Acta Polytechnica **62**(1):56–62, 2022
https://doi.org/10.14311/AP.2022.62.0056

TIME-DEPENDENT STEP-LIKE POTENTIAL WITH A FREEZABLE BOUND STATE IN THE CONTINUUM

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Abstract. In this work, we construct a time-dependent step-like potential supporting a normalizable state with energy embedded in the continuum. The potential is allowed to evolve until a stopping time t_i, where it becomes static. The normalizable state also evolves but remains localized at every fixed time up to t_i. After this time, the probability density of this state freezes becoming a Bound state In the Continuum. Closed expressions for the potential, the freezable bound state in the continuum, and scattering states are given.

Keywords: Bound states in the continuum, supersymmetric quantum mechanics, time-dependent quantum systems.

1. Introduction

The first discussion of Bound states In the Continuum (BICs) in quantum mechanics dates back to von Neumann and Wigner [1] who constructed normalizable states corresponding to an energy embedded in the continuum in a periodic potential $V(r) = E + \nabla^2\psi/\psi$ from a modulated free-particle wave function $\psi(r) = (\sin(r)/r)f(r)$, with twice the period of the potential. The localization of this state is interpreted as the result of its reflection in the Bragg mirror generated by the wrinkles of $V(r)$ as $r \to \infty$. The extended family of von-Neumann and Wigner potentials have been discussed and extended for many years [2–5] from different frameworks including the Gelfan-Levitan equation [6] also known as inverse scattering method [4, 7], Darboux transformations [8, 9] and supersymmetry (SUSY) [10–13], among others. Bound states In the Continuum are nowadays recognized as a general wave phenomenon and has been explored theoretically and experimentally in many different setups, see [14] for a recent review.

Exact solutions to the time-dependent Schrödinger equation are known only in a few cases, including the potential wells with moving walls [15, 16], which has been explored from several approaches (see, for instance, Ref. [17] and references therein) including the adiabatic approximation [18] and perturbation theory [16] and through point transformations [19–23], which combined with supersymmetry techniques allow to extend from the infinite potential well with a moving wall to the trigonometric Pöschl-Teller potential [24].

In this article, we present the construction of a time-dependent step-like potential. We depart from the standard stationary step potential and apply a second-order supersymmetric transformation to add a BIC. Then, by means of a point transformation, the potential and the state become dynamic and we allow them to evolve. After a certain time, we assume that all the time-dependence of the potential is frozen, such that the potential becomes stationary again and explore the behavior of the normalizable state. Intriguingly, it is seen that the freezable BIC is not an eigensolution of the stationary Schrödinger equation in the frozen potential, but rather solves an equation that includes a vector potential that does not generate a magnetic field whatsoever. Thus, by an appropriate gauge transformation, we gauge away the vector potential and observe the BIC that remains frozen as an eigenstate of the stationary Hamiltonian after the potential ceases to evolve in time.

In order to expose our results, we have organized the remaining of this article as follows: In Section 2 we describe the preliminaries of SUSY and a point transformation. Section 3 presents the construction of the time-dependent step-like potential and give explicit expressions for the freezable BIC and scattering states. Final remarks are presented in Section 4.

2. Supersymmetry and a point transformation

Point transformation is a successful technique to define a time-dependent Schrödinger equation with a full time-dependent potential from a known stationary problem [19, 20, 24]. In this section, we use a transformation of this kind in combination with a confluent supersymmetry transformation to obtain a time-dependent step-like potential from the stationary case.

2.1. Confluent supersymmetry

Darboux transformation, intertwining technique or supersymmetric quantum mechanics (SUSY) is a method to map solutions ψ of a Schrödinger equation into solutions $\bar{\psi}$ of another Schrödinger equation [25–29]. It is based on an intertwining relation where two Hamiltonians and a proposed operator $L^\dagger$ must fulfill the relation

$$\bar{H}L^\dagger = L^\dagger H, \tag{1}$$

where

$$H = -\frac{d^2}{dy^2} + V_0(y), \quad \bar{H} = -\frac{d^2}{dy^2} + \bar{V}(y). \tag{2}$$

The main ingredient of SUSY are the *seed solutions*, which correspond to solutions of the initial differential equation $Hu = \epsilon u$, where ϵ is a real constant called *factorization energy.* In this work we focus on the so called *confluent supersymmetry*, where $L^\dagger$ is a second-order differential operator. Once a seed solution and a factorization energy are chosen, the next step is to construct the following auxiliary function

$$v = \frac{1}{u}\left(\omega + \int u^2(y)dz\right), \tag{3}$$

where ω is a real constant to be fixed. Then, one way to fulfill (1) is by selecting

$$L^\dagger = \left(-\frac{d}{dy} + \frac{v'}{v}\right)\left(-\frac{d}{dy} + \frac{u'}{u}\right), \tag{4}$$

and the potential term in $\bar{H}$ as

$$\bar{V}(y) = V_0(y) - 2\frac{d^2}{dy^2}\ln\left(\omega + \int_{y_0}^{y} u^2 dz\right). \tag{5}$$

Then, solutions of the differential equation $H\psi = E\psi$, where E is energy, can be mapped using $L^\dagger$ and the intertwining relation as follows:

$$\begin{aligned}
H\psi &= E\psi, \\
&\Downarrow \qquad \text{times } L^\dagger \\
L^\dagger H\psi &= EL^\dagger\psi, \\
&\Downarrow \qquad \text{using (1)} \\
\bar{H}L^\dagger\psi &= EL^\dagger\psi, \\
&\Downarrow \qquad \text{defining } \bar{\psi} \propto L^\dagger\psi \\
\bar{H}\bar{\psi} &= E\bar{\psi}.
\end{aligned}$$

We define $\bar{\psi}$ as

$$\bar{\psi} = \frac{1}{E-\epsilon}L^\dagger\psi. \tag{6}$$

The factor $(E-\epsilon)^{-1}$ is introduced for normalization purposes. Moreover, $\bar{H}$ could have an extra eigenstate that cannot be written in the form (6). This state is called *missing state* and plays an important role in this work. The missing state is obtained as follows: First we have seen that $L^\dagger$ maps solutions of $H\psi = E\psi$ into solutions of $\bar{H}\bar{\psi} = E\bar{\psi}$, by obtaining the adjoint equation of (1) $HL = L\bar{H}$, where $L = (L^\dagger)^\dagger$ we can construct the inverse mapping, but there is a solution $\bar{\psi}_\epsilon$ such that $L\bar{\psi}_\epsilon = 0$. This solution is explicitly:

$$\bar{\psi}_\epsilon = C_\epsilon\frac{1}{v} = C_\epsilon\frac{u}{\omega + \int u^2(y)dy}, \tag{7}$$

where C_ϵ is a normalization constant if $\bar{\psi}_\epsilon$ is square integrable. This state fulfills $\bar{H}\bar{\psi}_\epsilon = \epsilon\bar{\psi}_\epsilon$. Notice that the selection of u, ϵ and ω is very relevant, we must choose these carefully to avoid the introduction of singularities in the potential $\bar{V}$ that lead to singularities also in $\bar{\psi}$. The function $\omega + \int u^2 dy$ must be nodeless. We can satisfy this if either $\lim_{y\to\infty} u(y) = 0$ or $\lim_{y\to-\infty} u(y) = 0$ and if ω is appropriately chosen.

2.2. Point transformation

Given that we know the solution of the time independent Schrödinger equation

$$\frac{d^2}{dy^2}\bar{\psi}(y) + \left[E - \bar{V}(y)\right]\bar{\psi}(y) = 0 \tag{8}$$

with a potential defined in $y \in (-\infty, \infty)$, let us consider the following change of variable

$$y(x,t) = \frac{x}{4t+1}, \tag{9}$$

where $x \in (-\infty, \infty)$ is considered as a spatial variable and $t \in [0, \infty)$ a temporal one. Then, the wavefunction

$$\phi(x,t) = \frac{1}{\sqrt{4t+1}}\exp\left\{\frac{i(x^2 + \frac{E}{4})}{4t+1}\right\}\bar{\psi}\left(\frac{x}{4t+1}\right), \tag{10}$$

solves the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\phi(x,t) + \frac{\partial^2}{\partial x^2}\phi(x,t) - V(x,t)\phi(x,t) = 0, \tag{11}$$

where the potential term is

$$V(x,t) = \frac{1}{(4t+1)^2}\bar{V}\left(\frac{x}{4t+1}\right). \tag{12}$$

In other words, the change of variable (9) together with the replacements $\bar{V} \to V$ and $\bar{\psi} \to \phi$ transform a stationary Schrödinger equation into a time dependent solvable one.

3. Time dependent step-like potential with a freezable bound state in the continuum

In this section, we depart from the well-known step potential $V(y) = \hat{V}\Theta(-y)$ as time independent system. Then, using confluent supersymmetry we will add a single BIC. Furthermore, with the point transformation previously introduced we transform the stationary system into a time-dependent system with an explicitly time-dependent potential. We will choose a stopping time or *freezing time* t_i after which the potential no longer evolves:

$$V_F(x,t) = \begin{cases} V(x,t) & 0 \le t < t_i, \\ V(x,t_i) & t \ge t_i. \end{cases} \tag{13}$$

Finally, the solutions of the Schrödinger equation will be presented.

Let us commence our discussion by considering the Step-Potential

$$V_0(y) = \begin{cases} \hat{V} & y \le 0, \\ 0 & y > 0, \end{cases} \tag{14}$$

defined along the axis $y \in (-\infty, \infty)$ and $\hat{V}$ is a positive constant. The solutions of this system are well known in the literature (see [30, 31]). Restricting ourselves to the case $0 < E_q < \hat{V}$, the solutions are:

$$\psi(y) = \begin{cases} \exp(\rho y) & y \le 0, \\ \cos(qy) + \frac{\kappa}{k}\sin(qy) & y > 0, \end{cases} \tag{15}$$

with energy $E_q = q^2$ and $\rho = \sqrt{\hat{V} - E_q}$.

Next, to perform the confluent supersymmetric transformation we choose a factorization energy such that $0 < \epsilon < \hat{V}$ and the corresponding seed solution $u(y)$ as

$$u(y) = \begin{cases} \exp(\kappa y) & y \le 0, \\ \cos(ky) + \frac{\kappa}{k}\sin(ky) & y > 0, \end{cases} \tag{16}$$

with $k^2 = \epsilon$ and $\kappa^2 = \hat{V} - \epsilon$. Note that $u(y) \to 0$ when $y \to -\infty$. Then, from (5) we obtain explicitly the SUSY partner $\bar{V}$:

$$\bar{V}(y) = \begin{cases} \hat{V} - \frac{16\exp(2\kappa y)\kappa^3\omega}{(\exp(2\kappa y)+2\kappa\omega)^2} & y \le 0 \\ 32k^2\left(k\cos(ky) + \kappa\sin(ky)\frac{\tilde{v}(y)}{\hat{v}(y)}\right) & y > 0, \end{cases} \tag{17}$$

where the functions $\tilde{v}(y)$ and $\hat{v}(y)$ are

$$\begin{aligned} \tilde{v}(y) = & \left[(k^2+\kappa^2)(k^2x+\kappa) + 2k^4\omega\right]\sin(ky) \\ & - k\left[(k^2+\kappa^2)(\kappa y+1) + 2k^2\kappa\omega\right], \\ \hat{v}(y) = & \left[2ky(k^2+\kappa^2) + 4k^3\omega - 2k\kappa\cos(2ky)\right. \\ & \left. +(k^2-\kappa^2)\sin(2ky)\right]^2. \end{aligned}$$

We can calculate directly from (7) the missing state associated to the factorization energy ϵ:

$$\bar{\psi}_\epsilon(y) = C_\epsilon \begin{cases} \frac{2\kappa\exp(\kappa y)}{2\kappa\omega+\exp(2\kappa y)} & y \le 0, \\ \frac{4k^3(\cos(ky)+\frac{\kappa}{k}\sin(ky))}{\hat{\psi}_\epsilon(y)} & y > 0, \end{cases} \tag{18}$$

where

$$\begin{aligned} \hat{\psi}_\epsilon(y) = & (k^2-\kappa^2)\sin(2ky) - 2\kappa k\cos(2ky) \\ & + 4\omega k^3 + 2ky\left(\kappa^2+k^2\right). \end{aligned}$$

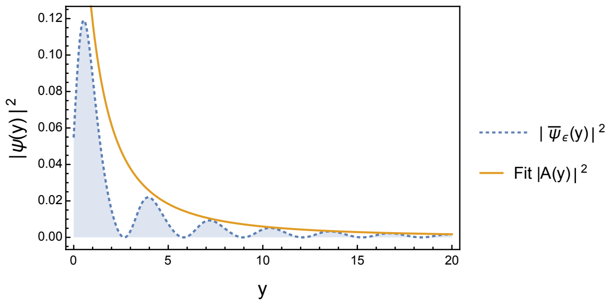

Figure 1. $|\bar{\psi}_\epsilon(y)|^2$ and an envelop function of the form $A(y) = \frac{a}{b+y}$, with $a = 2k(\kappa^2+k^2)^{-1/2}$, $b = 2\omega k^2(\kappa^2+k^2)^{-1}$. The scale of the graph is fixed with $\hat{V} = 5$, $k = 1$, $\kappa = 2$ and $C_\epsilon = 1$, in the appropriate units.

In order to confirm that $\bar{\psi}_\epsilon$ is square integrable, we proceed in the following way. First, we separate the integral $||\bar{\psi}_\epsilon||^2 = \int_{-\infty}^{\infty}|\bar{\psi}_\epsilon|^2 dy = \int_{-\infty}^{0}|\bar{\psi}_\epsilon|^2 dy + \int_0^{\infty}|\bar{\psi}_\epsilon|^2 dy$. The first integral can be directly calculated:

$$\int_{-\infty}^{0}|\bar{\psi}_\epsilon|^2 dy = |C_\epsilon|^2\sqrt{\frac{2}{\kappa\omega}}\tan^{-1}\left(\frac{1}{\sqrt{2\kappa\omega}}\right).$$

For the second integral, we can show that it is bounded by a square integrable function:

$$\begin{aligned} \frac{\int_0^\infty|\bar{\psi}_\epsilon|^2 dy}{|C_\epsilon|^2} &= \int_0^\infty\left|\frac{4k^3(\cos(ky)+\frac{\kappa}{k}\sin(ky))}{\hat{\psi}_\epsilon(y)}\right|^2 dy \\ &\le \int_0^\infty\left|\frac{4k^2\sqrt{k^2+\kappa^2}}{4\omega k^3 + 2ky\left(\kappa^2+k^2\right)}\right|^2 dy \\ &= \int_0^\infty\left|\frac{a}{b+y}\right|^2 dy = \frac{a^2}{b}, \end{aligned} \tag{19}$$

where $a = \frac{2k}{\sqrt{\kappa^2+k^2}}$, $b = \frac{2\omega k^2}{\kappa^2+k^2}$. Figure 1 shows a fair fit to the squared modulus of eq. (18) for $y > 0$.

For an energy $E = q^2 \neq \epsilon$, the wavefunction solving $\bar{H}\bar{\psi} = E\bar{\psi}$ is constructed using (6), and (15). It reads

$$\bar{\psi}(y) = \begin{cases} \left[\frac{(\kappa-\rho)\exp(\rho y)}{(q^2-k^2)}\right]\bar{\psi}_-(y) & y \le 0, \\ \frac{\bar{\psi}_+(y) - q^2\cos(qy) - q\rho\sin(qy)}{q^2-k^2} & y > 0, \end{cases} \tag{20}$$

where we abbreviated

$$\begin{aligned} \bar{\psi}_-(y) &= \frac{2\kappa\omega_0(\kappa+\rho) + (\rho-\kappa)\exp(2\kappa y)}{2\kappa\omega + \exp(2\kappa y)}, \\ \bar{\psi}_+(y) &= \frac{k^2(\rho\sin(qy) + q\cos(qy))}{q} \\ &+ \frac{4k(\kappa\sin(ky) + k\cos(ky))}{\hat{\psi}_\epsilon(y)} \\ &\times\left[\frac{k}{q}\left(\kappa\cos(ky) - k\sin(ky)\right)\left(\rho\sin(qy) + q\cos(qy)\right)\right. \\ &\left.+\left(\kappa\sin(ky) + k\cos(ky)\right)\left(q\sin(qy) - \rho\cos(qy)\right)\right]. \end{aligned}$$

In Figure 2 the potential $\bar{V}(y)$, along with the probability densities of the missing state $|\bar{\psi}_\epsilon(y)|^2$ and a scattering state $|\bar{\psi}(y)|^2$ are shown. We observe that the wavefunction of the BIC has an envelop function which tends to zero as $|y| \to \infty$, whereas the state $\bar{\psi}(y)$ is not localized.

The next step is to construct a time dependent potential from (17) using the point transformation presented in (9-12). Notice that $x = y$ at $t = 0$. Then $\bar{V}$ transforms as the piecewise potential:

$$V(x,t) = \frac{1}{(4t+1)^2}\left\{\hat{V} - \frac{16\kappa^3\omega\exp(\frac{2\kappa x}{4t+1})}{\left[2\kappa\omega + \exp(\frac{2\kappa x}{4t+1})\right]^2}\right\} \quad (21)$$

if $x \leq 0$, otherwise

$$V(x,t) = \frac{32k^2}{(4t+1)^2} \times \left[k\cos\left(\frac{kx}{4t+1}\right) + \kappa\sin\left(\frac{kx}{4t+1}\right)\frac{\tilde{v}(y(x,t))}{\hat{v}(y(x,t))}\right]. \quad (22)$$

In Figure 3 (top) we show the potential $V(x,t)$ at $t = 0$, $t = 0.1$ and $t = 0.2$. Its shape changes in time and its spatial profile oscillates as expected, vanishing as $x \to \infty$. Analogously, for the time-dependent BIC, the associated wavefunction for energy ϵ is explicitly

$$\phi_\epsilon(x,t) = \frac{1}{\sqrt{4t+1}}\exp\left\{\frac{i(x^2 + \frac{k^2}{4})}{4t+1}\right\}\bar{\psi}_\epsilon\left(\frac{x}{4t+1}\right), \quad (23)$$

This function solves the time-dependent Schrödinger equation $i\partial_t\phi_\epsilon + \partial_{xx}\phi_\epsilon - V\phi_\epsilon = 0$ and its square integrability is guaranteed since $\bar{\psi}_\epsilon(y)$ is a square integrable function:

$$\begin{aligned} ||\phi_\epsilon||^2 &= \int_{-\infty}^{\infty}|\phi_\epsilon(x,t)|^2 dx \\ &= \frac{1}{4t+1}\int_{-\infty}^{\infty}\left|\bar{\psi}_\epsilon\left(\frac{x}{4t+1}\right)\right|^2 dx \\ &= \int_{-\infty}^{\infty}|\bar{\psi}_\epsilon(y)|^2 dy = ||\bar{\psi}_\epsilon||^2. \end{aligned} \quad (24)$$

where we used the change of variable (9). Its probability density is shown in Figure 3 (center) at different times. This state is localized and the first peak in the probability density broadens and diminishes height as time increases.

For states with energy $E_q = q^2 \neq \epsilon$, the corresponding time-dependent wavefunction has the explicit form

$$\phi(x,t) = \frac{1}{\sqrt{4t+1}}\exp\left\{\frac{i(x^2 + \frac{q^2}{4})}{4t+1}\right\}\bar{\psi}\left(\frac{x}{4t+1}\right), \quad (25)$$

The behavior of the probability density $|\phi(x,t)|^2$, for $E = 2$ at different times is shown in Figure 3 (bottom). This state is unlocalized at any time.

Finally, we choose the freezing or stopping time t_i. Then, we can consider a charge particle in a potential:

$$V_F(x,t) = \begin{cases} V(x,t) & 0 \leq t < t_i, \\ V(x,t_i) & t \geq t_i. \end{cases} \quad (26)$$

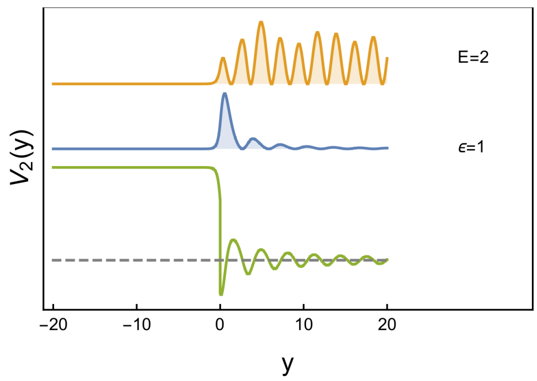

FIGURE 2. Potential $\bar{V}(y)$, along with the probability densities of the missing state $|\bar{\psi}_\epsilon(y)|^2$ and a scattering state $|\bar{\psi}(y)|^2$ are shown. The scale of the graph is fixed with $\hat{V} = 5$, $k = 1$, $\kappa = 2$, $q = \sqrt{2}$ and $\omega = 4$.

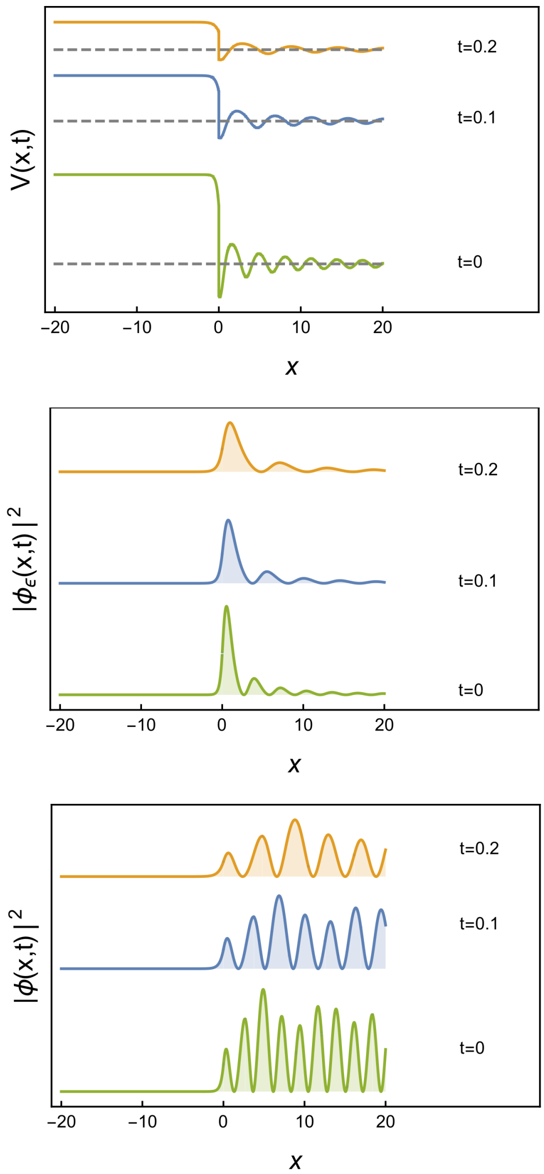

FIGURE 3. Behavior of the potential $V(x,t)$ (top), the BIC $\phi_\epsilon(x,t)$ (center) and the scattering state $\phi(x,t)$ (bottom) at the times $t = 0$, $t = 0.1$, and $t = 0.2$. The scale of the graphs is fixed by $\hat{V} = 5$, $k = 1$, $\kappa = 2$, $q = \sqrt{2}$ and $\omega = 4$.

where $V(x,t)$ is given by (21,22). Notice that when $t \in [0, t_i)$ the potential is changing in time, and when $t \geq t_i$ the potential is frozen. This potential is in fact a family, parametrized by $\omega > 0$, recall that ω was introduced by the confluent SUSY transformation.

Neither $\phi(x,t)$ nor $\phi_\epsilon(x,t)$ are stationary states, they evolve in time, and they are not eigenfunctions of the operator $-\partial_{xx} + V$. At any time $t \geq t_i$, the functions $\phi(x,t_i)$ and $\phi_\epsilon(x,t_i)$ satisfy the eigenvalue equation:

$$\left[\left(-\frac{\partial}{\partial x} + iA_x(x)\right)^2 + V(x,t_i)\right]\phi(x,t_i) = \frac{E}{(4t_i+1)^2}\phi(x,t_i), \quad t \geq t_i, \tag{27}$$

where $A_x(x) = -\partial_x\theta(x)$ and

$$\theta(x) = \frac{i}{4t_i+1}\left(x^2 + \frac{E}{4}\right). \tag{28}$$

Equation (27) is the Schrödinger equation for a charged particle under the influence of a vector potential $\mathbf{A} = (A_x, 0, 0)$ that, nevertheless, does not generate magnetic field since $\mathbf{B} = \nabla \times \mathbf{A} = 0$. Let us recall that the Schrödinger equation for a charged particle of charge q immersed in an external electromagnetic field is better written in terms of the scalar φ and vector potentials $\mathbf{A}$ through the Hamiltonian

$$H = (\hat{\mathbf{p}} + q\mathbf{A})^2 + q\varphi. \tag{29}$$

These electromagnetic potentials allow us to define the electric and magnetic fields as

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{30}$$

definition that does not change if the following transformations are performed simultaneously,

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\lambda, \qquad \varphi \to \varphi' = \varphi - \frac{\partial\lambda}{\partial t}, \tag{31}$$

where $\lambda = \lambda(x,t)$ is a scalar function. This is a statement of gauge invariance of Maxwell's equations. In quantum mechanics, the time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi \tag{32}$$

retains this feature if along the transformations in Eq. (31) in the Hamiltonian (29), the wavefunction changes according to the local phase transformation

$$\psi \to \psi' = e^{i\lambda}\psi. \tag{33}$$

In our example at hand, this freedom allows us to select λ in such a way that if at certain instant of time t_i the vector potential $\mathbf{A} \neq 0$ but before we had $\mathbf{A} = 0$, one can still have a Schrödinger equation without vector potential by tuning appropriately the scalar potential. In particular, by selecting

$$\lambda(x,t) = \ell(x)\Theta(t - t_i), \tag{34}$$

we can shift the scalar potential such that the time-dependent equation governing this state never develops a vector potential to begin with. Then, by choosing a vector potential $\mathbf{A}(x,t) = (A_x(x,t), 0, 0)$ where $A_x(x,t) = -\Theta(t-t_i)\partial_x\theta(x)$, we observe that the piecewise function

$$\phi_F(x,t) = \begin{cases} \phi(x,t) & 0 \leq t < t_i, \\ \bar{\psi}\left(\frac{x}{4t_i+1}\right) & t \geq t_i. \end{cases} \tag{35}$$

becomes a solution of

$$i\partial_t\phi_F(x,t) = \left[-\partial_{xx} + V_F(x,t)\right]\phi_F(x,t) = H\phi_F(x,t).$$

In particular, the function

$$\phi_{F\epsilon}(x,t) = \begin{cases} \phi_\epsilon(x,t) & 0 \leq t < t_i, \\ \bar{\psi}_\epsilon\left(\frac{x}{4t_i+1}\right) & t \geq t_i, \end{cases} \tag{36}$$

before the freezing time t_i is just a time dependent wave packet but for $t > t_i$ it becomes a Frozen Bound state In the Continuum satisfying the eigenvalue equation $H\phi_{F\epsilon} = \varepsilon\phi_{F\epsilon}$, where $\varepsilon = \epsilon/(4t_i+1)^2$. In Figure 4 we plot the potential V_F (top), the Freezable Bound State in the Continuum $\phi_{F\epsilon}$ (center) and a scattering state ϕ_F (bottom) at $t = 0.8$, $t = 1$ and $t = 1.8$, the freezing time is $t_i = 1$, note that after $t = 1$ neither the potential nor the wavefunctions evolve.

4. Final remarks

In this article, we apply a confluent supersymmetric transformation to the standard Step-Potential defined in the whole real axis. The seed solution that we use makes it possible to embed a localized squared integrable state in the continuum spectrum, a BIC. We have provided the system, potential, and states, with time evolution through a point transformation. Nevertheless, we notice that the wrinkles in the potential as $x \to \infty$ still localize a BIC at every fixed time.

Next, we allow the evolution of the system continue and at a given stopping time t_i, we freeze the potential and fix it stationary. Upon exploring the behavior of the BIC with this static potential after the freeze-out time, we surprisingly observe that it does not correspond to a solution of the stationary Schrödinger equation, but instead it develops a geometric phase encoded in a vector potential which does not generate any magnetic field. Thus, by gauging out this geometric phase, the resulting state becomes indeed an eigenstate of the frozen Hamiltonian. We call this state a Freezable Bound state In the Continuum.

Further examples are being examined under the strategy presented in this work, including vector potentials which might be relevant for pseudo-relativistic systems.

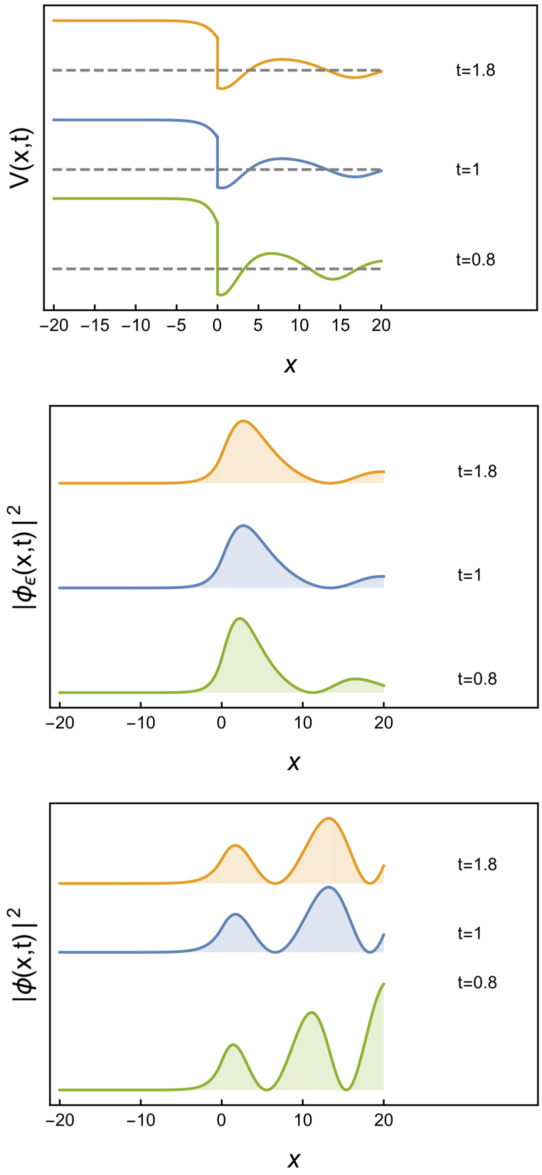

FIGURE 4. Behavior of the potential $V_F(x,t)$ (top), the FBIC $\phi_{F\epsilon}(x,t)$ (center) and the scattering state $\phi_F(x,t)$ (bottom) at the times $t = 0.8$, $t = 1$, and $t = 1.8$. The freezing time is $t_i = 1$. The scale of the graph is fixed by $\hat{V} = 5$, $k = 1$, $\kappa = 2$, $q = \sqrt{2}$ and $\omega = 4$.

Acknowledgements

The authors acknowledge Consejo Nacional de Ciencia y Tecnología (CONACyT-México) under grant FORDECYT-PRONACES/61533/2020. IG and AR acknowledge valuable discussions with Juan Angel Casimiro Olivares.

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