3½Lectures on Noncommutative Geometry

A. Sitarz

We present a short overview of noncommutative geometry. Starting with C^* algebras and noncommutative differential forms we pass to K-theory, K-homology and cyclic (co)homology, and we finish with the notion of spectral triples and the spectral action.

Keywords: Connes' noncommutative geometry, spectral triples (MCS 2000: 58B34, 46L85, 46L87, 81T75).

1 What is noncommutative geometry?

"... to savour the strange warm glow of being much more ignorant than ordinary people, who are only ignorant of ordinary things."

(Terry Pratchett, "Equal Rites")

Noncommutative geometry begins with classical geometry and extends into the realm of abstract algebras and operators. One may, of course, say that *noncommutative geometry* studies the geometry of *quantum spaces* – or, to be more explicit – the geometry of noncommutative algebras. Clearly, the word *quantum*, although at first only superficially related to *quantum mechanics* or *quantum field theory* might be the right one - both physics and mathematics are involved in many examples and there is a huge interplay between them. However, the notion of *quantum spaces* is a delicate one since the objects that noncommutative geometry attempts to study are (usually) not spaces – they cannot be visualized.

Then why study noncommutative geometry? First of all, it seems to be a natural and rich extension of the concept of spaces, one that can admit the notion of *geometry* in its various aspects. Moreover, within noncommutative geometry one has various objects on the same footing and one can do more than within classical geometry.

Last not least one should mention that many basic examples do arise from physics: the phase space in quantum mechanics, the Brillouin zone in the Quantum Hall Effect, the geometry of finite spaces in the noncommutative description of the Standard Model, quantum groups in integrable models or the quantized target space of string theory.

Before we begin this easy walk through the noncommutative reality, we need to mention that – like any subject, that is in the early stage of development, it has many branches. The approach presented here falls close to *Connes' noncommutative geometry* (It appears that the wording Connesian variety has also been used to describe this approach...) – (58B34, to give Mathematical Subject Classification number) and *noncommutative differential geometry* – (46L87). However, the topics that we shall mention range from C^*-algebras, differential algebras to ideals and exotic traces.

Let us also attempt to place the subject matter of noncommutative geometry in relation to other subjects in physics and mathematics. Clearly, in mathematics noncommutative geometry lies between algebra and geometry (meaning rather differential geometry), based on fundamental results of operator algebras. There are, however, many other, sometimes not evident connections – with topology, probability, measure theory, algebraic geometry, ring theory and also with number theory. In physics, it includes both classical field theory, quantum field theory, renormalization, quantum mechanics as well as gravity and string theory. Of course, one should be aware that we are far from certain whether the notion of space-time is indeed best described by noncommutative geometry and we still need some crucial theoretical steps to pursue this goal. Nevertheless some qualitative considerations and also the evidence that we already have from high energy physics make this line of research quite promising.

In this set of lectures we shall present an overview of mathematical objects and tools leading us towards the Noncommutative World. This set of lectures is by no means self-consistent and many statements are quoted without proofs. Some statements are also presented not in their most general form – we do this on purpose, having as a basic guide the need to explain the ideas rather than technical details. We assume some basic knowledge of algebra, differential geometry, operator algebras, Hilbert spaces as well as some knowledge of gauge theory and characteristic classes. Although not essential some knowledge in these topics is a big help when learning noncommutative geometry – a good example of a textbook that can be used as a starting point is the classical text [4].

Let us finish this introduction with the quotation from Alain Connes' interview with George Skandalis (Newsletter of the European Mathematical Society, No 3. (2007)).

> – What is noncommutative geometry? In your opinion, is "noncommutative geometry" simply a better name for operator algebras or is it a close but distinct field ?
>
> – Yes, it's important to be more precise. First, noncommutative geometry for me is this duality between geometry and algebra, with a striking coincidence between the algebraic rules and the linguistic ones. Ordinary language never uses parentheses inside the words. This means that associativity is taken into account, but not commutativity, which would permit permuting the letters freely.

2 Towards noncommutative topology

Mathemata mathematicis scribuntur.
(Mathematics is written for mathematicians.)
Nicolaus Copernicus

2.1 Where it all begins: Gelfand-Naimark

When we say *space*, we usually mean a *topological space*. It may, of course, have some additional properties and features but the basic ingredient, topology, is there. We usually assume that the topology is Hausdorff, which means that every two points can be separated by disjoint open sets. We know many examples of topological spaces and we have a good notion of continuous (complex valued) functions. One of the very basic observations (that we know almost intuitively) is that all continuous functions over a topological space form a complex vector space and that the product of two continuous functions is still a continuous function. This says that we have an *algebra* of continuous functions and, since we can take a complex conjugate of a function, it is a $*$-algebra. Let us assume for a while that our space is compact, then clearly to each function we can associate the supremum of its absolute value. Thus we arrive at the *norm* of a function. Going a step further we come to the notion of a C^*-algebra with the following formal definition:

Definition 2.1: An involutive Banach algebra $\mathcal{A}$ (that is a complex normed algebra, which is complete as a topological space in the norm) such that

$$\left\|aa^*\right\| = \|a\|^2,\ \forall a \in \mathcal{A}$$

is a C^*-algebra.

So, to cut the story short: the algebra of continuous functions on a compact Hausdorff space is a C^*-algebra with a unit! But does it work the other way round? Surprisingly (or not surprisingly) yes, as we can state in the Gelfand-Naimark theorem:

Theorem 2.2: Gelfand-Naimark A commutative unital C^*-algebra is an algebra of continuous functions on a compact Hausdorff space.

We shall not discuss the details of the proof as this can be found in numerous textbooks – and is (in principle) quite easy. The points of the space are provided by the characters of the algebra, which are continuous algebra morphisms from the algebra to the complex numbers.

This is the dawn of noncommutative geometry. Why? Note that C^* algebras might not be commutative. Indeed, take an example: the algebra $M_n(\mathbb{C})$ of matrices with complex entries, with hermitian conjugation and matrix multiplication is a very good and simple example of a noncommutative C^* algebra.

Then in the view of Gelfand-Naimark's theorem we can use noncommutative C^* algebras as the definition of *noncommutative Hausdorff compact spaces*. But these space have no *points* or have just a couple of *points*! Coming back to the matrix algebra $M_n(\mathbb{C})$ we see that for $n > 1$ there are no characters at all.

Another very good (and generic, as we shall see) example of a C^*-algebra comes from the theory of Hilbert spaces. The recipe is very easy: take a (separable) Hilbert space $\mathcal{H}$ and take an algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ with an operator norm. Then any norm closed subalgebra of $\mathcal{B}(\mathcal{H})$ is a (separable) C^* algebra. Of course, our toy-model algebra $M_n(\mathbb{C})$ is in fact of this type: just take $\mathcal{H} = \mathbb{C}^n$. The algebra of all bounded operators on it is nothing else but $M_n(\mathbb{C})$.

Of course, we have shown one of the most restrictive versions of the theorem. If we forget about *unital*, we still get Hausdorff spaces but the are only locally compact.

2.2. Into the C^*-world.

What are C^* algebras ? There are many (isomorphic) definitions out of which we have already presented one. We know that there are many C^* algebras, and what we would like is to have a description of them that would be on the same footing – independently, whether there are commutative or noncommutative. And we want not an abstract description but a concrete one. Again, Gelfand, Naimark and Segal come to our aid:

Theorem 2.3 (GNS): Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^* algebra of operators on a Hilbert space $\mathcal{H}$. If the algebra $\mathcal{A}$ is separable then we can take $\mathcal{H}$ to be separable.

Now we have a powerful tool: a description of all C^*-algebras as operators on the Hilbert space. This is a good starting point for noncommutative topology and towards many other notions like *measurable functions*, for instance. To summarize this section let us quote the dictionary, which establishes parallel notions between *standard* and *noncommutative* topology:

TOPOLOGY	ALGEBRA
(locally compact) topological space	commutative C^*-algebra
homeomorphism	automorphism
continuous proper map	morphism
compact space	unital -algebra
open (dense) subset	(essential) ideal
compactification	unitization
Stone-Čech compactification	multiplier algebra
Cartesian product	tensor product

2.3 The tricky bits and examples

Among the first problems that arise in the noncommutative world and have no classical correspondence, are some ambiguities in construction. For this reason, one should treat some "equivalences" from the above dictionary with due respect. A good example is the case of the tensor product of noncommutative C^* algebras, which might depend on the completion of the algebraic tensor product of two algebras. Let us first have a look at an example:

Example 2.4: Take an interval $I = (0,1)$ and the algebra of continuous functions on it, $C(I)$. Of course, we can interpret each element of the tensor product $C(I) \otimes C(I)$ as a continuous function on the Cartesian product $I^2 = I \times I$, but it clear that not all continuous functions on I^2 are of this form.

Of course, it is true that $C(I) \otimes C(I)$ is dense in $C(I^2)$, so, in order to work with tensor products of C^* we need to work out a way to complete the algebraic tensor product. This leads to a rather extended and not unique construction unlike the topological Cartesian product of spaces. This is rather bad news but we might find some comfort in the fact that many algebras (called *nuclear*) do have unique completions of tensor products with them. The list includes all commutative algebras, matrix algebras and – last but not least – the algebra of compact operators (which we shall define later). So in the end, there is no ambiguity in defining continuous functions on the square, but it might be a different story for a noncommutative square! Nevertheless, we shall always work with the algebraic tensor product, keeping in mind that some further details and more knowledge are needed when we try to think of C^* algebras.

We shall very often restrict ourselves to the C^* algebras generated by some concrete operators, which are defined by their actions on the orthonormal basis. So, if T is a bounded operator on a Hilbert space $\mathcal{H}$, then we shall denote by $C^*(T)$ the C^* algebra, which is the norm closure of the algebra of all polynomials in T, T^*. Let us consider a couple of examples, the first of which defines compact operators:

Example 2.5: Take P_n to be a one-dimensional projection on a Hilbert space with basis $\{e_k\}_{k\geq 0}$:

$$P_n e_k = \delta_{nk} e_k,\ n, k \geq 0.$$

Then the smallest C^* algebra that contains all projections P_n, $C^*(\{P_n\})$ is the algebra of compact operators $\mathcal{K}$.

Example 2.6: Let U be a unilateral shift on a Hilbert space with basis $\{e_k\}_{k\in\mathbb{Z}}$:

$$U\, e_n = e_{n+1},\ \ n \in \mathbb{Z}.$$

Then the algebra $C^*(U)$ is isomorphic to the algebra of continuous functions on the circle, $C(S^1)$.

Remark 2.7: Note that our intuition is that the operator U corresponds to the function $\phi \to e^{2\pi i\phi}$ on the circle, where ϕ is the angle. However, this may not be the case. Take, for instance, any monotonic, continuous function $\kappa : [0,1] \to [0,1]$. We can now replace one of the elements of the basis of the Hilbert space by $e^{2\pi i\kappa(\phi)}$ and using the standard procedure we can introduce a new orthonormal basis of the Hilbert space, in which one of the basis vectors is proportional to the chosen one. Certainly it will not be the basis we have in mind and the unilateral shift operator U cannot then be identified with $e^{2\pi i\phi}$.

Example 2.8: Take T to be a unilateral shift on a Hilbert space with basis $\{e_k\}_{k\geq 0}$:

$$T\, e_n = e_{n+1},\ \ n \geq 0.$$

Then the algebra $C^*(T)$ is isomorphic to the algebra $C(S^1) + \mathcal{K}$, in the following sense: there exist C^*-algebra morphisms i, π such that the following sequence is exact:

$$0 \to K \xrightarrow{\ i\ } C^*(T) \xrightarrow{\ \pi\ } C(S^1) \to 0.$$

Example 2.9: Take U, V to be the following unitary operators on the Hilbert space with basis $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$:

$$U\, e_{m,n} = e_{m+1,n}\,,\ V\, e_{m,n} = e^{-2\pi i\theta m} e_{m,n+1}\,,\ \ m, n \in \mathbb{Z},$$

where $\theta \in \mathbb{R}$.

If $\theta = 0$ we can identify the algebra with the continuous functions on the torus. If θ is irrational then the algebra $C^*(U,V)$ is the so-called irrational rotation algebra, aka functions on the Noncommutative Torus.

It is easy to see that U and V satisfy:

$$U\,V = e^{2\pi i\phi}\, VU$$

Clearly, the above presentation of the Noncommutative Torus is not unique. We can take as the Hilbert space $L^2(S^1)$ and as operators U and V:

$$U\, f(z) = z\, f(z),\, V\, f(z) = f(e^{2\pi i\theta} z).$$

Both operators are unitary and both satisfy the same commutation relation – is the C^*-algebra then the same? It is reassuring that the answer is positive (though it takes some time to prove it).

3 Differential geometry (noncommutative way)

Alice laughed: "There's no use trying," she said; "one can't believe impossible things."
– "I daresay you haven't had much practice," said the Queen. "When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

(Lewis Carroll, "Alice in Wonderland".)

Having started with topology we have established a nice setup for the discussion of noncommutative spaces. However, we are still very far from geometry as topology does not distinguish between a ball and a cube! Our task in this section is to carry out the parallels built up for C^*-algebras as noncommutative spaces for some more geometric notions. We shall begin with a pure algebraic setup of differential calculi.

3.1 Differential calculi

In the course of differential geometry one begins with the notion of a smooth manifold, C^∞ functions and vector fields. This is, however, reserved for a purely commutative world as can be immediately noticed when on takes the simplest example of a noncommutative space, described by the algebra of matrices $M_n(\mathbb{C})$, $n>0$. Suppose we want to have a vector field – what is a vector field then? A good answer is that a vector field is a *derivation* on the algebra of smooth functions:

Definition 3.1: A derivation δ on an algebra $\mathcal{A}$ is a map $\delta:\mathcal{A}\to\mathcal{A}$ satisfying the Leibniz rule:

$$\delta(ab)=\delta(a)b+a\delta(b),\ \ \forall a,b\in\mathcal{A}.$$

A derivation is inner if there exist an element$x\in\mathcal{A}$ such that for every $a\in\mathcal{A}$, $\delta(a)=[x,a]$. A derivation which is not inner is called an outer derivation.

Since for commutative algebras there are no inner derivations, vector fields are (in classical differential geometry) outer derivations. But let us look at $M_n(\mathbb{C})$ – where every derivation is in fact inner (which means that there are no outer derivations). So, can we call them vector fields?

We can even take a much simpler example: a commutative algebra of complex functions on two points. As a vector space it has two basis vectors: a unit (1) and the function which takes value 1 on the first point and −1 on the other, which we shall call e. Each function is a linear combination of these two, and the algebra structure is encoded in one simple identity $e^2=1$. Now what is the space of the derivations? Clearly, there are no inner ones, as the algebra is commutative. Assume that δ is a derivation. Then, using the Leibniz rule we have:

$$0=\delta(1)=2\delta(e)\,e,$$

which simply tells us that apart from the trivial derivation, $\delta\equiv 0$, there are no derivations at all.

Of course, a good lesson to learn is that we have chosen a bad object to start with. Instead of looking at the vector fields we need to look at differential algebras, which generalize nicely to the noncommutative world.

Definition 3.2: A differential graded algebra (DGA) over an algebra $\mathcal{A}$ is an $\mathbb{N}$-graded algebra, not necessarily finite, such that the 0-th grade is isomorphic with $\mathcal{A}$ and that is equipped with a degree linear map (grade increasing), which obeys the graded Leibniz rule:

$$d(\rho\,\omega)=d\rho\,\omega+(-1)^{|\rho|}\rho\,d\omega,$$

for any elements ω, ρ, where $|\rho|$ denotes the degree of the form ρ.

There is, however, no unique way to construct such an object in the noncommutative situation and one might have many different DGAs over a single algebra – even in the commutative case. Before we look at some interesting examples, let us define an important DGA, which can be canonically constructed for every algebra.

Proposition 3.3: We assume that the algebra $\mathcal{A}$ is unital. Let $\Omega^1_u(\mathcal{A})$ be the kernel of the multiplication map in $\mathcal{A}\otimes\mathcal{A}$:

$$\Omega^1_u(\mathcal{A})=\left\{\sum_i a_i\otimes b_i\right\}_{a_i,b_i\in\mathcal{A};\sum_i a_ib_i=0}$$

Let us take as $\Omega_u(\mathcal{A})$ the tensor algebra over $\mathcal{A}$ of $\Omega^1_u(\mathcal{A})$. Then the linear map defined on $\Omega^0_u(\mathcal{A})=\mathcal{A}$ as:

$$d_u(a)=a\otimes 1-1\otimes a,$$

extends in a unique way to a degree 1 linear operator on $\Omega_u(\mathcal{A})$, which satisfies the graded Leibniz rule and is nilpotent, $d_u^2=0$. This DGA is called a universal DGA and d_u is the universal external derivative.

Proof: It is a good exercise to have a look at the proof. Actually, most of the properties of d_u are used in the construction. For instance, we extend the definition of d_u in such a way that the Leibniz rule is assured. The universal one-forms could be always uniquely expressed as $\sum_i a_i\,d_ub_i$, for $a_i,b_i\in\mathcal{A}$, $\sum_i a_ib_i=0$. Indeed, an arbitrary universal one-form is of the type $\sum_i a_i\otimes b_i$ for some $a_i,b_i\in\mathcal{A}$ such that $\sum_i a_ib_i=0$. But:

$$\sum_i(-a_i)\,d_ub_i=\sum_i(-a_ib_i)\otimes 1+\sum_i a_i\otimes b_i=\sum_i a_i\otimes b_i.$$

We set:

$$d_u\left(\sum_i a_i\,d_ub_i\right)=\sum_i d_ua_i\otimes_{\mathcal{A}}d_ub_i.$$

This assures the graded Leibniz rule between elements of the algebra and one-forms, and makes sure that $d_u^2=0$ on $\mathcal{A}$. The rest is just the application of the Leibniz rule. We extend d_u to products of universal one-forms through:

$$d_u(\rho_1\otimes_{\mathcal{A}}\rho_2\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\rho_k)$$
$$=\sum_{j=1}^{k}(-1)^{j+1}\rho_1\otimes_{\mathcal{A}}\rho_2\otimes_{\mathcal{A}}\cdots d\rho_j\cdots\otimes_{\mathcal{A}}\rho_k$$

for each $\rho_i\in\Omega^1_u(\mathcal{A})$, $i=1,\ldots,k$. Of course, we need to check that the definition is compatible with the tensor product over $\mathcal{A}$, which mean that it gives the same result on $\omega\otimes_{\mathcal{A}}a\,\rho$ and $\omega\,a\otimes_{\mathcal{A}}\rho$:

$$\begin{aligned}
&d_u(\omega\otimes_{\mathcal{A}}a\,\rho)\\
&=d_u(\omega)\otimes_{\mathcal{A}}a\,\rho+(-1)^{|\omega|}\omega\otimes_{\mathcal{A}}d_u(a\,\rho)\\
&=d_u(\omega)\otimes_{\mathcal{A}}(a\,\rho)+(-1)^{|\omega|}\omega\otimes_{\mathcal{A}}(d_ua)\otimes_{\mathcal{A}}\rho+(-1)^{|\omega|}\omega\otimes_{\mathcal{A}}a(d_u\rho)\\
&=\left(d_u(\omega)a+(-1)^{|\omega|}\omega\otimes_{\mathcal{A}}d_ua\right)\otimes_{\mathcal{A}}\rho+(-1)^{|\omega|}\omega\,a\otimes_{\mathcal{A}}(d_u\rho)\\
&=d_u(\omega a)\otimes_{\mathcal{A}}\rho+(-1)^{|\omega|}\omega\,a\otimes_{\mathcal{A}}(d_u\rho)\\
&=d_u(\omega a\otimes_{\mathcal{A}}\rho)
\end{aligned}$$

It is now a matter of easy verification that $d_u^2=0$. □

From the above construction we obtain a very convenient presentation of universal differential forms: each form of degree k can be presented as a finite sum of elements of the type $a_0\,da_1\cdots da_k$. Moreover, having two forms $\omega_a=a_0\,da_1\cdots da_k$ and $\omega_b=a_0\,db_1\cdots db_k$ they are different unless for each $i = 0, 1, \ldots, k$, a_i and b_i are linearly dependent. This follows directly from the definition of $\Omega_u^k(\mathcal{A})$ as the products of k one-forms and the fact that each one-form is already in that form. The rest is the iterative application of the Leibniz rule.

Note that this hints at a very nice feature of the universal differential algebra: each space of forms of fixed grade is generated as a left-module over $\mathcal{A}$ by the image of d. We did not assume this in the definition of the DGA but we can use the (nonstandard) name of a *proper DGA* for those that have this property.

Finally we come to the question of the name. Universal differential algebra owes its name its nice property: it is indeed universal! What does this mean? The following is due to Karoubi:

Theorem 3.4: If $\Omega^*(\mathcal{B})$ is a graded differential algebra over $\mathcal{B}$ and $\phi : \mathcal{A} \to \mathcal{B}$ an algebra homomorphism, then there exists a unique extension of ϕ:

$$\widetilde{\phi} : \Omega_u^*(\mathcal{A}) \to \Omega^*(\mathcal{B}),$$

which is a morphism of graded differential algebras:

$$d \circ \phi = \phi \circ d.$$

For our purposes, we shall actually need a consequence of this statement,

Corollary 3.5: For any *proper DGA*, $\Omega(\mathcal{A})$, there exists a surjective morphism of differential graded algebras:

$$\pi : \Omega_u(\mathcal{A}) \to \Omega(\mathcal{A}).$$

Proof: We set:

$$\pi(a_0 d_u a_1 \cdots d_u a_k) = (a_0 da_1 \cdots da_k).$$

This is a well-defined morphism of differential algebras. But since the differential graded algebra $\Omega(\mathcal{A})$ is proper all its elements are of the form $a_0 da_1 \cdots da_k$ and hence in the image of π. □

Briefly, corollary 3.5 means that every proper differential graded algebra over the algebra $\mathcal{A}$ is isomorphic to a quotient of $\Omega_u(\mathcal{A})$ by a differential ideal, which is an ideal of the universal graded algebra, $\mathcal{I} \subset \Omega_u(\mathcal{A})$, such that $d_u(\mathcal{I}) \subset \mathcal{I}$.

To see how this works we need to look at a couple of examples.

Example 3.6: Let $\mathcal{A}$ be the algebra of functions on two points (described earlier). The bimodule of universal one-forms is generated by $d_u e$. Applying the Leibniz rule we see that:

$$e(d_u e) + (d_u e)e = 0,$$

so there are only two linearly independent one forms: de and ede. Similarly one constructs all higher-order forms, which (as in every universal calculus) can be of arbitrary order.

In fact, d_u is a derivation on the algebra $\mathcal{A}$, but a special one. It takes values not in $\mathcal{A}$ itself (it cannot, as we have shown before) but in a bimodule over $\mathcal{A}$.

Example 3.7: Let X be a space and $\mathcal{A}$ an algebra of functions on X (we shall not need anything more about this algebra, so we do not say whether the functions are measurable or smooth, or just arbitrary). Now, consider the following differential graded algebra:

$$\Omega^k(X) = \left\{ f : X^k \to \mathbb{C},\ f(x_1, \ldots, x_k) \equiv 0, \text{ if } \exists i \neq j : x_i = x_j \right\}$$

with the product:

$$(f \cdot g)(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+p}) = f(x_1, \ldots, x_k) g(x_{k+1}, \ldots, x_{k+p})$$
$$f \in \Omega^k(X),\ g \in \Omega^p(X).$$

The external derivative d is defined on the functions

$$f_0 \in \Omega^0(X)$$

as:

$$d(f_0)(x_1, x_2) = f0(x_1) - f0(x_2),$$

and then extended to forms of arbitrary order through:

$$(d f_n)(x_1, \ldots, x_{n+1}, x_{n+2}) = \sum_i (-1)^i f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+2}),$$

where $\hat{x}_i$ denotes that we just omit the i-th variable.

Exercise 3.8: Prove that d satisfies the graded Leibniz rule and is nilpotent, $d_u^2 = 0$. If X is a space consisting of a finite number of points, can we identify the differential algebra $\Omega(X)$?

Note that the differential graded algebra $\Omega(X)$ is in general not a proper one. This depends strongly on the class of functions that we consider, as we can easily see:

Exercise 3.9: Assume that X is the interval I and $\Omega^k(X)$ are polynomials (of variables) understood as functions on I^k. Show that the differential graded algebra is a proper one in this case (that is, for each k $\Omega^k(X)$ is generated by the image of d).

What happens if we take $\Omega^k(X)$ to be all continuous functions on I^k?

Although $\Omega(X)$ might not be proper, it appears to be a suitable generalization of the universal differential graded algebra, especially in the case, when the algebraic tensor products might not give us all elements to play with (as is the case with smooth or continuous functions).

Example 3.10: An interesting question is whether we can identify the standard de Rham differential graded algebra using the universal calculus and choosing a suitable quotient. The answer is positive but again we need to be careful about the tensor products. The construction should be similar as in example 3.7 with the algebra of smooth functions over X.

The de Rham differential calculus is obtained if we take the quotient of this differential graded algebra $\Omega_s(X)$ by the ideal described as follows. Let $\mathcal{I}$ be the space of all functions on X^2 such that:

$$\lim_{x \to y} \frac{f(x,y)}{x-y} = 0, \; x, y \in X.$$

It is not difficult to observe that multiplication by smooth functions from both sides does not lead outside $\mathcal{I}$. Taking the quotient $\Omega_s^1(I)/\mathcal{I}$ we first obtain all de Rham one-forms on X. The construction of higher order forms is then a formality.

We already know that there are many noncommutative differential calculi over one algebra. The universal one, apart from its universality property, is not actually very interesting. It carries very little information about the "space" underneath. The interesting examples are "smaller" calculi. Actually, even in the commutative case we might have many differential graded algebras, which are neither de Rham, nor universal. See the following example:

Example 3.11: Consider an algebra of smooth functions on the circle and the following differential graded algebra. The zero-forms are the smooth functions, $\Omega^k(S^1) = C^\infty(S^1)$. The bimodule of one-forms and the entire differential algebra is generated by two one-forms, ϑ and ω. The following gives the algebra rules and the action of the external derivative:

$$\begin{aligned} df &= (\partial f)\vartheta + (\partial 2 f)\omega, \\ d\vartheta &= 0, \\ \vartheta f &= f\vartheta + 2(\partial f)\omega, \\ d\omega &= \vartheta \wedge \vartheta, \\ \omega f &= f\omega, \\ \omega \wedge \vartheta &= -\vartheta \wedge \omega, \\ \omega \wedge \omega &= 0, \end{aligned}$$

where ∂ denotes the standard derivative of the function on the circle.

Note that a priori the differential algebra is infinite-dimensional, as we can construct product of arbitrary numbers of ϑ. However, since $\vartheta \wedge \vartheta$ generates a differential ideal, we might take a quotient and therefore set $\vartheta \wedge \vartheta = 0$. A point worth mentioning is that although the algebra is commutative the differential forms (which are not universal) do not commute with functions!

3.2 Involution, tensor products and representations

3.2.1 Involution

If the algebra $\mathcal{A}$ is equipped with an involution, we would like to have this operation extended onto the differential algebra. This is by no means a problem for the universal calculus, as on the zero-forms we take the assumed involution on the algebra and for higher-order universal forms we set:

$$(da)^* = -d(a^*).$$

So the problem, when reduced to the universal case, is trivial and the only thing we need to take care of is that the differential ideal $\mathcal{J} \subset \Omega_u(\mathcal{A})$ is involution invariant $\mathcal{J}^* = \mathcal{J}$.

Example 3.12: On the algebras of complex functions we have a natural involution – complex conjugation. Taking the trivial (but still interesting) example of two-point geometry, $e^* = e$, we have $de = -(de)^*$.

Exercise 3.13: Verify that in example 3.11 of the nonstandard differential algebra over the circle we have: $\vartheta^* = -\vartheta$ and $\omega^* = \omega$.

3.2.2 Tensor products

Next, let us discuss the procedure for constructing the differential graded algebra for the tensor product of two algebras $\mathcal{A}$ and $\mathcal{B}$. Assuming that the respective differential algebras $\Omega^*(\mathcal{A}), \Omega^*(\mathcal{B})$ are given, there exists a canonical procedure for creating a differential algebra over the tensor product, which uses the $\mathbb{Z}_2$-graded tensor product:

$$\Omega^*(\mathcal{A} \otimes \mathcal{B}) = \Omega^*(\mathcal{A}) \hat{\otimes} \Omega^*(\mathcal{B})$$

where $\hat{\otimes}$ means that we take the symmetric or antisymmetric part of the tensor product, with respect to the degree of forms. In other words the product of two forms of fixed degrees $\omega_{\mathcal{A}} \in \Omega^*(\mathcal{A})$ and $\omega_{\mathcal{B}} \in \Omega^*(\mathcal{B})$ is always commutative or anticommutative depending on their degrees:

$$\omega_{\mathcal{A}} \wedge \omega_{\mathcal{B}} = (-1)^{|\omega_{\mathcal{A}}||\omega_{\mathcal{B}}|} \omega_{\mathcal{B}} \wedge \omega_{\mathcal{A}}.$$

3.3 Representations of differential algebras

Finally, let us consider a specific way of obtaining differential graded algebras – connected with representations and commutators. Let $\mathcal{A}$ be an algebra and let π be its representation on a vector space (not necessarily finite dimensional). Let F be an endomorphism (a linear operator, in other words) of this vector space.

Lemma 3.14: If π is a representation of the algebra $\mathcal{A}$, then for each linear operator F the following gives a representation of the universal differential algebra $\Omega_u(\mathcal{A})$:

$$\begin{aligned} &\pi_F(a_0 da_1 da_2 \ldots da_n) \\ &\quad = \pi(a_0)[F, \pi(a_1)][F, \pi(a_2)] \ldots [F, \pi(a_n)] \end{aligned} \tag{1}$$

Note that π_F is nothing else but a representation of the algebra, and neither the grading nor the external derivative are in any way preserved. This follows from the fact that the kernel of π_F might not be a differential ideal. We also need to be careful while dealing with infinite dimensional representations, such as representations on the Hilbert space. In such a case, it is natural to assume that all operators $\pi(a)$ and the commutators $[F,\pi(a)]$ are bounded for all $a \in \mathcal{A}$. This naturally implies that the image of an arbitrary form is a bounded operator. F itself might not be bounded and we shall see the most natural examples when it is not the case.

There exists a canonical way to obtain a differential graded algebra through π_F: we have to take

$$\mathcal{J} = \ker \pi_F + d(\ker \pi_F).$$

This is a differential ideal within $\Omega_u(\mathcal{A})$ and then $\Omega^u(\mathcal{A})/\mathcal{J}$ will be a differential algebra.

Note that unless $d(\ker \pi_F) \subset \ker \pi_F$ the obtained differential algebra will not have a representation on the Hilbert space. One might always choose for a higher-order form its representative in the image $\pi_F(\Omega^*_u(\mathcal{A}))$, however, this cannot be done in a unique way.

3.4 From representations to differential calculi

In view of the previous section we might consider just a different way of obtaining differential graded algebras. Just start with a representation π of the algebra, choose a suitable operator F, and consider all commutators $[F,\pi(a)]$ as one-forms. With a bit of luck (and some additional assumptions) we shall the obtain a good example of a differential calculus.

We shall consider two canonical cases. First, let F be a selfadjoint $F = F^\dagger$. This assures that for an involutive algebra $\mathcal{A}$ and a $*$-representation we have $d(a^*) = -(da)^*$:

$$d(a^*) = ([F,\pi(a)])^\dagger = [\pi(a)^\dagger, F^\dagger] = [\pi(a^*), F] = -d(a^*).$$

Moreover, let us take $F^2 = 1$, which means that (as seen on a Hilbert space) F is a sign operator with eigenvalues being +1 and −1.

We have:

Lemma 3.15. Let $F = F^\dagger$ and $F^2 = 1$ be an operator on the Hilbert space $\mathcal{H}$ and let π be the representation of $\mathcal{A}$ as bounded operators on $\mathcal{H}$. Then π_u defined in 1 is a representation of the differential algebra, with:

$$\pi_F(d\omega) = \begin{cases} [F,\pi_u(\omega)] & |\omega| \text{ even} \\ [F,\pi_u(\omega)]_+ & |\omega| \text{ odd} \end{cases}$$

for any universal form ω, $[\cdot,\cdot]_+$ denotes anticommutator.

Proof: First observe:

$$[F,x]F = -F[F,x], \ \forall x \in B(\mathcal{H}).$$

Then:

$$\begin{aligned}
\pi_F(da_0\, da_1\, da_2 \ldots da_n) &= [F,\pi(a_0)][F,\pi(a_1)]\ldots[F,\pi(a_n)] \\
&= F\pi(a_0)[F,\pi(a_1)]\ldots[F,\pi(a_n)] \\
&\quad - \pi(a_0)[F,\pi(a_1)]\ldots[F,\pi(a_n)] \\
&= F(\pi(a_0)[F,\pi(a_1)]\ldots[F,\pi(a_n)]) \\
&\quad - (-1)^n(\pi(a_0)[F,\pi(a_1)]\ldots[F,\pi(a_n)])F \\
&= [F,\pi(a_0)[F,\pi(a_1)]\ldots[F,\pi(a_n)]]_\pm,
\end{aligned}$$

where the $\pm$ sign at the last bracket means that we take the commutator (or anticommutator) depending on n.

Clearly $d^2 = 0$, as:

$$[F,[F,x]_+] = [F,[F,x]]_+ = 0.$$

□

We have a first crude method for obtaining differential algebra through the representation of $\mathcal{A}$ on a Hilbert space. Of course, the differential calculi obtained in that way are not very nice. Even for the commutative algebras they are rather awkward and – in particular – infinite dimensional. Consider the example of a circle:

Example 3.16: Let $\mathcal{A}$ be the algebra of functions on the circle (for our purposes we shall take the only polynomials in U, with the representation on the Hilbert space with basis $\{e_n\}$, $n \in \mathbb{Z}$ given as $U e_n = e_{n+1}$. We take the operator F to be: $F e_n = sign(n + \frac{1}{2})e_n$. What are then the differential forms? We calculate dU:

$$\begin{aligned}
dU\, e_n &= [F,U]e_n = \left(sign\left(n + \frac{3}{2}\right) - sign\left(n + \frac{1}{2}\right)\right)e_{n+1} \\
&= 2\delta_{n,-1}\, e_{n+1}
\end{aligned}$$

The differential form dU is an operator of finite rank and, as we can easily see, $\frac{1}{2}U^* dU$ is a projection on the subspace spanned by e_{-1}.

Exercise 3.17: Show that every differential form over the algebra of polynomials in U and U^* is in fact a finite rank operator. Can we generalize this result to the one-forms constructed with F over the algebra of continuous function over the circle?

The obtained calculus is somehow strange – but we shall see that it plays an important role. Surely enough it is genuinely noncommutative even for such a commutative space as a circle. We may, however, look for some examples of the differential graded algebras, which are more "reasonable".

Example 3.18: Let $\mathcal{A}$ be the algebra of polynomial functions on the circle. We take the same representation on the Hilbert space as in example 3.16. But instead of taking the operator F let us consider an unbounded operator D, $De_n = n e_n$. Of course, D, as an unbounded operator is only densely defined, so we need to work with a dense subspace of $\mathcal{H}$. The one-forms are all generated by the commutator $[D,U]$:

$$[D,U]e_n = e_{n+1}$$

and dU is a bounded operator (and hence well-defined on the entire Hilbert space). Since all one-forms and higher order forms arise from products of the elements of the algebra and dU, the representation π_D is into bounded operators.

Observe that:

$$U\,dU = dU\,U, \ U^*\,dU = dU\,U^*,$$

so, the one-forms really do commute with the elements of the algebra. Since $D^2 \neq 1$ the image of the universal DGA is not a DGA itself. For instance, in the differential algebra we would have:

$$0 = d(U\,dU - dU\,U) = 2dU\,dU,$$

but on the other hand $\pi_D(dU)\pi_D(dU)$ is certainly a non-zero operator:

$$(\pi_D(dU)\pi_D(dU))\,e_n = e_{n+2}$$

To obtain a true differential graded algebra we need to quotient the algebra generated by U and dU by the differential ideal. The above calculation was not a coincidence, as it happens that $dU\,dU$ is exactly the element which should be added to the ideal generated by $\ker \pi_D$ in order to make it a differential one. As a result, we have a differential graded algebra with central one-forms (commuting with the elements of the algebra) and all higher-order forms vanishing. But this is nothing else than the de Rham differential algebra over a circle, when restricted to polynomial functions!

Finally, let us come back to the case of functions on two-points:

Exercise 3.19: Take the algebra of complex-valued functions on two points and its representation on the Hilbert space $\mathbb{C}^2$. Show that the universal differential calculus is isomorphic to the calculus given by the operator F:

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

4 The pleasures of geometry

"Geometry is the only science that it hath pleased God hitherto to bestow on mankind."

(Thomas Hobbes)

So far we have extended one geometric notion: that of differential algebras and differential forms. We still have many tasks ahead of us, at least from the practical point of view of applications to physics. We need to understand the noncommutative generalization of vector bundles (so we shall come back to the notion of vector fields in the end), connections on them, integration and, last not least, we need to recognize whether our constructions fall into the same classes from the topological point of view.

4.1. Projective modules

In the same manner as we have dealt with spaces, which we have replaced by (suitable) algebras we shall tackle vector bundles. So, instead of taking the vector bundle itself we take the linear space of all its sections. What is the structure of that space? Clearly, it is not an algebra but since each section might be multiplied by the function in the algebra we have the structure of a module.

Definition 4.1: $\mathcal{M}$ is a left-module over the algebra $\mathcal{A}$ if it is a linear space and there exists an associative action:

$$\mathcal{A} \otimes \mathcal{M} \ni m \otimes a \mapsto am \in \mathcal{M}$$

which satisfies:

$$a(m_1 + m_2) = am_1 + am_2,\ \ a(bm) = (ab)\,m,$$

for all $a, b, \in \mathcal{A}$ and $m, m_1, m_2 \in \mathcal{M}$.

However, for a vector bundle we have a condition of local triviality, which is an essential ingredient. Cutting the story short, it could also be nicely translated to the language of modules, based on the crucial result of Serre-Swan. First we need a definition:

Definition 4.2: The module $\mathcal{M}$ over an algebra $\mathcal{A}$ is projective if there exists a module $\mathcal{M}^\perp$ such that $\mathcal{M} \oplus \mathcal{M}^\perp \approx \mathcal{A}^n$ for some $n > 0$. The module is said to be finitely generated if there exist a finite number of elements $m_1, \ldots, m_k$ such that

$$\mathcal{M} = \left\{ \sum_{i=1}^{k} a_i\, m_i \right\}_{a_i \in \mathcal{A}}$$

Then we have Serre-Swan equivalence:

Theorem 4.3: The continuous sections of a vector bundle over a manifold form a finitely generated projective module over the algebra of continuous functions on the manifold. In turn, every finitely generated projective module over a commutative algebra of continuous functions is of that form.

Due to this theorem we have another straightforward generalization: sections of vector bundles are just elements of projective modules! Note that there are, of course, several equivalent but different definitions of projectivity in addition to 4.2.

Exercise 4.4: Let $\mathcal{M}$ be a left module over $\mathcal{A}$. Show that if any surjective module morphism $\pi : \mathcal{N} \to \mathcal{M}$ splits for any $\mathcal{A}$ left module $\mathcal{N}$, (which means that there exists a morphism $\rho : \mathcal{M} \to \mathcal{N}$, such that $\pi \circ \rho = id_{\mathcal{M}}$) then the module $\mathcal{M}$ is projective.

Another equivalent statement is that for any surjective module morphism $\pi : \mathcal{N}' \to \mathcal{N}$ every homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ can be lifted to a homomorphism $\rho' : \mathcal{M} \to \mathcal{N}'$ such that $\rho = \pi \circ \rho'$.

Why don we not play with projective modules, which are infinitely generated? Certainly, a good reason is that they do not correspond to locally trivial bundles. Moreover, the world of infinitely generated modules is a strange one. Look, for instance, at the so-called *Eilenberg swindle*. Take a projective module $\mathcal{M}$ and an infinitely generated free module $\mathcal{A}^\infty$. Then, if $\mathcal{M}^\perp$ is the completion of $\mathcal{M}$ to a free module we have:

$$\mathcal{M} \oplus \mathcal{A}^\infty = \mathcal{M} \oplus (\mathcal{M}^\perp \oplus \mathcal{M}) \oplus (\mathcal{M}^\perp \oplus \mathcal{M}) \oplus \cdots,$$

so:

$$\mathcal{M} \oplus \mathcal{A}^\infty = (\mathcal{M} \oplus \mathcal{M}^\perp) \oplus (\mathcal{M} \oplus \mathcal{M}^\perp) \oplus \cdots$$

and hence:

$$\mathcal{M} \oplus \mathcal{A}^{\infty} = \mathcal{A}^{\infty}.$$

In the next step we need to learn a bit more: how to construct projective modules and how to distinguish between different projective modules!

4.1.1 Modules and projections

Starting with an algebra $\mathcal{A}$ we already know how to construct a certain class of projective modules: $\mathcal{A}^n$, called *free modules*. Now, imagine we have a projection $p \in M_n(\mathcal{A})$, which is an $n \times n$ matrix with entries from the algebra $\mathcal{A}$, such that $p^2 = p$.

Lemma 4.5: Let $p \in M_n(\mathcal{A})$ be a projection. Let $\mathcal{M}_p$ be defined as a subspace of all elements in $\mathcal{A}^n$ such that $mp = m$:

$$\sum_{i=1}^{n} m_i\, p_{ji} = m_j$$

where we have explicitly taken $m \in \mathcal{A}^n$ as a collection of elements from $\mathcal{A}$, $m = \{m_1, \ldots, m_n\}$ and denoted the entries p_{ij} of the matrix p.

Then $\mathcal{M}_p$ is a finitely generated projective module.

Proof. We need to show that $\mathcal{M}_p$ is indeed a left-module over $\mathcal{A}$. This, however, follows immediately from the definition, if m satisfies $mp = m$ so does $a\, m$ (as the multiplication in $\mathcal{A}$ is associative). If e_i denotes the basis of $\mathcal{A}^n$, an element with zeroes at all entries apart from the i-th where it has 1, then all elements of $\mathcal{M}_p$ are generated by $\{e_i p\}_{i=1,\ldots,n}$. Moreover, because $1 - p$ is also a projection, $\mathcal{M}_p \oplus \mathcal{M}_{1-p} = \mathcal{A}^n$, so the module $\mathcal{M}_p$ is projective. □

Example 4.6: Take the algebra of functions over sphere S^2 and the following projection in $M_2(C(S^2))$:

$$p = \frac{1}{2}\begin{pmatrix} 1 + \cos\vartheta & \sin\vartheta e^{i\phi} \\ \sin\vartheta e^{-i\phi} & 1 - \cos\vartheta \end{pmatrix}.$$

The projective module defined by $\mathcal{M}_p$ is nontrivial (that is, it is not free) – and has a deep physical meaning. It is a projective module associated with the vector bundle of the magnetic monopole. The module with $1 - p$ is then the antimonopole.

The next example shows that we need to be careful about the algebra we take. In the commutative case this simply means that it *does* matter whether we take polynomials, smooth functions or continuous functions.

Example 4.7: Let $C(T^2)$ be the algebra of continuous functions over a torus. Consider the following projection:

$$p = \begin{pmatrix} f(\phi) & g(\phi) + h(\phi)e^{i\psi} \\ g(\phi) + h(\phi)e^{-i\psi} & 1 - f(\phi) \end{pmatrix},$$

where real-valued functions f, g, h satisfy: $gh = 0$ and $g^2 + h^2 = f - f^2$, and $0 \le \phi, \psi \le 2\pi$ denotes the usual coordinates on the torus. Although the verification that p is a projection is easy, we need to wait a while to show that it might correspond to a nontrivial line bundle (a 1-dimensional complex vector bundle) over the two-torus!

Actually, we might propose a different projection:

Example 4.8: Let $C(T^2)$ be again the algebra of continuous functions over torus. Let us take the following projection $\hat{p}$:

$$\begin{pmatrix} -\frac{1}{2}\sin^2\frac{\psi}{2}(\cos\phi + 1) + 1 & \frac{1}{2}i\sin\frac{\psi}{2}(\cos\frac{\psi}{2}(\cos\phi + 1) + i\sin\phi) \\ -\frac{1}{2}i\sin\frac{\psi}{2}(\cos\frac{\psi}{2}(\cos\phi + 1) - i\sin\phi) & \frac{1}{2}\sin^2\frac{\psi}{2}(\cos\phi + 1) \end{pmatrix}$$

which (certainly) is not of the above form. Again, to see what the vector bundle is we need to wait a bit.

An interesting observation is that while the projection p might be smooth (provided that f, g, h are smooth), $\hat{p}$ is only continuous! The advantage is, however, that $\hat{p}$ is close (of course, in a naive sense) to the matrix algebra over polynomials in $e^{i\psi}$ and $e^{i\phi}$.

The notion of equivalence of projective modules is a natural one, given through module isomorphisms. (To be more precise the notion could be slightly relaxed to a *stable isomorphism*: two modules are stably isomorphic, if they are isomorphic after adding a free module.) But how can we distinguish that two modules given by two different projections are equivalent? For this purpose we need a notion of the equivalence of projections, which is due to Murray-von Neumann.

Definition 4.9: We say that the projections p and q are equivalent, if there exists $u \in M_n(\mathcal{A})$ such that $u\, u^* = p$ and $u^*\, u = q$.

Note that any projection $p \in M_n(\mathcal{A})$ can always be embedded in $M_N(\mathcal{A})$, $N > n$, putting p in the upper left corner and filling the remaining entries with zeroes. This means that the equivalence of projections needs to be understood in $M_\infty(\mathcal{A})$ (seen as an inductive limit of $M_n(\mathcal{A})$, $n \to \infty$).

To finish this section let us mention what we can do with the newly established set of all equivalence classes of projections. First, a little help comes from the lemma:

Lemma 4.10: The space of equivalence classes of projections has the structure of a semigroup, with the addition:

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right].$$

And thus we have met the *K*-theory.

Definition 4.11: We define the K_0 group of the algebra $\mathcal{A}$ as the Groethendieck group associated with the semigroup $V(\mathcal{A})$. Formally we can speak of abstract classes of pairs with the equivalence relation:

$$([p],[q]) \sim ([p'],[q']) \Leftrightarrow \exists [r]:[p]+[q']+[r]=[p']+[q]+[r].$$

The origins of K-theory are based in the classification problems of (real) vector bundles over manifolds. The construction of the K_0-group in the seminal work of Atiyah was the breakthrough of topological K-theory – this together with the Serre-Swan theorem, which formulates the equivalence between vector bundles and finitely generated projective modules over commutative algebras enabled to push the theory to the C^*-algebraic setup.

We were very sloppy here on the details – for example, whether we use arbitrary algebras or C^*-algebras. For the purpose of the first K-theory group, K_0, this does not really matter (in the sense that for C^* algebras the *algebraic* K-theory we defined is the same as the *topological* K-theory. The difference arises later, when one turns to higher K-groups – but graciously we shall not touch this topic here.

In principle, K-theory of a C^*-algebra is a functor from this category to the category of abelian groups: K_0 is defined through equivalence classes of projections, whereas K_1 is the π_0 of the Gl_∞ group of the algebra (the inductive limit of invertible matrices over the algebra). We skip here the details of the construction and its properties, which can be found in many textbooks. The interested reader is recommended to consult them (see the list at the end of the notes).

For us, there are two important things: first of all, K-theory can be calculated (thanks to advanced tools such as excision and the connecting morphism between K_0 and K_1). Furthermore, it provides important information about the algebra itself, like the existence and classification of nontrivial (noncommutative) vector bundles. What is also significant, is that K-theory depends actually on the dense (and stable under holomorphic functional calculus) subalgebra; that is, in the commutative case one might work with continuous as well with smooth functions, and K-theory still does not change.

4.2 Connections on projective modules

Finally we can use what we have already learned and apply to both differential algebras and projective modules to construct a new object (known from differential geometry, of course): a connection.

In differential geometry we know that the theory of connections on vector bundles is equivalent to connections on principal fibre bundles. In noncommutative geometry we have all the necessary tools so we can start the theory in just the same way. Such objects are interesting from many points of view: first of all, they provide an element of the theory that enables some practical calculations of noncommutative Chern characters. This links K-theory with noncommutative differential forms. From the point of view of physics, connections are just a tool for the gauge theory and gravity. Therefore we can view this as a step towards the notion of noncommutative gauge theory!

Although the use of connections appears in many places, including Connes and Rieffel's seminal work on the Noncommutative Torus [24], it is worth mentioning that systematic analysis and proof of the relation between projectivity and the existence of universal connections is a great result of Cuntz and Quillen [33].

Let us start with a definition and some interesting properties.

Definition 4.12: Let $\mathcal{M}$ be a left projective module over an algebra $\mathcal{A}$ and let $\Omega^*(\mathcal{A})$ be a DGA over $\mathcal{A}$. A connection ∇ on $\mathcal{M}$ is a linear map:

$$\nabla : \mathcal{M} \mapsto \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M},$$

such that:

$$\nabla(a\, m) = a\nabla(m)a + da \otimes_{\mathcal{A}} m$$

where $m \in \mathcal{M}$, $a \in \mathcal{A}$.

Before we proceed further, let us pose the question of existence. Evidently, if $\mathcal{M}$ is a left projective module then we have a projection $p \in M_n(\mathcal{A})$ such that $\mathcal{M} = \mathcal{A}^n p$. Then one might always construct a canonical (so-called Grassmanian) connection. On a free module it is a trivial exercise to set:

$$\nabla(A) = dA,$$

where $A \in \mathcal{A}^n$ and the expression on the right-hand side is understood as an element of $\mathcal{A}^n$. Then using the inclusion of a projective module into $\mathcal{A}^n$ we construct the Grassmanian connection as a composition of the connection on the free module with the projection:

$$\nabla(Ap) = dA \otimes_{\mathcal{A}} p$$

It is a nice feature that the existence of *universal connections*, that is connections related with the universal differential algebra, is actually equivalent to the projectivity of the module.

Before we define the curvature of a connection let us observe that the space of connections is an affine space over $\mathcal{A}$-linear module endomorphisms of $\mathcal{M}$, that is every two connections differ by an element of $End_{\mathcal{A}}(\mathcal{M},\Omega^1(\mathcal{A})) \otimes_{\mathcal{A}} \mathcal{M}$.

The connection is also easily extended to a general degree-one linear map satisfying the graded Leibniz rule on $\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$. The difference of connections is always a $\Omega^*(\mathcal{A})$-module homomorphism.

In particular, for a free module the connection is always of the form:

$$\nabla(A) = dA + \alpha(A),$$

where $\alpha \in End_{\mathcal{A}}(\mathcal{A}^n, \Omega^1(\mathcal{A})) \otimes_{\mathcal{A}} \mathcal{A}^n$ can be understood as a matrix of one-forms over $\mathcal{A}$.

The curvature of a connection is its square, ∇^2, which itself is a degree two $\mathcal{A}$-linear module endomorphism of $\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$. For a free module we have:

$$\nabla^2(A) = (d\alpha + \alpha^2)(A),$$

which is a well-known formula from the gauge theory but now applicable to a general class of noncommutative objects!

We shall finish this short review of connections by mentioning that in case the module $\mathcal{M}$ has an $\mathcal{A}$-valued hermitian inner product one might require that the connection is *Hermitian*:

$$\langle \nabla m, m' \rangle + \langle m, \nabla m' \rangle = d\langle m, m' \rangle,$$

where the left-hand side is the natural extension of the inner product on the tensor products with one-forms. In the canonical example of the free module (and the canonical inner product) this translates to the requirement that $\alpha^* = -\alpha$ (as a matrix of differential forms).

To make the picture a bit more comprehensible, let us study two important examples.

Example 4.13: Take the algebra $\mathcal{A}$ of complex valued functions on the space of two points. It has (of course) only free modules and we take just the simple one, $\mathcal{A}$ itself. It is equipped with the standard hermitian product arising just from the complex conjugation and product in $\mathcal{A}$. Let us consider all hermitian connections with respect to universal differential calculi. We have, for $a \in \mathcal{A}$ (but $\mathcal{A}$ seen as a left-module)

$$\nabla(a) = da + \hat{\Phi} a,$$

where $\hat{\Phi}$ is an arbitrary one-form, so $\hat{\Phi} = \Phi\, de$, with Φ a complex function on two points. Since $\hat{\Phi}$ must be antiselfadjoint, when we recall all the rules of differential calculus (see examples 3.6 and 3.12), we have:

$$(\Phi\, de)^* = -de\, \Phi^* = \Phi\, de$$

Since $e\, de = -de\, e$ the function Φ is given by a complex number ϕ:

$$\Phi = (\phi - \phi^*)1 + (\phi + \phi^*)\, e.$$

The curvature of the connection ∇, ∇^2 is:

$$\nabla^2(a) = d\hat{\Phi} + \hat{\Phi}^2,$$

which rewritten in the language of forms de gives:

$$\nabla^2(a) = (\phi + \phi^* - 4\phi\phi^*)de\, de,$$

Next, by introducing $H = 1 - 4\phi$ we obtain:

$$\nabla^2(a) = \frac{1}{4}(1 - HH^*)de\, de.$$

Suppose now that we think of H as a gauge connection field and calculate the square of the curvature. Integrating this square over the space (two points) we get

$$\int F^2 = \frac{1}{2}(1 - HH^*).$$

If we construct a weird type of Kaluza-Klein theory with the extra space being the usual one, we might interpret this as an action for a field $H(x)$, which arises from the discrete geometry of two-points.

It is a surprise that this type of action is well known and has a very important role in physics as the action for the Higgs field!

5 Cycles and cyclic cohomology

"Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone."

(Albert Einstein)

Let us recall that the classical theory of connections on vector bundles leads to the notion of characteristic classes. The square of the connection, curvature, which is a differential two-form valued in the algebra of endomorphisms of the vector bundle is the basic building block of the theory.

In the noncommutative setup we have almost the same possibility, with the exception that the differential calculi are plentiful and therefore some of them (in particular the universal differential calculus) may carry no cohomology information, that can be used to construct characteristic classes. However, when we accept this approach we shall have no general principle in the theory: every single case needs to be studied separately! Moreover, it is rather unrealistic to study all possible differential calculi that can bring some new data. Again, the solution to the problem lies in the approach – we need to introduce a more general notion which replaces de Rham cohomology in the noncommutative setup.

Definition 5.1: Consider $C^n(\mathcal{A})$ – space of linear maps from $\mathcal{A}^{\otimes(n+1)}$ to $\mathbb{C}$, and the linear map $b: C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$:

$$\begin{aligned}(b\phi)(a_0, a_1, \ldots, a_n, a_{n+1}) :=\; & \phi(a_0 a_1, a_2, \ldots, a_{n+1}) \\ & -\phi(a_0, a_1 a_2, \ldots, a_{n+1}) \\ & + \ldots \\ & +(-1)^n \phi(a_0, a_1, \ldots, a_n a_{n+1}) \\ & +(-1)^{n+1} \phi(a_{n+1} a_0, a_1, \ldots, a_n),\end{aligned}$$

where ϕ is an $n+1$-linear functional and

$$a_0, a_1, \ldots, a_{n+1} \in \mathcal{A}.$$

Then $b^2 \equiv 0$, and we can define the cohomology of the complex $\{C^n, b\}_{n \in \mathbb{N}}$, which is called the Hochschild cohomology of A and denoted $HH^*(\mathcal{A})$:

$$HH^n(\mathcal{A}) = \frac{\ker b\big|_{C^n}}{\operatorname{Im} b\big|_{C^n}}.$$

Example 5.2: Let us see what is the zeroth Hochschild cohomology group for any algebra $\mathcal{A}$. From the definition, it consists of all linear functionals ϕ, which obey:

$$b\phi(a_0,a_1)=\phi(a_0a_1)-\phi(a_1a_0)=0.$$

Therefore $HH^0(\mathcal{A})$ is nothing else but the linear space of all traces on the algebra $\mathcal{A}$!

Higher Hochschild cohomology groups are more difficult to calculate (but still calculable in most examples). For the purpose of making connections with the commutative differential geometry let us quote:

Theorem 5.3: For an algebra of smooth functions on a manifold M, the continuous Hochschild cohomology group $HH^k(C^\infty(M))$ is canonically isomorphic with the space of de Rham currents on the manifold M (which are continuous linear functional on the space of de Rham forms).

For proof (and for more details) we refer to the seminal work of Connes [23].

So, we have a space, which corresponds (roughly) to the differential forms (although it is **not** a differential algebra)! What we need is to find a subcomplex which would give us in the commutative situation some information about the de Rham cohomology.

Definition 5.4: Consider $C^n_\lambda(\mathcal{A})$ - space of linear maps from $\mathcal{A}^{\otimes(n+1)}$ to $\mathbb{C}$, which are cyclic:

$$\phi(a_0,a_1,\ldots,a_n)=(-1)^n\phi(a_1,a_2,\ldots,a_n,a_0)$$

and the linear map $b:C^n_\lambda(\mathcal{A})\to C^{n+1}_\lambda(\mathcal{A})$, which is the restriction of the coboundary b defined above.
The homology of the cochain complex $(C^n_\lambda,b)_{n\in\mathbb{N}}$ is the cyclic cohomology of $\mathcal{A}$, denoted $HC^*(\mathcal{A})$.

Example 5.5: Let M be a manifold of dimension and $\Omega_{dR}(M)$ the differential algebra of de Rham forms over M. For smooth functions $a_0,a_1,\ldots,a_n$ we define:

$$\phi(a_0,a_1,\ldots,a_{n+1}):=\int a_0\,da_1\wedge\cdots\wedge da_n,$$

where $\int$ is the standard integral on the manifold. Then ϕ is a cyclic cocycle of dimension n. The cyclicity of ϕ follows from the Leibniz rule and the fact that the wedge product of forms is antisymmetric. We check explicitly that ϕ is a Hochschild cocycle:

$$\begin{aligned}
&b\phi(a_0,a_1,a_2,\ldots,a_{n+1})=\phi(a_0a_1,a_2,\ldots,a_{n+1})\\
&\quad-\phi(a_0,a_1a_2,\ldots,a_{n+1})+\ldots+(-1)^n\phi(a_0,\ldots,a_na_{n+1})\\
&\quad+(-1)^{n+1}\phi(a_{n+1}a_0,\ldots,a_n)\\
&\quad=\int(a_0a_1)da_2\wedge\cdots\wedge da_{n+1}-a_0d(a_1a_2)\wedge\cdots\wedge da_{n+1}+\ldots\\
&\qquad+(-1)^n a_0da_1\wedge\cdots\wedge d(a_na_{n+1})+(-1)^{n+1}a_{n+1}a_0da_1\wedge\ldots\\
&\qquad\wedge da_n=(-1)^n a_0da_1\wedge\cdots\wedge da_na_{n+1}\\
&\qquad+(-1)^{n+1}a_{n+1}a_0da_1\wedge\cdots\wedge da_n\\
&\quad=0
\end{aligned}$$

A *cycle* is a noncommutative generalization of what we have in differential geometry (as presented earlier): functions and forms together with the integral and the (inevitable) Stokes theorem. However, it is not at all that easy and if one goes noncommutative then, in general, there is no *default* construction of such a structure.

Definition 5.6: A cycle of dimension n over $\mathcal{A}$ is a graded differential algebra over $\mathcal{A}$ together with a closed graded trace $\int:\Omega(\mathcal{A})\to\mathbb{C}$:

$$\int\omega\rho=(-1)^{|\omega|}\int\rho\omega,\quad\rho\omega\in\Omega(\mathcal{A}),$$

$$\int d\rho=0,\quad\rho\in\Omega(\mathcal{A}).$$

Using cycles one might easily construct cyclic cocycles.

Lemma 5.7: Each cycle of dimension n over algebra $\mathcal{A}$ defines a class of a cyclic cocycle in $HC^n(\mathcal{A})$:

$$\rho(a_0,a_1,\ldots,a_n)=\int i(a_0)di(a_1)\ldots di(a_n).$$

For proof see [5], Proposition 4, p.186.

Recall that we have met a very nice prescription for the construction of differential graded algebras through the representation and commutators with an operator F, $F^2=1$. Can we get a *cycle* in this way? The answer is positive, provided that we are ready to add a couple of additional features.

Definition 5.8: If $\mathcal{A}$ is an algebra, π its representation as bounded operators on the Hilbert space, and F a selfadjoint operator such that $F^2=1$ and for every $a\in\mathcal{A}$ the commutators $[F,\pi(a)]$ are compact then we call $(\mathcal{A},\pi,F)$ a Fredholm module over $\mathcal{A}$. We say that the Fredholm module is even if there exists an operator $\gamma=\gamma^\dagger$, $\gamma^2=1$ which commutes with the representation π and anticommutes with F.

Actually we have already met an odd Fredholm module in the example of the "strange" differential algebra for the functions on the circle 3.16. And we have shown that indeed all commutators with the algebra elements were compact!

To define the graded trace on the cycle we need to know something about the summability of the Fredholm module, i.e. we need to assume that the products of commutators $[F,\pi(a)]$ fall into the ideal of trace class operators. Trace class operators are operators such that the series of partial sums of their eigenvalues is summable. More precisely, we say that the Fredholm module is $p+1$-summable, if for any $p+1$-elements the product of compact operators:

$$[F,a_1][F,a_2]\cdots[F,a_{p+1}]$$

is an operator of trace class.

Then we have:

Lemma 5.9: For a $(n+1)$ summable Fredholm module, the closed graded trace of dimension is given by:

$$\int\omega=\frac{1}{2}\mathrm{Tr}(\omega+F\omega F),$$

in the odd case, and for even Fredholm modules by:

$$\int \omega = \frac{1}{2} \operatorname{Tr} \gamma(\omega + F\omega F).$$

As a corollary we have:

Corollary 5.10: Each $n+1$-summable Fredholm module gives rise to an n-cyclic cocycle.

To exploit relations with K-theory and formulate the Chern map using connections one needs an additional ingredient. This is cyclic cohomology (or rather a version of it, so-called periodic cyclic cohomology) that is the natural receptor for the Chern map. We will see how differential graded algebras can help us to construct cyclic cohomology elements.

Example 5.11: Recall again the construction of the differential algebra for the circle, with the operator F, $F^2 = 1$ in 3.16. The commutators of with the elements of the algebra are compact and in particular, commutators with polynomials are finite rank operators. Therefore they are trace class and we can easily define a 1-cyclic cocycle on the polynomial algebra:

$$\phi(a_0, a_1) = \operatorname{Tr} \frac{1}{2} \operatorname{Tr}(a_0[F, a_1] + Fa_0[F, a_1]F).$$

We calculate this explicitly for homogeneous polynomials. First,

$$U^n [F, U^k] e_p = \left(sign(p + k + \frac{1}{2}) - sign(p + \frac{1}{2}) \right) e_{p+k+n}.$$

Therefore:

$$\operatorname{Tr} U^n [F, U^k] = \begin{cases} 2k & \text{if } n + k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then:

$$\phi(U^n, U^k) = \delta_{n+k,0} 2k.$$

Let us verify that it is a cyclic cocycle. First of all, observe that it is cyclic:

$$\phi(U^n, U^k) = \delta_{n+k,0} k = -\delta_{n+k,0} n = -\phi(U^k, U^n).$$

Furthermore:

$$\begin{aligned} b\phi(U^k, U^m, U^n) &= \phi(U^{k+m}, U^n) - \phi(U^k, U^{m+n}) + \phi(U^{k+n}, U^m) \\ &= 2\delta_{k+m+n}(n - (m+n) + m) \\ &= 0 \end{aligned}$$

Note that up to some rescaling we obtain the same cocycle as the one arising from classical de Rham forms and the standard integration:

$$\phi_{dR}(U^n, U^k) = \int_0^1 d\theta e^{2\pi i n\theta} d e^{2\pi i k\theta} = (2\pi i) k \delta_{n+k,0}.$$

Now we can turn to the truly noncommutative world.

Example 5.12: Do you remember the Noncommutative Torus $UV = e^{2\pi i\theta} VU$? We shall construct a two-dimensional cycle over it. First, the differential forms. We take two generating one-forms ω_U and ω_V, which are central. The external derivative becomes:

$$da = \delta_U(a)\omega_U + \delta_V(a)\omega_V,$$

where δ_U and δ_V are two outer derivations on the algebra of the Noncommutative Torus:

$$\delta_n(U^n, V^m) = nU^n V^m, \;\; \delta_m(U^n, V^m) = mU^n V^m.$$

The two-forms are generated by the wedge product of the one-forms:

$$\omega_U \wedge \omega_V = -\omega_V \wedge \omega_U.$$

For the trace we take:

$$\int a\omega_U \wedge \omega_V = \operatorname{Tr} a,$$

where Tr is the standard trace on the algebra, defined on the polynomials as:

$$\operatorname{Tr} U^n V^m = \delta_{n,0}\, \delta_{m,0}.$$

Clearly $\int$ is a graded trace (since the basic one-forms anticommute and Tr is a trace), so we need only to show that it is closed:

$$\begin{aligned} \int d(U^n V^m \omega_U + U^k V^l \omega_V) &= \int (-mU^n V^m + kU^k V^l)\omega_U \wedge \omega_V \\ &= \operatorname{Tr}(-mU^n V^m + kU^k V^l) \\ &= -m\delta_{n,0}\delta_{m,0} + k\delta_{k,0}\delta_{l,0} = 0. \end{aligned}$$

The resulting two-cyclic cocycle is:

$$\begin{aligned} \phi(a_0, a_1, a_2) &= \int a_0\, da_1\, da_2 \\ &= \int a_0 \,(\delta_U(a_1)\omega_U + \delta_V(a_1)\omega_V) \wedge (\delta_U(a_2)\omega_U + \delta_V(a_2)\omega_V) \\ &= \int a_0 \,(\delta_U(a_1)\delta_V(a_2) - \delta_U(a_2)\delta_V(a_1))\omega_U \wedge \omega_V \\ &= \operatorname{Tr}(a_0 \,(\delta_U(a_1)\delta_V(a_2) - \delta_U(a_2)\delta_V(a_1))). \end{aligned}$$

We shall use this construction later – also in the case $\theta = 0$, which is the usual commutative torus.

5.1 Chern-Connes pairing

In this section we shall use the cyclic cohomology to produce invariants of projective modules. We must be aware that in doing so we cut a really long story short. Details and more aspects of the theory are left for the intrigued reader to further self-study.

Assume now we have an algebra $\mathcal{A}$. Let p be a projection in $M_n(\mathcal{A})$ which is associated to a finitely generated projective module $\mathcal{M}_p$. On the other side, let ϕ be an even cyclic cocycle over $\mathcal{A}$. Even without knowing much about the theory we can construct the following pairing:

$$\langle \phi, p \rangle = \sum_{i_1, \ldots, i_k} \phi(p_{i_1 i_2}, p_{i_2 i_3}, \ldots, p_{i_k i_1}), \tag{2}$$

which (for brevity) we shall always write as $\phi(p, p, \ldots, p)$. As such, it is only a number. However, the following theorem makes it a really important number:

Theorem 5.13: The pairing (refChern depends only on the equivalence class of the of the projection p and also only on the class of the cocycle ϕ within the cyclic cohomology group $HC^{even}(\mathcal{A})$.

For proof we refer to Connes [5].

What we have gained? An extremely useful tool of noncommutative geometry applicable to noncommutative topology: we have a very precise way of distinguishing different topological classes of projective modules. Let us test this knowledge on some examples.

Example 5.14: Let us take the two-dimensional sphere and the projection of the magnetic monopole from example 4.6. Taking the standard cyclic cocycle that arises from de Rham differential forms and the standard integration, we have:

$$
\begin{aligned}
pdpdp &= \frac{1}{8}\begin{pmatrix} 1+\cos\vartheta & \sin\vartheta e^{i\phi} \\ \sin\vartheta e^{-i\phi} & 1-\cos\vartheta \end{pmatrix} \\
&\cdot\begin{pmatrix} -\sin\vartheta\, d\vartheta & \cos\vartheta e^{i\phi} d\vartheta + i\sin\vartheta e^{i\phi} d\phi \\ \cos\vartheta e^{i\phi} d\vartheta - i\sin\vartheta e^{i\phi} d\phi & \sin\vartheta\, d\vartheta \end{pmatrix} \\
&\cdot\begin{pmatrix} -\sin\vartheta\, d\vartheta & \cos\vartheta e^{i\phi} d\vartheta + i\sin\vartheta e^{i\phi} d\phi \\ \cos\vartheta e^{i\phi} d\vartheta - i\sin\vartheta e^{i\phi} d\phi & \sin\vartheta\, d\vartheta \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} i\sin\vartheta(1+\cos\vartheta) & i\sin^2\vartheta e^{i\phi} \\ -i\sin^2\vartheta e^{i\phi} & i\sin\vartheta(1-\cos\vartheta) \end{pmatrix} d\vartheta \wedge d\phi.
\end{aligned}
$$

The integral over S^2 of the trace of the above expression is:

$$\int_0^{\pi} d\vartheta \int_0^{2\pi} d\phi \left(\frac{1}{2} i \sin\vartheta\, d\vartheta \wedge d\phi\right) = 2\pi i.$$

Since the trivial line bundle over the sphere S^2 given by the trivial projection ($p = 1$) has a trivial pairing with the two-cyclic cocycle (for the obvious reason that $p = 1$ and hence $dp = 0$) – we have the proof (based, of course, on theorem 5.13) that the projective module of the magnetic monopole is not trivial.

Example 5.15: Let us take the commutative two-torus, with the cyclic cocycle given by the de Rham forms and integration. First, we start with the smooth projection 4.7. Skipping the easy calculation we show the result:

$$
\begin{aligned}
\mathrm{Tr}\, p\, dp\, dp = {} & 2i(g(t)(2f(t)g'(t) - 2f'(t)g(t) - g(t)') \\
& + \cos(s)g(t)(2f(t)h'(t) - 2h(t)f'(t) - h'(t))).
\end{aligned}
$$

Since the integral of $\cos(s)$ vanishes, only the first component contributes to the pairing. We rewrite it further as:

$$(f(t)g^2(t))' - 3f'(t)g^2(t) - \frac{1}{2}(g^2(t))'.$$

Integrating by parts, and using the fact that all functions are smooth and periodic we get the result for the pairing as:

$$(4\pi i)\int_0^{2\pi} dt(3f'(t)g^2(t)).$$

Recall that the function f, g, h must obey:

$$g(t)h(t) = 0,\ g(t)^2 + h(t)^2 = f(t) - f(t)^2.$$

Then:

$$
\begin{aligned}
(4\pi i)\int_0^{2\pi} dt(3f'(t)g^2(t)) &= (4\pi i)\int_{supp\, g(t)} 3f'(t)(f(t) - f^2(t)) \\
&= (4\pi i)\sum_{j=1}^{n}(-1)^{j+1}\left(\frac{3}{2}f^2(x_i) - f^3(x_i)\right).
\end{aligned}
$$

where the points x_i are such that:

$$
\begin{aligned}
supp\ g(t) &= (0, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_n, 2\pi) \\
&= \{0 \le t \le 2\pi : g(t) \ne 0\}.
\end{aligned}
$$

Since points x_i are the boundary points of a set where g (and thus h) do not vanish, by continuity both g and h must vanish each x_i. So, f satisfies $f^2(x_i) = f(x_i)$ and therefore it is either 0 or 1 at each point. Hence, we introduce

$$\eta_j = 2\left(\frac{3}{2}f^2(x_i) - f^3(x_i)\right) = \begin{cases} 0 \\ 1 \end{cases}$$

Since f is periodic, $f(0) = f(2\pi)$ and the points 0 and 2π cancel each other in the sum. In the end we have:

$$(2\pi i) = \sum_{j=1}^{n}(-1)^{j+1}\eta_j.$$

Therefore, independently of the choice of f the result is an integral multiple of $2\pi i$!

Example 5.16: Let us turn to the second presentation of the (supposedly) nontrivial projection on the torus. We use the cyclic cocycle as derived in in example 5.12, keeping in mind that the functions that we use in the projection are in fact only continuous and are not differentiable at one point ($t = 0$, identified with $t = 1$).

We calculate the pairing (again skipping the tedious but uncomplicated algebraic manipulations), obtaining:

$$\phi_2(p, p, p) = -\mathrm{Tr}\frac{1}{4} i \sin\left(\frac{\Psi}{2}\right)(\cos(\Phi) + 1) = -(2\pi i),$$

where we have taken the not-normalized trace coming from the integration ($\mathrm{Tr}1 = (2\pi)^2$).

Again, we obtain a result different from zero, which ensures that the projective module is nontrivial. Shall we worry that the projection was not actually a smooth one? Yes, but only a little bit. Note that each function in the matrix elements of the projection has a well-defined left derivative at each point. However, even though this derivative is not continuous at one point, we can multiply functions with such discontinuities without problems. Since the integration is also well-defined fur such functions, we see that no significant problems arise.

Finally, we turn to a very nontrivial (and very) noncommutative example.

Example 5.17: Consider the following family of projective (and smooth) modules the over the Noncommutative Torus:

$$\mathcal{M}_q = S(\mathbb{R}) \otimes \mathbb{C}^q,\ q \in \mathbb{N},$$

where $S(\mathbb{R})$ denotes the space of Schwartz functions (rapidly decreasing functions on $\mathbb{R}$) with the module structure:

$$U(\xi(s)\otimes v) = e^{2\pi i s}\xi(s)\otimes(w_u v),$$

$$V(\xi(s)\otimes v) = \xi\left(s-\frac{p}{q}+\theta\right)\otimes(w_v v),$$

where w_u, w_v are matrices satisfying

$$w_u w_v = e^{2\pi i\frac{p}{q}}\, w_v w_u.$$

It is not evident that these modules are projective and that every projective module over the NC Torus is either free or is of this form. Details are to be found in papers by Rieffel and Connes [24]. Here we shall just use this knowledge to find the pairing between the standard two-cyclic cocycle over the Noncommutative Torus and the module. The drawback is that we do not know the projection – and although one can construct it (surprisingly enough it is a projection in the algebra of the Noncommutative Torus itself!) – it is sufficiently complicated to make this form of approach rather difficult.

In turn we shall construct the connections and curvature over the module. Then using a function of the curvature (like in the case of characteristic classes) we shall recover the pairing.

As the connection on the module we take:

$$(\nabla\xi)(s) = 2\pi i\frac{sq}{p-\theta q}\omega_U\otimes\xi(s)+\omega_V\otimes\frac{d\xi(s)}{ds},$$

where we use the forms ω_U, ω_V introduced earlier. The verification that ∇ is the connection is explicit. The curvature is

$$F = -\frac{2\pi i q}{p-\theta q}\omega_U\wedge\omega_V.$$

The pairing, in this two-dimensional case, is given by applying the closed graded trace in the differential algebra to the trace of the curvature two-form (remember that the curvature is a two-form but with values in the endomorphism of the projective module):

$$\frac{1}{2\pi i}\frac{2\pi i q}{p-\theta q}\mathrm{Tr}(id_{\mathcal{M}}),$$

and since:

$$\mathrm{Tr}(id_{\mathcal{M}}) = p-\theta q,$$

we get the integer q as the value of the pairing.

Here we smuggled in the information about the dimension of the projective module (or whatever we might call it) – which could be defined as the trace of the identity endomorphism. In fact, one may use the pairing and some facts about its integrality (which we do not mention in these notes) to get this value. Nevertheless, although we are dealing with a very strange family of projective modules in a pure noncommutative setup, we can still say that we can distinguish between those which are not equivalent to each other with the help of the Chern-Connes pairing.

5.2 Summary

We may now summarize all the tools and construction that we have learned in this *crash course* on noncommutative geometry. We just extend the dictionary, which we constructed first for *noncommutative topology*:

GEOMETRY	ALGEBRA
vector bundles	finitely generated projective modules
differential forms	differential graded algebra
integration of differential forms	closed graded trace on DGA
simplicial (de Rham) homology	cyclic homology
infinitesimals	compact operators
integral	trace (exotic trace)
connection on a vector bundle	connection on a projective module
characteristic classes	Chern-Connes character

6 Spectral geometry and its applications

Ubi materia, ibi geometria.
(Where there is matter, there is geometry.)
(Johannes Kepler)

In this last part of the lectures we shall use (and probably overuse) the word *spectral*. Its sense will be described in the definition of properties of spectral triples – a concise proposition for *noncommutative spin manifolds*. The clue is that (almost) everything is set by the Dirac operator and it, in turn, is defined through its set of discrete eigenvalues with multiplicities. We briefly touch the main proposition, which links the theory with physics: the construction of gauge theories and the spectral action principle.

6.1 Enter: the Dirac operator

In the example of the differential graded algebra on the circle we tested a peculiar unbounded operator D (example 3.18). Of course, this was not a coincidence and the story could have been repeated for any compact spin manifold. Taking as the Hilbert space the square-summable sections of the spinor bundle $L^2(M, S)$ and D as the true Dirac operator (for a given Riemannian metric) we shall always (in a similar manner) recover the de Rham differential algebra. The Dirac operator on a compact spin manifolds is indeed a very elegant object: an unbounded, self-adjoint operator, with a discrete spectrum and with the growth of eigenvalues governed by the dimension of the manifold.

The Dirac operator is also a very useful tool: it encodes a lot of topological and geometrical information about the manifold, in particular about the differential algebra and the metric. We shall mimic this construction in the noncommutative world, assuming that it is the basic data that makes noncommutative geometry *the geometry*. But to do this we shall learn one more tool: the construction of exotic traces.

6.2 Exotic traces and residues

Let us start with a definition:

Definition 6.1. Assume we have a positive, compact operator T on a Hilbert space, with the eigenvalues $\mu_i(T)$, $i = 1, 2, \ldots$ Suppose that the eigenvalues decrease to 0 so fast that the following expression makes sense:

$$\mathrm{Tr}_\omega(T) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{i=1}^{N} \mu_i(T).$$

If it exists then we call it a Dixmier trace.

Example 6.2: Let us take, for instance, the operator $(|D|+1)^{-1}$, where D is the Dirac operator on the circle, mentioned in the previous example. Since each positive integer (apart from 1) enters into the spectrum of $|D|+1$ exactly twice, we have:

$$\begin{aligned}\mathrm{Tr}_\omega(|D|+1)^{-1} &= \lim_{N\to\infty} \frac{1}{\log N}\left(\sum_{i=1}^{N} \frac{2}{i}\right) \\ &= 2 \lim_{N\to\infty} \frac{1}{\log N}(\log(N) + \gamma + o(N)) = 2,\end{aligned}$$

where we have used the formula for the asymptotic form of the harmonic numbers, and γ is the Euler constant.

Theorem 6.3: The space of all operators for which the Dixmier trace can be calculated is a two-sided ideal in the algebra of bounded operators, moreover for R bounded and T in this ideal:

$$\mathrm{Tr}_\omega RT = \mathrm{Tr}_\omega TR.$$

For the proof of the theorem and also for verification that the Dixmier trace is well defined and is indeed a trace, we refer to Connes' book [5].

Assume now that we have an unbounded operator D such that $|D|^{-1}$ is compact (for simplicity we eliminate zeroes from the spectrum of D). For sufficiently large $s > 0$ the operator $|D|^{-s}$ will be a trace class and thus the function:

$$\zeta_D(s) = \mathrm{Tr}\,|D|^{-s},$$

makes sense. Using the functional calculus on a Hilbert space we can define this function at least on the part of the complex plane for $\mathrm{Re}(s)$ big enough. Suppose now that we make an analytic continuation of the function ζ_D and that it extends to the entire complex plane with the exception of several isolated points. Then at each of these points we can calculate the residue of $\zeta_D(s)$ – just as a functional.

We have:

Theorem 6.4: If $|D|^{-1}$ is an operator of Dixmier class then:

$$\mathrm{Tr}_\omega |D|^{-s} = Res_{z=1} |D|^{-z}.$$

Example 6.5: An elliptic differential operator on a manifold is just a special case of an unbounded operator. For instance, taking the Laplace operator Δ on a compact manifold of dimension n, it appears that $\Delta^{-\frac{n}{2}}$ is in the Dixmier class and its Dixmier trace is related to the Wodzicki residue (which is a functional on the space of differential operators given explicitly by the integral of the principal symbol).

As a specific example take (again) the 2-dimensional torus and its Laplace operator $\Delta = \partial_\Psi^2 + \partial_\Phi^2$, where $0 \geq \Phi, \Psi < 2\pi$ are again the standard coordinates on the torus.

The principal symbol of Δ (and so Δ^{-1}) is constant, so the Wodzicki residue, which is (in two dimensions) its integral over the sphere bundle and the manifold, is:

$$Wres(\Delta^{-1}) = 4\pi(2\pi)^2.$$

To calculate the Dixmier trace of we need to calculate the following limit:

$$\lim_{N\to\infty} \frac{1}{\log N} \frac{1}{(2\pi)^2} \left(\sum_{m^2+n^2 \leq N^2} \frac{1}{m^2+n^2} \right).$$

Leaving the evaluation of the asymptotics of the sum as an exercise (see [2], for hints) let us state the result:

$$\mathrm{Tr}_\omega(\Delta^{-1}) = \frac{1}{2\pi}.$$

The ratio:

$$\frac{Wres(\Delta^{-1})}{\mathrm{Tr}_\omega(\Delta^{-1})} = 2(2\pi)^2,$$

and is, in fact, universal (in this dimension 2).

6.3 Spectral triples

Imagine we want to encompass everything we have learned in one compact definition. So, we work with a suitable algebra, which is a subalgebra of a C^* algebra. It is represented (faithfully) on a Hilbert space. Further, we need to have a suitable definition of a differential algebra – and here we need just to choose a suitable unbounded operator D on the Hilbert space so that the differential one-forms, the commutators of D with the elements of the algebra remain bounded. The sign of D will define for us a Fredholm module, and thus a cyclic cocycle. Moreover, suitably chosen D will allow us to introduce noncommutative integrations through the residua of ζ_D. So, we are ready for the formal definition:

Definition 6.6: Let us have an algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint unbounded operator D with compact resolvent, such that

$$\forall a \in \mathcal{A},\ [D, \pi(a)] \in B(\mathcal{H}),$$

then we call $(\mathcal{A}, \pi, D)$ a spectral triple.

Since the definition is very basic, we shall need (in most cases) some additional structures. We say that the spectral triple is *even* if there exists an operator γ such that $\gamma = \gamma^{\dagger}$, $\gamma\pi(a) = \pi(a)\gamma$ and $\gamma D + D\gamma = 0$. We say that the spectral triple is finitely summable if the operator D^{-1} has eigenvalues of the order $o(n^{-p})$ for some $p \geq 0$. If the growth of eigenvalues of D is exactly of the order n^p, we say that the spectral triple is of metric dimension p. For such a triple we might introduce the noncommutative integral:

$$⨍ T = \mathrm{Res}_{z=0} \mathrm{Tr}(T|D|^{-z}).$$

This exists for all operators T, which are products of $\pi(a)$, powers of D and their commutators with D and $|D|$.

Definition 6.7: A real spectral triple of *KO*-dimension n mod 8 is a spectral triple with an antilinear unitary operator J, $J\,J^* = 1$, such that:

$$DJ = \epsilon JD,\ \ J^2 = \epsilon',\ \ J\gamma = \epsilon''\gamma\, J. \tag{3}$$

and

$$[J\pi(a)J, \pi(b)] = 0, [[D, a], J\pi(b)J] = 0, \tag{4}$$

for all $a, b \in \mathcal{A}$, and where the signs ϵ, ϵ', ϵ'' depend on the *KO*-dimension modulo 8 according to the following rules:

n mod 8	0	1	2	3	4	5	6	7
ϵ	+	−	+	+	+	−	+	+
ϵ'	+	+	−	−	−	−	+	+
ϵ''	+		−		+		−	

The first of conditions 4 states that conjugation by J maps the algebra to its commutant, whereas the second condition, the so-called order-one condition, states that the one forms commute as well with the commutant of $\mathcal{A}$.

The following establishes (precisely) the relation of spectral triples to classical differential geometry (for proof see [2]).

Theorem 6.8: If $\mathcal{A} = C^{\infty}(M)$, M is a spin Riemannian compact manifold, S is a spinor bundle over M, $\mathcal{H} = L^2(S)$ (summable sections of spinor bundle) and D is the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure), of *KO* and metric dimension dim(M).

Even more interesting is the Reconstruction Theorem, which states the inverse:

Theorem 6.9: If $\mathcal{A}$ is a commutative algebra and $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple satisfying all required conditions then M is a spin manifold and $\mathcal{A} = C^{\infty}(M)$, $\mathcal{H} = L^2(S)$, and D is the Dirac operator on M.

A weaker version of the theorem is due to Bondia & Varilly [2], a proof of the full version was proposed by Varilly & Rennie [44] and then improved by Connes.

We did not present here any further requirements for the spectral triples. This includes further algebraic conditions, like the existence of a certain Hochschild cycle, which is mapped to γ or 1 (depending on the dimension), or the Poincaré duality in *K*-theory. There are also some very restrictive conditions, more of the "analysis" type. They ensure, for instance, that the algebra is *smooth* with respect to the derivation given by the commutators with D and $|D|$, and the suitable domain of definition of these operators on the Hilbert space is a projective module over this algebra. Although all of this plays a crucial role in the reconstruction theorem, it is not certain that the formulation of these requirements is in its final form in the noncommutative situation. In fact, some of the recent examples coming from q-deformed spaces can hardly meet these requirement, yet they have reasonable Dirac operators.

6.3.1 The use of spectral geometries and examples

If spectral geometries are applicable to noncommutative algebras then we should learn how to extract the geometric data. We already know how to obtain a differential graded algebra. But, how can we get the metric? It appears that a simple formula allows us to recover the distances on the manifold (and in general, also introduce a metric on the space of states):

$$d(x, y) = \sup_{\|[D,a]\| \leq 1} |x(a) - y(a)|, \tag{5}$$

where $x, y \in M$ and $a \in C^{\infty}(M)$. We already know that

$$\int a = \mathrm{Tr}_{\omega} \pi(a)|D|^{-n}$$

is well-defined. Once we are sure that the Dixmier trace of $|D|^{-n}$ exists we can use this as a definition of the integral on the manifold. Summarizing – all *practical* information of geometry is encoded in the datum of the spectral triple.

Example 6.10: We might come back to the canonical example of two points. As the Dirac operator we take an arbitrary self-adjoint off-diagonal operator on $\mathbb{C}^2$. Clearly all conditions of the spectral triple are fulfilled: even more, we can construct a real spectral triple, of course, of dimension 0.

For this, we double the Hilbert space to $\mathbb{C}^2 \otimes \mathbb{C}^2$. It is now convenient to write every operator in a block diagonal form:

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & J_0 \\ J_0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & D_0 \\ D_0^{\dagger} & 0 \end{pmatrix},$$

where J_0 is the standard complex conjugation on $\mathbb{C}^2$.

The condition $D\,J = J\,D$ gives:

$$D_0 = D_0^T,$$

whereas the order-one condition is satisfied for any D. Therefore the spectral triple is set by a symmetric two-by-two matrix D_0. Of course, since the diagonal entries do not contribute to the commutators, it is the off-diagonal term that matters and that fixes, for instance, the metric. Indeed, we might ask what the distance is between these two points?

Using the formula 5, we need to calculate $|f(+) - f(-)|$, where $+, -$ denotes the two points of the space, for any function such that $\|[D, f]\| \leq 1$. First, take $f = z + we$, where $z, w \in \mathbb{C}$ and e is the generating function of the algebra $e^2 = 1$. We calculate:

$$[D, f] = \begin{pmatrix} 0 & w[D_0, e] \\ w[D_0^\dagger, e] & 0 \end{pmatrix}.$$

Next:

$$[D_0, e] = 2\begin{pmatrix} 0 & -(D_0)_{12} \\ (D_0)_{12} & 0 \end{pmatrix}.$$

where we have used that $(D_0)_{12} = (D_0)_{21}$ (*symmetric matrix* D_0). The norm of $[D, f]$ is:

$$\|[D, f]\| = 4|(D_0)_{12}w|.$$

For the function f given above $f(+) - f(-) = 2w$, therefore the distance between the two points, in the noncommutative geometry given by the Dirac operator D, is:

$$\mathrm{dist}(+,-) = \sup_{4|(D_0)_{12}w| \leq 1} 2|w| = \frac{1}{2}\frac{1}{|(D_0)_{12}|}.$$

It is a more difficult task to calculate the distance for the circle. There, we need to consider all smooth functions on S^1, represent them on the Hilbert space and calculate their operator norm!

Finally, let us study the best known example of the spectral triple: that of the Noncommutative Torus. There are several ways to guess or derive the construction – we shall, however, be very minimalistic and just provide it.

Example 6.11: We take the algebra of the Noncommutative Torus as generated by U, V, with the relation $UV = e^{2\pi i\theta}VU$, together with the faithful representation on the Hilbert space $\mathcal{H}$ (presented in example 2.9). We double the Hilbert space, take the diagonal representation of the algebra and introduce the operators J, D, γ as block-type operators. First we set:

$$J_0 e_{m,n} = e^{-2\pi i\theta mn} e_{-m,-n},$$

$$\delta e_{m,n} = (m + in)e_{m,n}.$$

Then:

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & \delta \\ \delta^\dagger & 0 \end{pmatrix}.$$

gives the data of the real spectral triple of KO and metric dimension 2 on the Noncommutative Torus.

Exercise 6.12: Verify that all properties of spectral triples are satisfied!

6.4 Making (noncommutative) physics

Suppose we accept that spectral triples do describe *noncommutative manifolds*. Is there any physical contents in them? Can we use them to describe some noncommutative physics? The answer is yes and, indeed, we shall be able to provide – at least – some partial answers.

6.4.1 Gauge theory and gravity

If we have a spectral triple $(\mathcal{A}, \pi, D)$ we may always wonder whether the Dirac operator we have chosen is a good one. Certainly nothing (apart from some symmetries) guarantees it and, in fact, a simple transformation

$$D \mapsto D + A,$$

where A is a self-adjoint one-form $A = \sum \pi(a_i)[D, \pi(b_i)]$ allows us to construct a spectral triple with the same algebra and Hilbert space but a – slightly – modified Dirac operator. Moreover, when we look at it, we see that in this way we reconstruct the gauge theory and is nothing else but the gauge potential. We call this *inner fluctuations of the Dirac operator*. It is a small step to calculate D^2, recover the curvature of the gauge potential and construct the action. In this approach there are, however, some hidden obstacles, which have origin in the fact that D defines a differential algebra only after we quotient out an additional ideal. Then, there are many non-equivalent ways of embedding the two-forms into the algebra of operators on the Hilbert space (which we need if we want to use noncommutative integrals to calculate the action).

Clearly, the information that is encoded in D includes also the metric and hence the Riemannian connection. Are we able to construct the gravity action as well? A partial answer was provided some time ago by Kastler, Kalau and Walze, who proved that the Einstein-Hilbert functional (the integral of the scalar of curvature, in other words) on manifolds can be expressed as a Wodzicki residue of a certain power of the operator D [38, 39].

Now, we are ready to see the proposition that encompasses both contributions.

6.4.2 The spectral action

Again, assume that we have a spectral triple, that is $(\mathcal{A}, \pi, D)$. The following defines a functional on the space of all admissible Dirac operators:

$$S(D) = \mathrm{Tr}\, f(D^2),$$

where f is a cut-off function, which, for instance, vanishes for arguments bigger than a certain number Λ. This idea appeared for the first time (in a similar phrasing) in the work of Sakharov [41] in 1965.

Of course, the action functional depends on the choice of f. However, it appears that the dependence is not as significant as we might suspect, as we shall see later.

Let us consider now the easy example of two-points and the spectral action in this case.

Example 6.13: Recall the construction of the spectral triple for the algebra of two-points and its Dirac operator D. The only free parameter in D was a complex number $(D_0)_{12}$. A most general self-adjoint one-form A is given as:

$$A = \begin{pmatrix} 0 & \omega \\ \omega^\dagger & 0 \end{pmatrix},$$

where

$$\omega = (D_0)_{12} \begin{pmatrix} 0 & w \\ z & 0 \end{pmatrix},$$

for arbitrary $w, z \in \mathbb{C}$. Since our triple is a real one and $JD = DJ$, we need to require the same for the gauge potential A and thus we have $z = w$. Then the spectral action is:

$$\mathrm{Tr}(D + A)^2 = 4|(D_0)_{12}|^2 (1 + z)(1 + \bar{z}).$$

It is not an exciting answer but, in fact, we did not expect anything exciting here. More interesting things happen when we consider the continuous geometry of the C^∞-functions on a manifold and the spectral action of the true Dirac operator!

Lemma 6.14: For the spectral triple $(C^\infty(M), L^2(M,S), D)$ over a 4-dimensional manifold, the spectral action (modulo topological and boundary terms) has the following asymptotic expansion:

$$\begin{aligned}
\mathrm{Tr}\, f\left(\frac{D^2}{\Lambda^2}\right) &= \frac{1}{16\pi^2}\left(f_4\Lambda^4 + f_2\Lambda^2 + \frac{f_0}{2}\right)\int \sqrt{g}\, d^4x \\
&- \frac{1}{96\pi^2}(f_2\Lambda^2 + f_0)\int R\sqrt{g}\, d^4x \\
&+ \frac{1}{4\pi^2}\frac{1}{360} \\
&\cdot f_0 \int (5R^2 - 2R_{\mu\nu}R^{\mu\nu} - 12R^{\mu}_{;\mu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}),
\end{aligned}$$

where $f_k = \int_0^\infty f(u) u^{k-1} du$ for $k > 0$ and $f_0 = f(0)$ are the moments of f.

This is a *pure gravity* action, which includes the cosmological constant, the Einstein-Hilbert action and some additional term, which depend on the Riemannian curvature tensor, the Ricci tensor and the scalar curvature.

If we introduce just a bit of noncommutative geometry, by taking as the algebra not $C^\infty(M)$ but $C^\infty(M) \otimes M_N(\mathbb{C})$, the algebra of matrix valued functions on M we obtain the possibility of constructing an $SU(N)$ gauge theory. The gauge connection one-form $A = A_\mu dx^\mu$ (in local coordinates) will enter the spectral action as well and we shall additionally obtain (apart from some change in the coefficients in the two first terms) another term in the Λ-independent part of the expansion: the Yang-Mills action:

$$\frac{1}{4\pi^2}\frac{1}{120} \cdot f_0 \int \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}).$$

6.5 The standard model

So far we have recovered important parts of theoretical physics all encoded in one simple action. There is, however, more to it as we shall finally see in this section. The crucial point is to take the geometries of the type $M \times F$, where M is a Riemannian manifold and F is a discrete geometry. It is like a Kaluza-Klein model but with the extra dimensions being in fact of (classical dimension) zero.

We shall study here a toy model of the construction, referring to the recent papers by Connes [31, 32] for the detailed advanced construction, which makes contact with the real physical Standard Model.

Example 6.15: Let us begin with the construction of the (real) spectral triple for the algebra of functions on $M \times F_2$, with F_2 the space of two points. The standard procedure is to take the spectral triples on both spaces and construct their tensor product. However, we shall simplify the construction: we shall just postulate the Dirac operator. Another simplification that we make is that we skip the requirement of the reality conditions (in the sense of J-reality) and take the manifold to be flat (say a 4-torus!).

As a Hilbert space we take the two copies of the spinorial Hilbert space over manifold M : $\mathcal{H} = L^2(M,S) \oplus L^2(M,S)$. The functions act on $\mathcal{H}$ so that

$$f(h_+ \oplus h_-) = f(+)h_+ \oplus f(-)h_-.$$

As the Dirac operator we take:

$$D = \begin{pmatrix} \gamma^\mu\partial_\mu + & \gamma^5 \\ \gamma^5 & \gamma^\mu\partial_\mu \end{pmatrix},$$

where by γ^5 we denote the $\mathbb{Z}_2$-grading of the standard spectral triple over M (which in physical notation is γ^5) and $\gamma^\mu(\partial_\mu - \omega_\mu)$ is the standard Dirac operator expressed in local coordinates with the spin connection ω_μ. The most general inner fluctuations of the Dirac operator are:

$$A = \begin{pmatrix} \gamma^\mu A_\mu(x)^+ + & \gamma^5 H(x) \\ \gamma^5 H^*(x) & \gamma^\mu A_\mu(x)^- \end{pmatrix}$$

where $A_\mu(x)^\pm$ are two copies of the $U(1)$ gauge potential and $H(x)$ is a complex-valued field!

We calculate the square of the Dirac operator paying particular attention to the terms that involve H. The rest, depending only on $A_\mu(x,\pm)$, does not differ from the usual classical theory. The terms that depend on H in the square of the Dirac operator $D + A$ are:

$$\begin{pmatrix} H(x)H(x)^* & \gamma^\mu\gamma^5(\partial_\mu + A_\mu(x)^+ - A_\mu(x)^-)H(x) - 1 \\ \gamma^\mu\gamma^5(\partial_\mu + A_\mu(x)^- - A_\mu(x)^+)H(x)^* & H(x)H(x)^* \end{pmatrix}$$

The next step is a not a trivial one – but using the knowledge of the asymptotic expansion and Seeley-de Witt coefficients (see [22, 25] for explanation and details) we find out what will change in the expansion of the spectral action. We skip the exact calculation (which we recommend, however, as a good exercise!)

It is no surprise that the additional terms involve certain covariant functionals of H:

- the vacuum energy of field $H(x)$ (both in Λ^2 and Λ^0 parts)
$$\int \sqrt{g}\,d^4x\, H(x)H(x)^*,$$
- the coupling of H to the scalar of curvature (Λ^0 part)
$$\int \sqrt{g}\,d^4x\, H(x)H(x)^* R,$$
- the kinetic term for the $H(x)$ field:
$$\int \sqrt{g}\,d^4x\,(D_\mu H(x) D^\mu H(x)^*).$$
- the potential of the $H(x)$ field:
$$\int \sqrt{g}\,d^4x\,|H(x)|^4.$$

Here D_μ denotes there the $U(1)\times U(1)$-*covariant derivative*. We can interpret these contributions in physical terms as the kinetic action of the Higgs field $H(x)$ and the Higgs potential, which after some rescaling can be written as:

$$\int \sqrt{g}\,d^4x\,(\rho^2 \pm H(x)H(x)^*)^2.$$

The crucial information (from the physical point of view) lies actually not in the exact values of the coefficients but their relative signs. If $|H|^2$ and $|H|^4$ appear with opposite signs we have the standard Higgs potential leading to the symmetry breaking mechanism.

Since (a priori) Λ and all coefficients f_k from the cutoff function are free parameters we can actually fix them so that the signs are correct.

For the more realistic approach to the Standard Model we need to take a slightly complicated model, with the finite algebra being $\mathbb{C} \oplus \mathcal{H} \oplus M_3(\mathbb{C})$. It comes as no surprise that this construction leads to the full gauge group of the Standard Model: $U(1)\times SU(2)\times SU(3)$. Taking an appropriate spectral triple over the finite algebra (which is actually of KO-dimension 6) and tensoring it with the usual spectral triple over a manifold, one obtains as a spectral action:

$$\begin{aligned}
S = {} & \frac{1}{\pi^2}\left(48 f_4\Lambda^4 - f_2\Lambda^2 c + \frac{f_0}{4} d\right)\int \sqrt{g}\,d^4x \\
& + \frac{96 f_2\Lambda^2 f_0 c}{24\pi^2}\int R\sqrt{g}\,d^4x \\
& + \frac{f_0}{10\pi^2}\int\left(\frac{11}{6}R^*R^* - 3C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)\sqrt{g}\,d^4x \\
& + \frac{(-2a\, f_2\Lambda^2 + e\, f_0)}{\pi^2}\int |\varphi|^2\sqrt{g}\,d^4x \\
& + \frac{f_0}{2\pi^2}\int a\left|D_\mu\varphi\right|^2\sqrt{g}\,d^4x \\
& - \frac{f_0}{12\pi^2}\int aR|\varphi|^2\sqrt{g}\,d^4x \\
& + \frac{f_0}{2\pi^2}\int\left(g_3^2 G^i_{\mu\nu}G^{\mu\nu i} + g_2^2 F^\alpha_{\mu\nu}F^{\mu\nu\alpha} + \frac{5}{3}g_1^2 B_{\mu\nu}B^{\mu\nu}\right)\sqrt{g}\,d^4x \\
& + \frac{f_0}{2\pi^2}\int b|\varphi|^4\sqrt{g}\,d^4x,
\end{aligned}$$

where F, B, G are curvatures of the electro-weak and strong gauge fields and ϕ is the SM Higgs doublet (seen as a quaternionic field).

For more details, please consult [6].

6.6 Where and why learn more?

In these three lectures we have tried to give a glimpse of noncommutative geometry – a theory, which, motivated by examples, extends the notion of *geometry* into the algebraic world. What we still need to supply is a word about prospects: first of learning (where to learn more) but also the prospects of the field (why learn it).

6.6.1 The sources

For the more interested reader we recommend further reading. First of all, there is an excellent introduction to almost all of the topics of these lectures: "Elements of Noncommutative Geometry" [2] by José Gracia-Bondia, Joseph Várilly and Hector Figueroa. It offers a comprehensive and detailed course in noncommutative geometry reviewing also the most recent trends and links with physics. There are, of course, the books by Alain Connes: the seminal work "Noncommutative Geometry" [5] and the more recent book with Mathilde Marcolli [6]. Other textbooks which give a review of selected topics (and are written more from the perspective of mathematical physicist) are "An Introduction to Noncommutative Differential Geometry and its Physical Applications", [17], by John Madore and "An introduction to noncommutative spaces and their geometry", [14], by Giovanni Landi.

There are, of course, numerous books on some mathematical aspects of noncommutative geometry, e.g. *K*-theory, for instance. We shall not list here all possible available books and monographs but will give only single examples. First of all, there is an excellent monograph of Jean-Louis Loday [16] on cyclic homology, Hochschild and related subjects. A concise and useful review of the topis is given by Husemoller [12]. Cyclic homology within noncommutative geometry is presented in [8]. A very good and comprehensible introduction to *K*-theory can be found in a friendly approach to *K*-theory by Wegge-Olsen [19] and also in the book by Blackaddar [1]. The overview of the link between cyclic cohomology, *K*-theory and Chern parings is nicely explained in the book of Jacek Brodzki [3]. Almost everything on operator algebras can be found in the excellent monograph by Richard Kadison, John Ringrose [13]. A review of differential graded algebras can be found in the lecture notes by Michel Dubois-Violette [35, 36]. One of the basic classical texts on all aspects of topology, differential geometry, gauge theories and characteristic classes is the textbook "Analysis, Manifolds and Physics", [4], by Choquet-Bruhat and DeWitt-Morette. Everything one wants to know about spin geometry is in the book (surprise, surprise): "Spin Geometry", [15], by Lawson and Michelsohn. All properties of Dirac operators are explained in the work of Thomas Friedrich "Dirac Operator" [10]. The most recent concise introduction to spectral triples (treating the classical and noncommutative examples) is contained in the book by Joseph Varilly, [18], "An Introduction to Noncommutative Geometry".

There are, of course, numerous reviews and lecture notes from courses at institutions and schools (for example: [7, 9, 11]). Written for different purposes and by different authors, they offer views of the topic from many differen angles. A selection can also be found found on the internet, on the web pages of *noncommutative geometers* or common sites like the „Noncommutative blog".

6.6.2 The outlook

It is hard to see at the moment whether Noncommutative Geometry will become the right tool for describing the physics: both known physics and physics yet to be discovered. We have mentioned that NCG finds applications in a range of topics, from the Quantum Hall Effect [20], Standard Model [43, 40] up to string theory [42, 28]. There are other branches of noncommutative geometry that we have not even touched: Hopf algebras, quantum groups and quantum deformations, deformation quantization, noncommutative field theory, Hopf algebras in renormalization – to list only those, which (more or less) have some links to physics.

One may say that we are just at the dawn of noncommutative geometry: it is a world still to be discovered. Whether this geometry will be the *geometry* used to describe the world is not known. But we might soon find out.

Acknowledgement

Partially supported by Polish Government grants 115/E-343/SPB/6.PRUE/DIE 50/2005–2008 and 189/6.PRUE/2007/7

References

[1] Blackadar, B.: *K-theory for Operator Algebras*. Cambridge: Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, 1998.

[2] Bondia, José M., Várilly, J., Figueroa, H.: *Elements of Noncommutative Geometry*. Boston (MA): Birkhauser Advanced Texts, Birkhauser Boston, Inc., 2001.

[3] Brodzki, J.: *An Introduction to K-theory and Cyclic Cohomology. Advanced Topics in Mathematics*. Warsaw: PWN – Polish Scientific Publishers, 1998.

[4] Choquet-Bruhat, Y., C. DeWitt-Morette, C.: *Analysis, Manifolds and Physics*. Amsterdam: North-Holland Publishing Co., 1989.

[5] Connes, A.: *Noncommutative Geometry and Physics*. San Diego: Academic Press, Inc., 1994.

[6] Connes, A., Marcolli, M.: *Noncommutative Geometry, Quantum Fields and Motives*. (Colloquium Publications), Providence: American Mathematical Society, 2008.

[7] Cuntz, J., Khalkhali, M. (eds.): *Cyclic cohomology and noncommutative geometry*. (Proceedings of a workshop, Fields Institute, Waterloo). Providence: American Mathematical Society (AMS), 1997.

[8] Cuntz, J., Skandalis, G., Tsygan, B.: *Cyclic homology in non-commutative geometry*. (Encyclopaedia of Mathematical Sciences 121(II)). Berlin: Springer, 2004.

[9] Doplicher, S., Longo, R. (eds.): *Noncommutative geometry*. (Lectures given at the C. I. M. E. summer school, Martina Franca), Berlin: Springer, 2004.

[10] Friedrich, T.: *Dirac Operators in Riemannian Geometry*. Providence: American Mathematical Society (AMS), 2000.

[11] Higson, N., Nigel, J. Roe (eds.): *Surveys in Noncommutative Geometry*. (Proceedings from the Clay Mathematics Institute instructional symposium). Providence: American Mathematical Society, 2006.

[12] Husemoller, D.: *Lectures on Cyclic Homology*. Berlin: Springer-Verlag, 1991.

[13] Kadison, R., Ringrose, J.: *Fundamentals of the Theory of Operator Algebras*. Advanced Theory, Vol. 1–2 , American Mathematical Society, 1997.

[14] Landi, G.: *An Introduction to Noncommutative Spaces and Their Geometry*. Lecture Notes in Physics. Berlin: Springer-Verlag, 1997.

[15] Lawson, H., Michelsohn, M-L.: *Spin Geometry*. Princeton, NJ: Princeton Mathematical Series, 38. Princeton University Press, 1989.

[16] Loday, J-L.: *Cyclic Homology*. Grundlehren der Mathematischen Wissenschaften, 301. Berlin: Springer-Verlag, 1998.

[17] Madore, J.: *An Introduction to Noncommutative Differential Geometry and its Physical Applications*. Cambridge: Cambridge University Press, 1995, 1999.

[18] Várily, J.: *An Introduction to Noncommutative Geometry*. (EMS Series of Lectures in Mathematics), Zürich: European Mathematical Society Publishing House, 2006.

[19] Wegge-Olsen, N.: *K-theory and C^*-algebras. A friendly Approach*. Oxford Science Publications. Oxford, New York: The Clarendon Press, 1993.

[20] Bellissard, J.: Noncommutative Geometry and Quantum Hall Effect. In *Proceedings of the International Congress of Mathematicians*, Vol. **1**, **2** Zürich: 1994, p. 1238–1246, Basel: Birkhauser, 1995.

[21] Chamseddine, A. H., Connes, A.: An Universal Action Formula, *Phys. Rev. Lett*., Vol. **77** (1996), p. 4868.

[22] Chamseddine, A. H., Connes, A.: The Spectral Action Principle. *Commun. Math. Phys*., Vol. **186** (1997), p. 731.

[23] Connes, A.: Noncommutative Differential Geometry. *Inst. Hautes Études Sci. Publ. Math*., (1985), No. 62, p. 257–360.

[24] Connes, A., Rieffel, M.: Yang-Mills for Noncommutative Two-Tori. *Operator Algebras and Mathematical Physics* (1985), p. 237–266, Contemp. Math., 62.

[25] Connes, A.: The Action Functional In Noncommutative Geometry. *Commun. Math. Phys*., Vol. **117** (1988), p. 673.

[26] Connes, A., Lott, J.: Particle Models and Noncommutative Geometry. *Nucl. Phys. Proc*., Suppl. **18B** (1991), p. 29.

[27] Connes, A.: Noncommutative Geometry and Reality. *J. Math. Phys.*, Vol. **36** (1995), p. 6194.

[28] Connes, A., Douglas, M. R., Schwarz, A.: Noncommutative Geometry and Matrix Theory: Compactification on Tori. JHEP **9802** (1998), 003.

[29] Connes, A.: A Short Survey of Noncommutative Geometry. *J. Math. Phys.*, Vol. **41** (2000), p. 3832.

[30] Connes, A.: Noncommutative Geometry: Year 2000. arXiv:math.qa/0011193.

[31] Connes, A., Chamseddine, A. H., Marcolli, M.: Gravity and the Standard Model with Neutrino Mixing. arXiv:hep-th/0610241

[32] Connes, A., Chamseddine, A. H.: A Dress for SM the Beggar. arXive:math/07063690.

[33] Cuntz, J., Quillen, D.: Algebra Extensions and Nonsingularity. *J. Amer. Math. Soc*., Vol. **8** (1995), No. 2, p. 251–289.

[34] Doplicher, S., Fredenhagen, K., Roberts, J.: The Quantum Structure of Spacetime at the Planck Scale and Quantum Fields. *Comm. Math. Phys*., Vol. **172** (1995), no. 1, p. 187–220.

[35] Dubois-Violette, M.: Lectures on Graded Differential Algebras and Noncommutative Geometry. arXiv:math.qa/9912017.

[36] Dubois-Violette, M.: Lectures on Differentials, Generalized Differentials and on Some Examples Related to Theoretical Physics. arXiv:math.qa/0005256.

[37] Gracia-Bondia, J. M.: Noncommutative Geometry and the Standard Model: An Overview. arXiv:hep-th/9602134.

[38] Kalau, W., Walze, M.: Gravity, Non-Commutative Geometry and the Wodzicki Residue. *J. Geom. Phys*., Vol. **16** (1995), No. 4, p. 327–344.

[39] Kastler, D.: The Dirac Operator and Gravitation. *Commun. Math. Phys.*, Vol. **166** (1995), p. 633.

[40] Kastler, D.: Noncommutative Geometry and Fundamental Physical Interactions: The Lagrangian Level: Historical Sketch and Description of the Present Situation. *J. Math. Phys*., Vol. **41** (2000), p. 3867.

[41] Sakharov, A.: Vacuum Quantum Fluctuations in Curved Space and the Theory of Gravitation. Dokl. Akad. Nauk Ser. Fiz. **177** (1967), p. 70–71.

[42] Seiberg, N., Witten, E.: String Theory and Noncommutative Geometry. *J. High Energy Phys.*, (1999), No. 9, Paper 32.

[43] Varilly, J., Gracia-Bondia, J. M.: Connes' Noncommutative Differential Geometry and the Standard Model. *J. Geom. Phys*., Vol. **12** (1993), p. 223.

[44] Varilly, J., Rennie, A.: Reconstruction of Manifolds in Noncommutative Geometry. arXiv:math/0610418

Andrzej Sitarz
e-mail: sitarz@if.uj.edu.pl

Institute of Physics
JagiellonianUniversity
Reymonta 4
30-059 Kraków, Poland