep of the corresponding non-maximal Clifford algebra. To be specific, e.g., the 6 mod 8 non-maximal Clifford algebra $C(6,0)$ is obtained from the maximal Clifford algebra $C(9,0)$, whose matrix size is 16 according to (4.33), by deleting three gamma matrices.

To complete our discussion what it remains to specify the construction of the primitive maximal Clifford algebras for both $C(0,3+8n)$ series (which can be named as "quaternionic series", due to its connection with this division algebra, as we will see in the next section), and also the "octonionic" series $C(0,7+8n)$. The answer can be provided with the help of the three Pauli matrices (4.32). We first construct the 4×4 matrices realizing the Clifford algebra $C(0,3)$ and the 8×8 matrices realizing the Clifford algebra $C(0,7)$. They are given, respectively, by

$$
C(0,3) \equiv \begin{array}{l} \tau_A \otimes \tau_1, \\ \tau_A \otimes \tau_2, \\ \mathbf{1}_2 \otimes \tau_A. \end{array} \quad (4.36)
$$

and

$$
C(0,7) \equiv \begin{array}{l} \tau_A \otimes \tau_1 \otimes \mathbf{1}_2, \\ \tau_A \otimes \tau_2 \otimes \mathbf{1}_2, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_1, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_2, \\ \tau_1 \otimes \mathbf{1}_2 \otimes \tau_A, \\ \tau_2 \otimes \mathbf{1}_2 \otimes \tau_A, \\ \tau_A \otimes \tau_A \otimes \tau_A. \end{array} \quad (4.37)
$$

The three matrices of $C(0,3)$ will be denoted as $\overline{\tau}_i = 1,2,3$. The seven matrices of $C(0,7)$ will be denoted as $\widetilde{\tau}_i = 1,2,\ldots,7$.

In order to construct the remaining Clifford algebras of the two series we first need to apply the (4.30) algorithm to $C(0,7)$ and construct the 16×16 matrices realizing $C(1,8)$ (the matrix with a positive signature is denoted as γ_9, $\gamma_9^2 = \mathbf{1}$, while the eight matrices with negative signatures are denoted as γ_j, $j = 1,2,\ldots,8$, with $\gamma_j^2 = -\mathbf{1}$). We are now in the position to explicitly construct the whole series of primitive maximal Clifford algebras $C(0,3+8n)$, $C(0,7+8n)$ through the formulas

$$
C(0,3+8n) \equiv \begin{array}{lll}
\overline{\tau}_i \otimes \gamma_9 \otimes \ldots & \ldots & \otimes\gamma_9, \\
\mathbf{1}_4 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \otimes \mathbf{1}_{16}, \\
\mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \otimes \mathbf{1}_{16}, \\
\mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \otimes \mathbf{1}_{16}, \\
\ldots & \ldots & \ldots, \\
\mathbf{1}_4 \otimes \gamma_9 \otimes \ldots & \ldots & \otimes\gamma_9 \otimes \gamma_j,
\end{array} \quad (4.38)
$$

and similarly

$$
C(0,7+8n) \equiv \begin{array}{lll}
\widetilde{\tau}_i \otimes \gamma_9 \otimes \ldots & \ldots & \otimes\gamma_9, \\
\mathbf{1}_8 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \otimes \mathbf{1}_{16}, \\
\mathbf{1}_8 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \otimes \mathbf{1}_{16}, \\
\mathbf{1}_8 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots & \otimes \mathbf{1}_{16}, \\
\ldots & \ldots & \ldots, \\
\mathbf{1}_8 \otimes \gamma_9 \otimes \ldots & \ldots & \otimes\gamma_9 \otimes \gamma_j.
\end{array} \quad (4.39)
$$

Please note that the tensor product of the 16-dimensional representation is taken n times. The total size of the (4.38) matrix representations is then 4×16^n, while the total size of (4.39) is 8×16^n.

With the help of the formulas presented in this section we are able to systematically construct a set of representatives of the real irreducible representations of Clifford algebras in arbitrary space-times and signatures. It is also convenient to explicitly present of Clifford algebras with the division algebras of the quaternions (and of the octonions).

This relation can be understood as follows. First we note that the three matrices appearing in C(0, 3) can also be expressed in terms of the imaginary quaternions τ_i satisfying

$$
\tau_i \cdot \tau_j = -\delta_{ij} + \epsilon_{ijk}\tau_k . \quad (4.40)
$$

As a consequence, the whole set of maximal primitive Clifford algebras $C(0,3+8n)$, as well as their maximal descendants, can be represented with quaternionic-valued matrices. In its turn the spinors now have to be interpreted as quaternionic-valued column vectors.

Similarly, there exists an alternative realization of the basic relations of the generators of the Euclidean Clifford algebra , obtained by identifying its seven generators with the seven imaginary octonions (for a review on octonions see e.g. [24]) satisfying the algebraic relation

$$
\tau_i \cdot \tau_j = -\delta_{ij} + C_{ijk}\tau_k \quad (4.41)
$$

for $i,j,k = 1,\ldots,7$ and C_{ijk} the totally antisymmetric octonionic structure constants given by

$$
C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \quad (4.42)
$$

and vanishing otherwise. The octonions are non-associative and cannot be represented in matrix form with the usual matrix multiplication. On the other hand, a construction due to Dixon allows us to produce the seven 8×8 matrix generators of the C(0, 7) Clifford algebras in terms of the octonionic structure constants. Given a real octonion $x = x_0 + \sum_i x_i\tau_i$, with real coefficients x_0, x_i, for $i = 1,\dots,7$, the left action of the imaginary octonions τ_i ($x' = \tau_i \cdot x$) is reproduced in terms of the 8×8 Clifford gamma matrix γ_i, linearly acting on x_0, x_i's.

5 The field content of irreducible representations

It is now possible to plug the information contained in Clifford algebras and apply the construction outlined in Section 3 to compute the admissible field content for the length-4 multiplets for arbitrary values of N. This construction was done in [14]. We present here the list of length-4 field content up to $N \le 11$.

Up to $N = 8$ we have

$N = 1$	NO
$N = 2$	NO
$N = 3$	(1, 3, 3, 1)
$N = 4$	NO
$N = 5$	(1, 5, 7, 3), (3, 7, 5, 1), (1, 6, 7, 2), (2, 7, 6, 1), (2, 6, 6, 2), (1, 7, 7, 1)
$N = 6$	(1, 6, 7, 2), (2, 7, 6, 1), (2, 6, 6, 2), (1, 7, 7, 1)
$N = 7$	(1, 7, 7, 1)
$N = 8$	NO

(5.43)

Since there are no length-l irreps with $l \ge 5$ for $N \le 9$, the above list, together with the already known length-2 and length-3 irreps, provides the complete classification of the admissible field content of the irreducible representations for $N \le 8$.

Please note that the length-4 irrep of $N = 3$, (1, 3, 3, 1), is self-dual under the [14] *high* $\Leftrightarrow$ *low* spin duality, while two of the non-equivalent length-4 $N = 5$ irreps are self-dual, (2, 6, 6, 2) and (1, 7, 7, 1). The remaining ones are pair-wise dually related ($(1,5,7,3) \Leftrightarrow (3,7,5,1)$ and $(1,6,7,2) \Leftrightarrow (2,7,6,1)$).

The $N = 9$ length-4 irreducible multiplet (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k \equiv d_4$, since $d_3 = 16 - h$, $d_4 = 16 - k$. The complete list of $N = 9$ length-4 fields content is expressed by h, k satisfying the constraint

$$h + k \le 8. \tag{5.44}$$

$N = 10$ is the lowest supersymmery admitting length-5 irreps. The field content of its length-4 irreps is given by (d_1, d_2, d_3, d_4), expressed in terms of the two positive integers $h \equiv d_1$, $k \equiv d_4$, since $d_3 = 32 - h$, $d_4 = 32 - k$. If we set

$$r = \min(h, k) \tag{5.45}$$

the non-equivalent length-4 field content is given by the ordered pair of positive integers h, k satisfying the constraint

$$h + k + r \le 24. \tag{5.46}$$

For $N = 11$ the length-4 fields content (d_1, d_2, d_3, d_4) is expressed in terms of the two positive integers $h \equiv d_1$, $k \equiv d_4$, since $d_3 = 64 - h$, $d_2 = 64 - k$. Setting as before $r = \min(h, k)$ and introducing the $s(r)$ function defined through

$$s(r) = \begin{Bmatrix} 8 - r & \text{for } r = 1,\dots,7 \\ 0 & \text{otherwise} \end{Bmatrix} \tag{5.47}$$

we can express the constraints on h, k as

$$h + k + r - s(r) \le 48. \tag{5.48}$$

6 The off-shell invariant actions of the $N = 4$ sigma models

In the late 1980's and early 1990's, the whole set of off-shell invariant actions of the $N = 4$ supersymmetries were produced ([5] and references therein), by making use of the superfield formalism. This result was reached after slowly recognizing the multiplets carrying a representation of the one-dimensional $N = 4$ supersymmetry. The results discussed here allow us to reconstruct, in a unified framework, all off-shell invariant actions of the correct mass-dimension (the mass-dimension $d = 2$ of the kinetic energy) for the whole set of $N = 4$ irreducible multiplets. They are given by the (4, 4), (3, 4, 1), (2, 4, 2) and (1, 3, 4) multiplets.

We are able to construct the invariants without using a superfield formalism. We use instead a construction which can be extended, how we will prove later, even for large values of N, in the cases where the superfield formalism is not available. We will use the fact that the supersymmetry generators Q_i's act as graded Leibniz derivatives. Manifestly invariant actions of the N-extended supersymmetry can be obtained by expressing them as

$$I = \int dt(Q_1 \cdot \ldots \cdot Q_N f(x_1, x_2, \ldots, x_k)) \tag{6.49}$$

with the supersymmetry transformations applied to an arbitrary function of the 0-dimensional fields x_i's, $i = 1,\dots,k$ entering an irreducible multiplet of the N-extended supersymmetry. Since the supersymmetry generators admit mass-dimension $= \frac{1}{2}$ (being the "square roots" of the Hamiltonian), we have that (6.49) is a manifestly supersymmetric invariant whose lagrangian density $Q_1 \cdot \ldots \cdot Q_N f(x_1, x_2, \ldots, x_k)$ has a dimension $d = \frac{N}{2}$. For $N = 4$ the lagrangian density has the correct dimension of a kinetic term.

The k variables x_i's can be regarded as coordinates of a k-dimensional manifold. The corresponding actions can therefore be seen as $N = 4$ supersymmetric one-dimensional sigma models evolving in a k-dimensional target manifold. For each $N = 4$ irrep we get the following results. In all cases below the arbitrary $\alpha(x_i)$ function is given by $\alpha = \Box f(x)$. We get the following list.

i) The $N = 4$ (4, 4) case. We have:

$$Q_i(x, x_j; \psi, \psi_j) = (-\psi_i, \delta_{ij}\psi - \varepsilon_{ijk}\psi_k; \dot{x}_i, -\delta_{ij}\dot{x} + \varepsilon_{ijk}\dot{x}_k),$$
$$Q_4(x, x_j; \psi, \psi_j) = (\psi, \psi_j; \dot{x}, \dot{x}_j). \tag{6.50}$$

The most general invariant Lagrangian L of dimension $d = 2$ is given by

$$\begin{aligned} L &= \alpha(\vec{x})[\dot{x}^2 + \dot{x}_j^2 - \psi\dot{\psi} - \psi_j\dot{\psi}_j] \\ &+ \partial_x\alpha[\psi\psi_j\dot{x}_j - \frac{1}{2}\varepsilon_{ijk}\psi_i\psi_j\dot{x}_k] \\ &+ \partial_l\alpha[\psi_l\psi\dot{x} + \psi_l\psi_j\dot{x}_j + \frac{1}{2}\varepsilon_{ljk}\psi_j\psi_k\dot{x} - \varepsilon_{ljk}\psi_j\dot{x}_k\psi] \\ &- \Box\alpha\frac{1}{6}\varepsilon_{ljk}\psi\psi_l\psi_k\psi_k. \end{aligned} \tag{6.51}$$

ii) The $N = 4$ (3, 4, 1) case. We have:

$$Q_i(x_j; \psi, \psi_j; g) = (\delta_{ij}\psi - \varepsilon_{ijk}\psi_k; \dot{x}_i; -\delta_{ij}g + \varepsilon_{ijk}\dot{x}_k; -\psi_i),$$
$$Q_4(x_j; \psi, \psi_j; g_j) = (\psi_j; g, \dot{x}_j; \psi). \tag{6.52}$$

The most general invariant Lagrangian L of dimension $d = 2$ is given by

$$\begin{aligned} L &= \alpha(\vec{x})[\dot{x}_j^2 + g^2 - \psi\dot{\psi} - \psi_j\dot{\psi}_j] \\ &+ \partial_i\alpha[\varepsilon_{ijk}(\psi\psi_j\dot{x}_k + \frac{1}{2}g\psi_j\psi_k) - g\psi\psi_i + \psi_i\psi_j\dot{x}_j] \\ &- \frac{\Box\alpha}{6}\varepsilon_{ijk}\psi\psi_i\psi_j\psi_k. \end{aligned} \tag{6.53}$$

iii) The $N = 4$ (2, 4, 2) case. We have:

$$\begin{aligned} Q_1(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) &= (\psi_0, \psi_3; \dot{x}, -g, h, -\dot{y}; -\dot{\psi}_1, \dot{\psi}_2), \\ Q_2(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) &= (\psi_3, \psi_0; \dot{y}, -h, -g, \dot{x}; -\dot{\psi}_2, -\dot{\psi}_1), \\ Q_3(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) &= (-\psi_2, \psi_1; h, \dot{y} - \dot{x}, -g; -\dot{\psi}_3, \dot{\psi}_0), \\ Q_4(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) &= (\psi_1, \psi_2; g, \dot{x}, \dot{y}, h; \dot{\psi}_0, \dot{\psi}_3). \end{aligned} \tag{6.54}$$

The most general invariant Lagrangian L of dimension $d = 2$ is given by

$$\begin{aligned} L &= \alpha(x, y)[\dot{x}^2 + \dot{y}^2 + g^2 + h^2 - \psi\dot{\psi} - \psi_j\dot{\psi}_j] \\ &+ \partial_x\alpha[\dot{y}(\psi_1\psi_2 - \psi_0\psi_3) + g(\psi_2\psi_3 - \psi_0\psi_1) + h(\psi_1\psi_3 - \psi_0\psi_2)] \\ &+ \partial_y\alpha[-\dot{x}(\psi_1\psi_2 - \psi_0\psi_3) - g(\psi_1\psi_3 + \psi_0\psi_2) + h(\psi_2\psi_3 - \psi_0\psi_1)] \\ &- \Box\alpha\,\psi_0\psi_1\psi_2\psi_3. \end{aligned} \tag{6.55}$$

iv) The $N = 4$ (1, 4, 3) case. We have:

$$Q_i(x; \psi, \psi_j; g_j) = (-\psi_i; g_i, -\delta_{ij}\dot{x} + \varepsilon_{ijk}g_k; \delta_{ij}\dot{\psi} - \varepsilon_{ijk}\dot{\psi}_k),$$
$$Q_4(x; \psi, \psi_j; g_j) = (\psi; \dot{x}, g_j; \dot{\psi}_j). \tag{6.56}$$

The most general invariant Lagrangian L of dimension $d = 2$ is given by

$$\begin{aligned} L &= \alpha(x)[\dot{x}^2 - \psi\dot{\psi} - \psi_i\dot{\psi}_i + g_i^2] \\ &+ \alpha'(x)[\psi g_i\psi_i - \frac{1}{2}\varepsilon_{ijk}g_i\psi_j\psi_k] - \frac{\alpha''(x)}{6}[\varepsilon_{ijk}\psi\psi_i\psi_j\psi_k]. \end{aligned} \tag{6.57}$$

It is worth recalling that $N = 4$ is associated, as we have discussed, to the algebra of the quaternions. This is why in cases (4, 4), (3, 4, 1) and (1, 4, 3) the invariant actions can be written by making use of the quaternionic tensors δ_{ij} and ε_{ijk}. In the (2, 4, 2) case two fields are dressed to be auxiliary fields and this spoils the quaternionic covariance property.

7 Octonions and $N = 8$ sigma-models

Just as the $N = 4$ supersymmetry is related with the algebra of quaternions, the $N = 8$ supersymmetry is related with the algebra of the octonions. More specifically, it can be proven that the $N = 8$ supersymmetry can be produced from the lifting of the $Cl(0, 7)$ Clifford algebra to $Cl(0, 9)$. On the other hand, it is well-known, as we have discussed before, that the seven 8×8 antisymmetric gamma matrices of $Cl(0, 7)$ can be recovered by the left-action of the imaginary octonions on the octonionic space. As a result, the entries of the seven antisymmetric gamma-matrices of $Cl(0, 7)$ can be expressed in terms of the totally antisymmetric octonionic structure constants C_{ijk}'s. The non-vanishing C_{ijk}'s are given by

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{7.58}$$

The non-vanishing octonionic structure constants are associated with the seven lines of the Fano projective plane, the smallest example of a finite projective geometry, see [24]. The $N = 8$ supersymmetry transformations of the various irreps can, as a consequence, be expressed in terms of octonionic structure constants. This is in particular true for the dressed (1, 8, 7) multiplet, admitting seven fields which are "dressed" to become auxiliary fields. This is an example of a multiplet which preserves the octonionic structure since the seven dressed fields are related to the seven imaginary octonions. We have that the supersymmetry transformations are given by

$$Q_i(x; \psi, \psi_j; g_j) = (-\psi_i; g_i, -\delta_{ij}\dot{x} + C_{ijk}g_k; \delta_{ij}\dot{\psi} - C_{ijk}\dot{\psi}_k),$$
$$Q_8(x; \psi, \psi_j; g_j) = (\psi; \dot{x}, g_j; \dot{\psi}_j). \tag{7.59}$$

for $i, j, k = 1, \ldots, 7$. The strategy for constructing the most general $N = 8$ off-shell invariant action of the (1, 8, 7) multiplet makes use of the octonionic covariantization principle. When restricted to an $N = 4$ subalgebra, the invariant actions should have the form of the $N = 4$ (1, 4, 3) action (6.57). This restriction can be made in seven non-equivalent ways (the seven lines of the Fano plane). The general $N = 8$ action should be expressed in terms of octonionic structure constants. With respect to (6.57), an extra-term could in principle be present. It is given by $\int dt\beta(x)C_{ijk}\psi_i\psi_j\psi_k\psi_l$ and is constructed in terms of the octonionic tensor of rank 4

$$C_{ijkl} = \frac{1}{6}\varepsilon_{ijklmnp}C_{mnp} \tag{7.60}$$

(where $\varepsilon_{ijklmnp}$ is the seven-index totally antisymmetric tensor). Please note that the rank-4 tensor is obviously vanishing when restricting to the quaternionic subspace. One immediately verifies that the term $\int dt\beta(x)C_{ijk}\psi_i\psi_j\psi_k\psi_l$ breaks the $N = 8$ supersymmetries and cannot enter the invariant action. As concerns the other terms, starting from the general action (with $i, j, k = 1, \ldots, 7$)

$$\begin{aligned} S = \int dt\{&\alpha(x)[\dot{x}^2 - \psi\dot{\psi} - \psi_i\dot{\psi}_i + g_i^2] \\ &+ \alpha'(x)[\psi g_i\psi_i - \frac{1}{2}C_{ijk}g_i\psi_j\psi_k] \\ &- \frac{\alpha''(x)}{6}[C_{ijk}\psi\psi_i\psi_j\psi_k]\} \end{aligned} \tag{7.61}$$

we can prove that the invariance under the Q_i generator $(= 1, \ldots, 7)$ is broken by terms which, after integration by parts, contain at least a second derivative α''. We obtain, e.g., a non-vanishing term of the type $\int dt\alpha'' \frac{\psi}{2} C_{ijk}g_i\psi_k\psi_l$. In order to guarantee the full $N = 8$ invariance (the invariance under Q_8 is automatically guaranteed) we have therefore to set $\alpha''(x) = 0$, leaving α a linear function in x. As a result, the most general $N = 8$ off-shell invariant action of the (1, 8, 7) multiplet is given by

$$\begin{aligned} S = \int dt\{&(ax + b)[\dot{x}^2 - \psi\dot{\psi} - \psi_i\dot{\psi}_i + g_i^2] \\ &+ a[\psi g_i\psi_i - \frac{1}{2}C_{ijk}g_i\psi_j\psi_k]\}. \end{aligned} \tag{7.62}$$

We can express this result in the following terms: the association of the $N = 8$ supersymmetry with the octonions implies that the octonionic structure constants enter as coupling constants in the $N = 8$ invariant actions. The situation w.r.t. the other $N = 8$ multiplets is more complicated. This is due to the fact that the dressing of some of the bosonic fields to auxiliary fields does not respect octonionic covariance. The construction of the invariant actions can however be performed along similar lines, the octonionic structure constants being replaced by the "dressed" structure constants. The procedure for a generic irrep is more involved than in the(1, 8, 7) case. The full list of invariant actions for the $N = 8$ irreps is currently being written. The results will be reported elsewhere. The method proposed is quite interesting because it allows us in principle to construct the most general invariant actions. It is worth mentioning that various groups, using $N = 8$ superfield formalism, are still working on the problem of constructing the most general invariant actions.

Let us close this section by pointing out that the only sign of the octonions is through their structure constants entering as parameters in the (7.62) $N = 8$ off-shell invariant action. (7.62) is an ordinary action, in terms of ordinary associative bosonic and fermionic fields closing an ordinary $N = 8$ supersymmetry algebra.

8 Non-equivalent representations with the same field content

The irreducible representations of the N-extended supersymmetry algebra are nicely presented in terms of N-colored graphs with arrows (we will explain below how to draw the graphs). The existence of irreducible representations admitting the same field content, but non-equivalent graphs was pointed out in [25]. In [15] the non-equivalent graphs associated to irreducible representations up to $N \le 8$ were classified. We discuss here both construction of [15] and also its main results. Since it can be quite easily proved that non-equivalent graphs are not encountered for $N \le 4$, it is sufficient to discuss the irreducible representations of $N = 5, 6, 7, 8$, which are obtained through a dressing of the $N = 8$ length-2 root multiplet of type (8, 8) (see the previous discussion). Inequivalent graphs (see [15]) are described by the so-called connectivity of the irreps. Connectivity can be understood as follows. For the class of irreducible representations under consideration any given field of dimension d is mapped, under a supersymmetry transformation, either

i) to a field of dimension $d + \frac{1}{2}$ belonging to the multiplet (or to its opposite, the sign of the transformation being irrelevant for our purposes) or,

ii) to the time-derivative of a field of dimension $d - \frac{1}{2}$.

If the given field belongs to an irrep of the N-extended one-dimensional supersymmetry algebra, therefore $k \le N$ of its transformations are of type i), while the $N - k$ remaining ones are of type ii). Let us now specialize our discussion to a length-3 irrep (the interesting case for us). Its field content is given by $(n_1, n, n - n_1)$, while the set of its fields is expressed by $(x_i; \psi_j; g_k)$, with $i = 1, \ldots, n_1$, $j = 1, \ldots, n$, $k = 1, \ldots, n - n_1$. The x_i's are 0-dimensional fields (the ψ_j are $\frac{1}{2}$-dimensional fields and g_k are 1-dimensional fields, respectively). The connectivity associated to the given multiplet is defined in terms of the ψ_g symbol. It encodes the following information. The n $\frac{1}{2}$-dimensional fields ψ_j are partitioned in the subsets of m_r fields admitting k_r supersymmetry transformations of type i) (k_r can take the 0 value). We have $\sum_r m_r = n$. The ψ_g symbol is expressed as

$$\psi_g \equiv m_{1k_1} + m_{2k_2} + \ldots \tag{8.63}$$

As an example, the $N = 7$ (6, 8, 2) multiplet admits connectivity $\psi_g = 6_2 + 2_1$ (see (9.68)). This means that there are two types of fields ψ_j. Six of them are mapped, under supersymmetry transformations, into the two auxiliary fields g_k. The two remaining fields ψ_j are only mapped into a single auxiliary field.

An analogous symbol, x_ψ, can be introduced. It describes the supersymmetry transformations of the x_i fields into the ψ_j

fields. This symbol is, however, always trivial. An N-irrep with $(n_1, n, n-n_1)$ field content always produces $x_\psi \equiv n_{1N}$. Let us now discuss how to compute the connectivities. (8, 8) involves 8 bosonic and 8 fermionic fields entering a column vector (the bosonic fields are accommodated in the upper part). The 8 supersymmetry operators $\hat{Q}_i$ $(i = 1, \ldots, 8)$ in the (8, 8) $N = 8$ irrep are given by the matrices

$$\hat{Q}_j = \begin{pmatrix} 0 & \gamma_j \\ -\gamma_j \cdot H & 0 \end{pmatrix}, \quad \hat{Q}_8 = \begin{pmatrix} 0 & \mathbf{1}_8 \\ \mathbf{1}_8 \cdot H & 0 \end{pmatrix}, \tag{8.64}$$

where the γ_j matrices $(j = 1, \ldots, 7)$ are the 8×8 generators of the $Cl(0, 7)$ Clifford algebra and $H = i\frac{\mathrm{d}}{\mathrm{d}t}$ is the Hamiltonian. The $Cl(0, 7)$ Clifford irrep is uniquely defined up to similarity transformations and an overall sign flipping [22]. Without loss of generality we can unambiguously fix the γ_j matrices to be given as in the Appendix. Each γ_j matrix (and the $\mathbf{1}_8$ identity) possesses 8 non-vanishing entries, one in each column and one in each row. The whole set of non-vanishing entries of the eight (A.1) matrices fills the entire $8 \times 8 = 64$ squares of a "chessboard". The chessboard appears in the upper right block of (8.64).

The length-3 and length-4 $N = 5, 6, 7, 8$ irreps (no irrep with length $l > 4$ exists for $N \leq 9$, see [14]) are acted upon by the Q_i's supersymmetry transformations, obtained from the original $\hat{Q}_i$ operators through a dressing,

$$\hat{Q}_i \to Q_i = D\hat{Q}_i D^{-1}, \tag{8.65}$$

realized by a diagonal dressing matrix D. It should be noted that only the subset of "regular" dressed operators Q_i (i.e., having no $\frac{1}{H}$ or higher poles in its entries) act on the new irreducible multiplet. Apart from the self-dual (4, 8, 4) $N = 5, 6$ irreps, without loss of generality, for our purpose of computing the irrep connectivities, the diagonal dressing matrix D which produces an irrep with $(n_1, n, n-n_1)$ fields content can be chosen to have its non-vanishing diagonal entries given by $\delta_{pq} d_q$, with $d_q = 1$ for $q = 1, \ldots, n_1$ and $q = n+1, \ldots, 2n$, while $d_q = H$ for $q = n_1 + 1, \ldots, n$. Any permutation of the first n entries produces a dressing which is equivalent, for computing both the field content and the ψ_g connectivity, to D. The only exceptions correspond to the $N = 5$ (4, 8, 4) and $N = 6$ (4, 8, 4) irreps. Besides the diagonal matrix D as above, non-equivalent irreps can be obtained by a diagonal dressing D' with diagonal entries $\delta_{pq} d'_q$, with $d'_q = H$ for $q = 4, 6, 7, 8$ and $d'_q = 1$ for the remaining values of q.

Similarly, the $(n_1, n_2, n-n_1, n-n_2)$ length-4 multiplets are acted upon by the Q_i operators dressed by D, whose non-vanishing diagonal entries are now given by $\delta_{pq} d_q$, with $d_q = 1$ for $q = 1, \ldots, n_1$ and $q = 2n - n_2 + 1, \ldots, 2n$, while $d_q = H$ for $q = n_1 + 1, \ldots, 2n - n_2$.

The $N = 5, 6, 7, 8$ length-2 (8, 8) irreps are unique (for the given value of N), see [26].

It is also easily recognized that all $N = 8$ length-3 irreps of a given field content produce the same value of ψ_g connectivity (8.63). As concerns the length-3 $N = 5, 6, 7$ irreps the situation is as follows. Let us consider the irreps with $(k, 8, 8-k)$ field content. Their supersymmetry transformations are defined by picking an $N < 8$ subset from the complete set of 8 dressed Qi operators. It is easily recognized that for $N = 7$, no matter which supersymmetry operator is discarded, any choice of the seven operators produces the same value for the ψ_g connectivity. Irreps with different connectivity can therefore only be found for $N = 5, 6$. The $\begin{pmatrix} 8 \\ 6 \end{pmatrix} = 28$ choices of $N = 6$ operators fall into two classes, denoted as A and B, which can, potentially, produce $(k, 8, 8-k)$ irreps with different connectivity. Similarly, the $\begin{pmatrix} 8 \\ 5 \end{pmatrix} = 56$ choices of $N = 5$ operators fall into two A and B classes which can, potentially, produce irreps of different connectivity. For some given $(k, 8, 8-k)$ irrep, the value of ψ_g connectivity computed in both $N = 5$ (as well as $N = 6$) classes can actually coincide. In the next Section we will show when this feature does indeed happen.

To be specific, we present a list of representatives of the supersymmetry operators for each N and in each $N = 5, 6$ A, B class. We have, with diagonal dressing D,

$$\begin{array}{lcl} N = 8 & \equiv & Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8 \\ N = 7 & \equiv & Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7 \\ N = 6 \text{ (case } A) & \equiv & Q_1, Q_3, Q_4, Q_5, Q_6, Q_7 \\ N = 6 \text{ (case } B) & \equiv & Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \\ N = 5 \text{ (case } A) & \equiv & Q_3, Q_4, Q_5, Q_6, Q_7 \\ N = 5 \text{ (case } B) & \equiv & Q_2, Q_3, Q_4, Q_5, Q_6 \end{array} \tag{8.66}$$

and, with diagonal dressing D' for the (4, 8, 4) irreps,

$$\begin{array}{lcl} N = 6 \text{ (case } A') & \equiv & Q_1, Q_3, Q_4, Q_5, Q_6, Q_7 \\ N = 5 \text{ (case } A') & \equiv & Q_3, Q_4, Q_5, Q_6, Q_7 \end{array} \tag{8.67}$$

We are now in a position to compute the connectivities of the irreps (the results are furnished in the next Section). Quite literally, the computations can be performed by filling a chessboard with pawns representing the allowed configurations.

9 Classification of the irrep connectivities

In this Section we report the results of the computation of the allowed connectivities for the $N = 5, 6, 7$ length-3 irreps. It turns out that the only values of $N \leq 8$ allowing the existence of multiplets with the same field content but non-equivalent connectivities are $N = 5$ and $N = 6$. The results concerning the allowed ψ_g connectivities of the length-3 irreps are reported in the following table (the A, A', B cases of $N = 5, 6$ are specified)

$$
\begin{array}{|l|l|l|l|}
\hline
\text{lenght-3} & N=7 & N=6 & N=5 \\
\hline
(7,8,1) & 7_1+1_0 & 6_1+2_0 & 5_1+3_0 \\
\hline
(6,8,2) & 6_2+2_1 & \begin{array}{l} 6_2+2_0 \;\; (A) \\ 4_2+4_1 \;\; (B) \end{array} & \begin{array}{l} 4_2+2_1+2_0 \;\; (A) \\ 2_2+6_1 \;\; (B) \end{array} \\
\hline
(5,8,3) & 5_3+3_2 & \begin{array}{l} 4_3+2_2+2_1 \;\; (A) \\ 2_3+6_2 \;\; (B) \end{array} & \begin{array}{l} 4_3+3_1+1_0 \;\; (A) \\ 1_3+5_2+2_1 \;\; (B) \end{array} \\
\hline
(4,8,4) & 4_4+4_3 & \begin{array}{l} 4_4+4_2 \;\; (A \\ 2_4+4_3+2_2 \;\; (A') \\ 8_3 \;\; (B) \end{array} & \begin{array}{l} 4_4+4_1 \;\; (A) \\ 1_4+3_3+3_2+1_1 \;\; (A') \\ 4_3+4_2 \;\; (B) \end{array} \\
\hline
(3,8,5) & 3_5+5_4 & \begin{array}{l} 2_5+2_4+4_3 \;\; (A) \\ 6_4+2_3 \;\; (B) \end{array} & \begin{array}{l} 1_5+3_4+4_2 \;\; (A) \\ 2_4+5_3+1_2 \;\; (B) \end{array} \\
\hline
(2,8,6) & 2_6+6_2 & \begin{array}{l} 2_6+6_4 \;\; (A) \\ 4_5+4_4 \;\; (B) \end{array} & \begin{array}{l} 2_5+2_4+4_3 \;\; (A) \\ 6_4+2_3 \;\; (B) \end{array} \\
\hline
(1,8,7) & 1_7+7_6 & 2_6+6_5 & 3_5+5_4 \\
\hline
\end{array}
\qquad (9.68)
$$

It is useful to explicitly present, in at least one pair of examples, the supersymmetry transformations (depending on the ε_i global fermionic parameters) for multiplets admitting different connectivities and the same field content. We write below a pair of $N=5$ irreps (the $(4,8,4)_A$ and the $(4,8,4)_B$ multiplets) differing by connectivity. It is also convenient to visualize them graphically. The graphical presentation at the end of this Section is given as follows. Three rows of (from bottom to up) 4, 8 and 4 dots are associated with the x_i, ψ_j and g_k fields, respectively. Supersymmetry transformations are represented by lines of 5 different colors (since $N=5$). Solid lines are associated to transformations with a positive sign, and dashed lines to transformations with a negative sign. It is easily recognized that in the type A graph there are 4 ψ_j points with four colored lines connecting them to the g_k points, while the 4 remaining ψ_j points admit a single line connecting them to the g_k points. In the type B graph we have 4 ψ_j points with three colored lines and the 4 remaining ψ_j points with two colored lines connecting them to the g_k points.

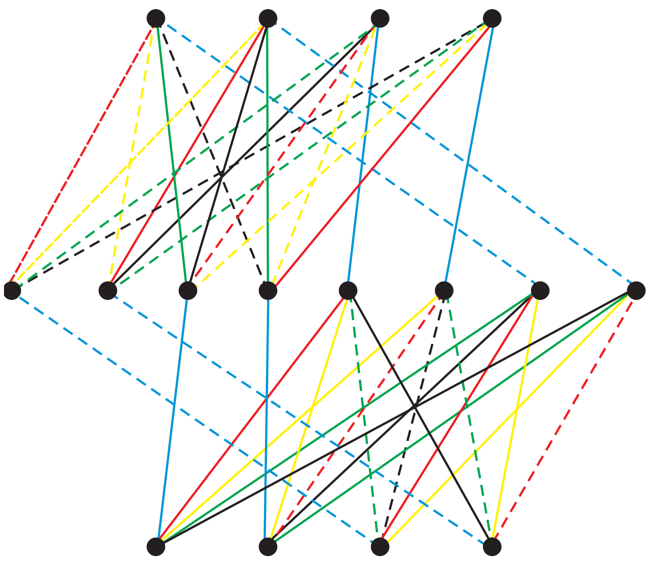

Fig. 1: Graph of the $N=5\,(4,8,4)$ multiplet of 4_4+4_1 connectivity (type A)

The supersymmetry transformations are explicitly given by

i) The $N=5\;(4,8,4)_A$ transformations:

$$
\begin{aligned}
\delta x_1 &= \varepsilon_2\psi_3+\varepsilon_4\psi_5+\varepsilon_3\psi_6+\varepsilon_1\psi_7+\varepsilon_5\psi_8\\
\delta x_2 &= \varepsilon_2\psi_4+\varepsilon_3\psi_5+\varepsilon_4\psi_6-\varepsilon_5\psi_7+\varepsilon_1\psi_8\\
\delta x_3 &= -\varepsilon_2\psi_1-\varepsilon_1\psi_5-\varepsilon_5\psi_6+\varepsilon_4\psi_7+\varepsilon_3\psi_8\\
\delta x_4 &= -\varepsilon_2\psi_2+\varepsilon_5\psi_5-\varepsilon_1\psi_6+\varepsilon_3\psi_7-\varepsilon_4\psi_8\\
\delta\psi_1 &= -i\varepsilon_2\dot{x}_3-\varepsilon_4 g_1-\varepsilon_3 g_2-\varepsilon_1 g_3-\varepsilon_5 g_4\\
\delta\psi_2 &= -i\varepsilon_2\dot{x}_4-\varepsilon_3 g_1+\varepsilon_4 g_2+\varepsilon_5 g_3-\varepsilon_1 g_4\\
\delta\psi_3 &= i\varepsilon_2\dot{x}_1+\varepsilon_1 g_1+\varepsilon_5 g_2-\varepsilon_4 g_3-\varepsilon_3 g_4\\
\delta\psi_4 &= i\varepsilon_2\dot{x}_2-\varepsilon_5 g_1+\varepsilon_1 g_2-\varepsilon_3 g_3+\varepsilon_4 g_4\\
\delta\psi_5 &= i\varepsilon_4\dot{x}_1+i\varepsilon_3\dot{x}_2-i\varepsilon_1\dot{x}_3+i\varepsilon_5\dot{x}_4+\varepsilon_2 g_3\\
\delta\psi_6 &= i\varepsilon_3\dot{x}_1-i\varepsilon_4\dot{x}_2-i\varepsilon_5\dot{x}_3-i\varepsilon_1\dot{x}_4+\varepsilon_2 g_4\\
\delta\psi_7 &= i\varepsilon_1\dot{x}_1-i\varepsilon_5\dot{x}_2+i\varepsilon_4\dot{x}_3+i\varepsilon_3\dot{x}_4-\varepsilon_2 g_1\\
\delta\psi_8 &= i\varepsilon_5\dot{x}_1+i\varepsilon_1\dot{x}_2+i\varepsilon_3\dot{x}_3-i\varepsilon_4\dot{x}_4-\varepsilon_2 g_2\\
\delta g_1 &= -i\varepsilon_4\dot{\psi}_1-i\varepsilon_3\dot{\psi}_2+i\varepsilon_1\dot{\psi}_3-i\varepsilon_5\dot{\psi}_4-i\varepsilon_2\dot{\psi}_7\\
\delta g_2 &= -i\varepsilon_3\dot{\psi}_1+i\varepsilon_4\dot{\psi}_2+i\varepsilon_5\dot{\psi}_3+i\varepsilon_1\dot{\psi}_4-i\varepsilon_2\dot{\psi}_8\\
\delta g_3 &= -i\varepsilon_1\dot{\psi}_1+i\varepsilon_5\dot{\psi}_2-i\varepsilon_4\dot{\psi}_3-i\varepsilon_3\dot{\psi}_4+i\varepsilon_2\dot{\psi}_5\\
\delta g_4 &= -i\varepsilon_5\dot{\psi}_1-i\varepsilon_1\dot{\psi}_2-i\varepsilon_3\dot{\psi}_3+i\varepsilon_4\dot{\psi}_4+i\varepsilon_2\dot{\psi}_6
\end{aligned}
\qquad (9.69)
$$

ii) The $N=5\;(4,8,4)_B$ transformations:

$$
\begin{aligned}
\delta x_1 &= \varepsilon_5\psi_2+\varepsilon_2\psi_3+\varepsilon_4\psi_5+\varepsilon_3\psi_6+\varepsilon_1\psi_7\\
\delta x_2 &= -\varepsilon_5\psi_1+\varepsilon_2\psi_4+\varepsilon_3\psi_5-\varepsilon_4\psi_6+\varepsilon_1\psi_8\\
\delta x_3 &= -\varepsilon_2\psi_1-\varepsilon_5\psi_4-\varepsilon_1\psi_5+\varepsilon_4\psi_7+\varepsilon_3\psi_8\\
\delta x_4 &= -\varepsilon_2\psi_2+\varepsilon_5\psi_3-\varepsilon_1\psi_6+\varepsilon_3\psi_7-\varepsilon_4\psi_8\\
\delta\psi_1 &= -i\varepsilon_5\dot{x}_2-i\varepsilon_2\dot{x}_3-\varepsilon_4 g_1-\varepsilon_3 g_2-\varepsilon_1 g_3\\
\delta\psi_2 &= i\varepsilon_5\dot{x}_1-i\varepsilon_2\dot{x}_4-\varepsilon_3 g_1+\varepsilon_4 g_2-\varepsilon_1 g_4\\
\delta\psi_3 &= i\varepsilon_2\dot{x}_1+i\varepsilon_5\dot{x}_4+\varepsilon_1 g_1-\varepsilon_4 g_3-\varepsilon_3 g_4\\
\delta\psi_4 &= i\varepsilon_2\dot{x}_2-i\varepsilon_5\dot{x}_3+\varepsilon_1 g_2-\varepsilon_3 g_3+\varepsilon_4 g_4\\
\delta\psi_5 &= i\varepsilon_4\dot{x}_1+i\varepsilon_3\dot{x}_2-i\varepsilon_1\dot{x}_3-\varepsilon_5 g_2+\varepsilon_2 g_3\\
\delta\psi_6 &= i\varepsilon_3\dot{x}_1-i\varepsilon_4\dot{x}_2-i\varepsilon_1\dot{x}_4+\varepsilon_5 g_1+\varepsilon_2 g_4\\
\delta\psi_7 &= i\varepsilon_1\dot{x}_1+i\varepsilon_4\dot{x}_3+i\varepsilon_3\dot{x}_4-\varepsilon_2 g_1+\varepsilon_5 g_4\\
\delta\psi_8 &= i\varepsilon_1\dot{x}_2+i\varepsilon_3\dot{x}_3-i\varepsilon_4\dot{x}_4-\varepsilon_2 g_2-\varepsilon_5 g_3
\end{aligned}
\qquad (9.70)
$$

$$\delta g_1 = -i\varepsilon_4\dot{\psi}_1 - i\varepsilon_3\dot{\psi}_2 + i\varepsilon_1\dot{\psi}_3 + i\varepsilon_5\dot{\psi}_6 - i\varepsilon_2\dot{\psi}_7$$
$$\delta g_2 = -i\varepsilon_3\dot{\psi}_1 + i\varepsilon_4\dot{\psi}_2 + i\varepsilon_1\dot{\psi}_4 - i\varepsilon_5\dot{\psi}_5 - i\varepsilon_2\dot{\psi}_8$$
$$\delta g_3 = -i\varepsilon_1\dot{\psi}_1 - i\varepsilon_4\dot{\psi}_3 - i\varepsilon_3\dot{\psi}_4 + i\varepsilon_2\dot{\psi}_5 - i\varepsilon_5\dot{\psi}_8$$
$$\delta g_4 = -i\varepsilon_1\dot{\psi}_2 - i\varepsilon_3\dot{\psi}_3 + i\varepsilon_4\dot{\psi}_4 + i\varepsilon_2\dot{\psi}_6 + i\varepsilon_5\dot{\psi}_7$$

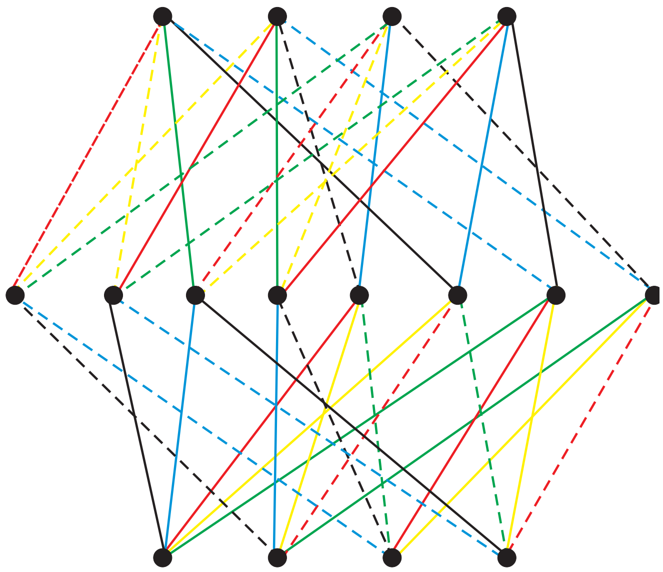

Fig. 2: Graph of the $N = 5\,(4, 8, 4)$ multiplet of $4_3 + 4_2$ connectivity (type$B$)

10 Tensoring irreducible representations: their fusion algebras and the associated graphs

The tensor product of linear irreducible representations can be decomposed into their irreducible constituents. This decomposition contains useful information in the construction of bilinear (in general, multilinear) terms entering a supersymmetric invariant action. We recall that the auxiliary fields in a given representation transform as a total derivative (a time derivative in one dimension). Useful information concerning the decomposition of the tensor products of the irreducible representations can be encoded in the so-called fusion algebra of the irreps and their supersymmetric vacua. The notion of a fusion algebra of the supersymmetric vacua of the N-extended one dimensional supersymmetry, introduced in [14], is constructed by analogy with the fusion algebra for rational conformal field theories. Fusion algebras can also be nicely presented in terms of their associated graphs. We explicitly present here the $N=1$ and $N=2$ fusion graphs (with two subcases for each N, according to whether or not the irreps are distinguished w.r.t. their bosonic/fermionic statistics). Let us discuss here how to present the [14] results in graphical form. The irreps correspond to points. N_{ij}^k oriented lines (with arrows) connect the $[j]$ and the $[k]$ irrep if the decomposition $[i]\times[j] = N_{ij}^k[k]$ holds. The arrows are dropped from the lines if the $[j]$ and $[k]$ irreps can be interchanged. The $[i]$ irrep should correspond to a generator of the fusion algebra. This means that the whole set of $N_l = N_{lj}^k$ fusion matrices is produced as the sum of powers of the $N_i = N_{ij}^k$ fusion matrix.

Let us discuss explicitly the $N=2$ case. We obtain the following list of four irreps (if we discriminate their statistics):

$$\begin{aligned}
[1] &\equiv (2,2)_{Bos};\\
[2] &\equiv (1,2,1)_{Bos};\\
[3] &\equiv (2,2)_{Fer};\\
[4] &\equiv (1,2,1)_{Fer}
\end{aligned}\qquad (10.71)$$

The corresponding $N=2$ fusion algebra is realized in terms of four 4×4, mutually commuting, matrices given by

$$N_1 = \begin{pmatrix} 1 & 2 & 1 & 0\\ 0 & 2 & 0 & 2\\ 1 & 0 & 1 & 2\\ 0 & 2 & 0 & 2\end{pmatrix} \equiv X;$$
$$N_2 = N_4 = \begin{pmatrix} 0 & 2 & 0 & 2\\ 0 & 2 & 0 & 2\\ 0 & 2 & 0 & 2\\ 0 & 2 & 0 & 2\end{pmatrix} \equiv Y; \qquad (10.72)$$
$$N_3 = \begin{pmatrix} 1 & 0 & 1 & 2\\ 0 & 2 & 0 & 2\\ 1 & 2 & 1 & 0\\ 0 & 2 & 0 & 2\end{pmatrix} \equiv Z.$$

The fusion algebra admits three distinct elements, X, Y, Z and one generator (we can choose either X or Z), due to the relations

$$Y = \frac{1}{8}(X^3 - 2X),\quad Z = -\frac{1}{4}(X^3 - 6X^2 + 4X). \quad (10.73)$$

The vector space spanned by X, Y, Z is closed under multiplication

$$\begin{aligned}
X^2 = Z^2 = ZX &= X + 2Y + Z,\\
XY = Y^2 = YZ &= 4Y
\end{aligned}\qquad (10.74)$$

This fusion algebra corresponds to the “smiling face” graph below. We obtain the following four tables for the fusion graphs of the $N=1$ and $N=2$ supersymmetric quantum mechanics irreps. The “A” cases below correspond to ignore the statistics (bosonic/fermionic) of the given irreps. In the “B” cases, the number of fundamental irreps is doubled w.r.t. the previous ones, in order to take the statistics of the irreps into account. We have

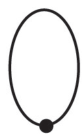

Fig. 3: Fusion graph of the $N = 1$ superalgebra (A case, 1 irrep, no bosons/fermions distinction)

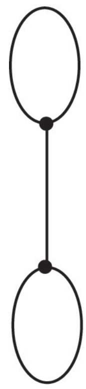

Fig. 4: Fusion graph of the $N = 1$ superalgebra (B case, 2 irreps, bosons/fermions distinction)

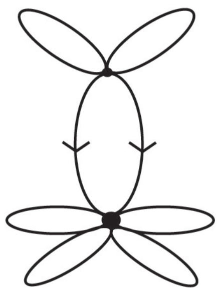

Fig. 5: Fusion graph of the $N = 2$ superalgebra (A case, 2 irreps, no bosons/fermions distinction)

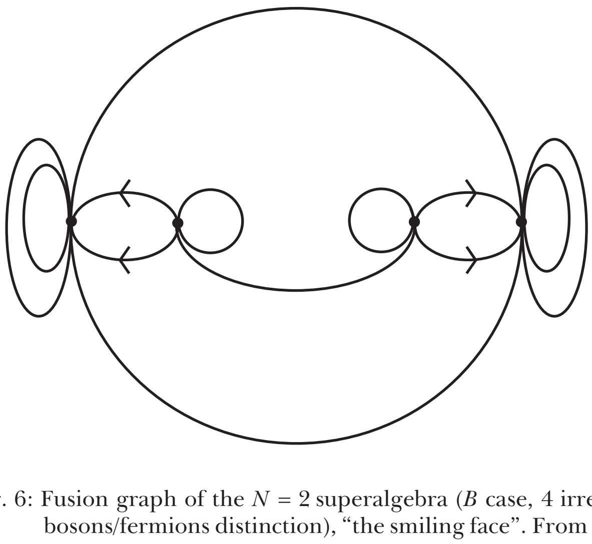

Fig. 6: Fusion graph of the $N = 2$ superalgebra (B case, 4 irreps, bosons/fermions distinction), "the smiling face". From left to right the four points correspond to the [2] – [1] – [3] – [4] irreps, respectively. The lines are generated by the $N_1 \equiv X$ fusion matrix, see (10.72)

11 Conclusions

Supersymmetric quantum mechanics is a fascinating subject with several open problems. The potentially most interesting one concern the construction of off-shell invariant actions with the dimension of a kinetic term for large values of N (let us say $N > 8$). They could provide a hint towards an off-shell formulation of higher dimensional supergravity and M-theory. Other important topics concern the nature of the non-linear realizations of the supersymmetry and their connection with linear representations. We have here presented the rich mathematics underlying the linear irreducible representations realized on a finite number of time-dependent fields. We have shown how to use this information to construct supersymmetric invariant one-dimensional sigma models. We have seen that behind supersymmetric quantum mechanics there exists an interlacing of several mathematical structures, Clifford algebras, division algebras, graph theory. Further mathematical structures seem to enter the picture (Cayley-Dickson algebras, exceptional Lie algebras, etc.). The theory of supersymmetric quantum mechanics is rich in surprises and seems to lie at the crossroads of various mathematical disciplines. We have just given a taste of it here.

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Appendix

We present here for completeness the set (unique up to similarity transformations and an overall sign flipping) of the seven 8×8 gamma matrices γ_i which generate the $Cl(0,7)$ Clifford algebra. The seven gamma matrices, together with the 8-dimensional identity $\mathbf{1}_8$, are used in constructing the $N = 5, 6, 7, 8$ supersymmetry irreps, as explained in the main text.

$$
\begin{aligned}
\gamma_1 &= \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0 \end{pmatrix}
&\quad
\gamma_2 &= \begin{pmatrix} 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0 \end{pmatrix}
\\
\gamma_3 &= \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0 \end{pmatrix}
&\quad
\gamma_4 &= \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}
\\
\gamma_5 &= \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&-1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix}
&\quad
\gamma_6 &= \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1\\ -1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix}
\\
\gamma_7 &= \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0 \end{pmatrix}
&\quad
\mathbf{1}_8 &= \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}
\end{aligned}
\tag{A.1}
$$

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Francesco Toppan
e-mail: toppan@cbpf.br

CBPF
Rua Dr. Xavier Sigaud 150
cep 22290-180, Rio de Janeiro (RJ), Brazil

