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LINEARISED COHERENT STATES FOR NON-RATIONAL SUSY EXTENSIONS OF THE HARMONIC OSCILLATOR

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Abstract. In this work, we derive two equivalent non-rational extensions of the quantum harmonic oscillator using two different supersymmetric transformations. For these extensions, we built ladder operators as the product of the intertwining operators related with these equivalent supersymmetric transformations, which results in two-step ladder operators. We linearised these operators to obtain operators of the same nature that follow a linear commutation relation. After the linearisation, we derive coherent states as eigenstates of the annigilation operator and analyse some relevant mathematical and physical properties, such as the completeness relation, mean-energy values, temporal stability, time evolution of the probability densities, and Wigner distributions. From these properties, we conclude that these coherent states present both classical and quantum behaviour.

Keywords: Supersymmetric quantum mechanics, non-rational extensions, linearised ladder operators, coherent states.

1. Introduction

In quantum physics, supersymmetric quantum mechanics (SUSY) is considered the most efficient technique to generate new quantum potentials from an initial solvable one (see [1–5] for reviews on the topic). This method allows modifying the energy spectrum of an initial Hamiltonian to obtain new Hamiltonians with known eigenstates and eigenvalues. These potentials obtained with SUSY are known as extensions or SUSY partners of the considered initial potential. Moreover, when two different SUSY transformations lead to the same potential (up to an additive constant), it can be said that the extensions are equivalent [6, 7].

Equivalent rational extensions of the quantum harmonic oscillator are very attractive in mathematical physics since its eigenstates are written in terms of exceptional orthogonal polynomials and the results are useful for studying superintegrable systems or generating solutions to the Painlevé equations [8–10]. In a recent work of the authors [11], it was shown that the equivalence between SUSY transformations goes beyond rational extensions and can be extended to non-rational extensions of the harmonic oscillator, i.e. extensions whose potentials cannot be written as the quotient of two polynomials, by considering not only polynomial solutions but also general solutions of the Schrödinger equation.

However, since the birth of quantum theory, it has been relevant to study the quantum states at the border between classical and quantum regimes. In this sense, it is well-known that Schrödinger, in 1926 [12], derived quantum states of the harmonic oscillator that resemble classical behaviour on the phase-space as the classical oscillator does. Later on, in 1962, Glauber rediscovered these states, known as coherent states, and found that they provided the quantum description of coherent light [13]. Since then, there has been a continuous research activity in quantum physics looking for quantum states with a behaviour at the border between classical and quantum regimes by examining semi-classical phase-space properties, in particular, by systems generated by SUSY [4, 14–20].

The coherent states of the harmonic oscillator are Gaussian states, labeled by a complex number z, that minimize the Heisenberg uncertainty relation. They can be constructed either as displaced versions of the ground state or as eigenvectors of the annihilation operator. Moreover, they form an overcomplete set in the sense that

$$\frac{1}{\pi}\int_{\mathbb{C}} |z\rangle \langle z| \, d^2 z = \mathbb{1}. \tag{1}$$

These four properties are commonly used as definitions of coherent states when we have a potential different from the harmonic oscillator, see for example [21–25]. Each definition gives, in general, different sets of coherent states. In this work, we obtain coherent states of non-rational extensions of the harmonic oscillator as eigenvectors of the annihilation operator.

For this purpose, we need to find ladder operators of the system.

The outline of the work is the following: In the next section, we present a short summary of SUSY. In Section 3, we generate two equivalent non-rational extensions of the harmonic oscillator. Then, we construct ladder operators as the product of the intertwining operators of the SUSY transformations. In the Section 4, we linearise the ladder operators to obtain a linear commutation relationship, then, we derive coherent states as eigenstates of the annihilation operator and study some of their properties. Our conclusions are presented in the last section.

2. Supersymmetric quantum mechanics

With this technique, we start with two Hamiltonians

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+V(x),\quad \widetilde{H}=-\frac{1}{2}\frac{d^2}{dx^2}+\widetilde{V}(x), \quad (2)$$

where H is the initial Hamiltonian with known eigenfunctions $\psi_n(x)$ and eigenvalues E_n, $n=0,1,2,\ldots$, whereas $\widetilde{H}$ is the Hamiltonian under construction. The potential $\widetilde{V}$ is known as the *extension* or *supersymmetric partner* of V. Now, we propose the existence of k-th order differential operators B, B^+ that intertwine H and $\widetilde{H}$ as

$$\widetilde{H}B^+=B^+H,\quad B\widetilde{H}=HB. \quad (3)$$

By properly choosing k general solutions u_j $(j=1,2,\ldots,k)$ of the stationary Schrödinger equation $Hu_j=\epsilon_j u_j$, with corresponding energies ϵ_j, the SUSY partner potential $\widetilde{V}(x)$ reads

$$\widetilde{V}(x)=V(x)-[\ln W(u_1,u_2,\ldots,u_k)]'', \quad (4)$$

where $W(f_1,f_2,\ldots,f_k)$ denotes the Wronskian of the functions in its argument. The functions u_j are usually referred to as *seed solutions* and the constant ϵ_j as *factorization energies*. Be aware that to have a regular potential, we must choose the seed solutions in such a way the Wronskian has no zeroes.

If $B^+\psi_n\neq 0$, the eigenfunctions $\widetilde{\psi}_n$, $n=0,1,\ldots$, of $\widetilde{H}$ can be computed with the relation

$$\widetilde{\psi}_n(x)=\frac{B^+\psi_n(x)}{\sqrt{(E_n-\epsilon_1)\ldots(E_n-\epsilon_k)}} =\frac{1}{\sqrt{(E_n-\epsilon_1)\ldots(E_n-\epsilon_k)}}\frac{W(u_1,u_2,\ldots,u_k,\psi_n)}{W(u_1,u_2,\ldots,u_k)}. \quad (5)$$

The constructed Hamiltonian $\widetilde{H}$ may contain additional eigenfunctions $\widetilde{\psi}_{\epsilon_i}$, known as *missing states*, for some of the factorization energies ϵ_i, given by

$$\widetilde{\psi}_{\epsilon_i}\propto\frac{W(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_k)}{W(u_1,\ldots,u_k)}. \quad (6)$$

If $\widetilde{\psi}_{\epsilon_j}$ fullfills the boundary conditions of the quantum problem, then ϵ_j must be included in the spectrum of $\widetilde{H}$.

In particular, for second-order supersymmetric quantum mechanics, the intertwining operators have the explicit form [26]

$$B=\frac{1}{2}\left[\frac{d^2}{dx^2}+g(x)\frac{d}{dx}+g'(x)+h(x)\right], \quad (7)$$

$$B^+=\frac{1}{2}\left[\frac{d^2}{dx^2}-g(x)\frac{d}{dx}+h(x)\right]. \quad (8)$$

where the functions $g(x),h(x)$ are found in terms of the only two seed solutions u_1,u_2 with the corresponding factorization energies ϵ_1,ϵ_2, as

$$g=\frac{W'(u_1,u_2)}{W(u_1,u_2)},\quad h=\frac{g'}{2}+\frac{g^2}{2}-2V+\frac{\epsilon_1+\epsilon_2}{2}. \quad (9)$$

Finally, the intertwining operators B and B^+ fulfill the following factorization relations:

$$B^+B=(\widetilde{H}-\epsilon_1)\ldots(\widetilde{H}-\epsilon_k), \quad (10)$$

$$BB^+=(H-\epsilon_1)\ldots(H-\epsilon_k), \quad (11)$$

i.e., the product of B^+ and B are polynomials of the Hamiltonians H and $\widetilde{H}$.

3. Non-rational extensions of the quantum harmonic oscillator and their ladder operators

Let us consider the harmonic oscillator potential $V=\frac{1}{2}x^2$ and the Hamiltonian H as

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2, \quad (12)$$

whose eigenfunctions and eigenvalues are

$$\psi_n(x)=\sqrt{\frac{1}{2^n\sqrt{\pi}n!}}e^{-\frac{x^2}{2}}H_n(x),\quad E_n=n+\frac{1}{2},$$

where $n=0,\ 1,\ 2,\ldots$ and $H_n(x)$ are Hermite polynomials [27].

When eigenfunctions of a Hamiltonian are employed as seed functions to generate its SUSY partner, the results are rational extensions and the transformation is called Krein-Adler transformation [6, 7, 28].

Moreover, rational extensions can also be built by employing the polynomial non-normalizable solutions of the Schrödinger equation

$$\varphi_m(x)=e^{\frac{x^2}{2}}\mathcal{H}_m(x),\quad E_{-m-1}=-\left(m+\frac{1}{2}\right),$$

where $m=0,1,2,\ldots$, and $\mathcal{H}_m(x)=(-i)^m H_m(ix)$ are the modified Hermite polynomials [29], which are free of nodes for even m and possess a single node at $x=0$ for m odd. In the case of m even, the reciprocal of these solutions are square-integrable functions [6].

We can generate non-rational extensions of the harmonic oscillator potential using non-polynomial solutions of the Schrödinger equation as seed functions in a SUSY transformation. Let us write down the general solution of the stationary Schrödinger equation, with an arbitrary factorization energy denoted by $\mathcal{E} = \lambda + 1/2$, as

$$u(x) = e^{-\frac{x^2}{2}}[H_\lambda(x) + \gamma H_\lambda(-x)], \tag{13}$$

where

$$H_\lambda(x) \equiv \frac{2^\lambda \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\lambda}{2}\right)} {}_1F_1\left(-\frac{\lambda}{2}; \frac{1}{2}; x^2\right) + \frac{2^\lambda \Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} x\, {}_1F_1\left(\frac{1-\lambda}{2}; \frac{3}{2}; x^2\right), \tag{14}$$

are defined as Hermite functions [30, 31],

$${}_1F_1(a;b;z) \equiv \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{z^n}{n!}, \tag{15}$$

is the confluent hypergeometric function, and γ is a real parameter. If $\gamma > 0$, the solution will have an even number of zeroes and for $\gamma < 0$, an odd number of nodes.

3.1. First SUSY transformation

As the first non-rational extension of the harmonic oscillator, we perform a second-order SUSY transformation where we add two new levels with factorization energies $-3/2 < \mathcal{E}_1 < 1/2$ and $\mathcal{E}_2 = E_{-2} = -3/2$, both below the ground state energy. We start by choosing the seed solutions as

$$u_1^{(1)}(x) = e^{-\frac{x^2}{2}}[H_{\lambda_1}(x) + \gamma H_{\lambda_1}(-x)],$$
$$u_2^{(1)}(x) = \varphi_1(x), \tag{16}$$

where $\lambda_1 = \mathcal{E}_1 - 1/2$. To obtain a nodeless Wronskian $W(u_1^{(1)}, u_2^{(1)})$, we take $\gamma > 0$. Notice that $\mathcal{E}_1$ is an arbitrary energy between $E_0 = 1/2$ and $E_{-2} = -3/2$. By following the relation (8), we can define a set of second-order intertwining operators $B^{(1)}, B^{(1)+}$ which satisfy the relations

$$\widetilde{H}^{(1)} B^{(1)+} = B^{(1)+} H, \tag{17}$$

and its adjoint. The SUSY partner potential is

$$\widetilde{V}^{(1)} = \frac{1}{2}x^2 - \left[\ln W(u_1^{(1)}, u_2^{(1)})\right]''. \tag{18}$$

Since $u_1^{(1)}$ is an infinite series, the potential $\widetilde{V}^{(1)}$ is a non-rational extension of V. To find the eigenfunctions of the Hamiltonian $\widetilde{H}^{(1)}$, we use the operator $B^{(1)+}$ as

$$\widetilde{\psi}_n^{(1)} = \frac{B^{(1)+}\psi_n}{\sqrt{(E_n - \mathcal{E}_1)(E_n - \mathcal{E}_2)}}, \quad n = 0, 2, 3, \dots \tag{19}$$

Regarding both missing states of this extension

$$\widetilde{\psi}_{\mathcal{E}_1}^{(1)} \propto \frac{u_2^{(1)}}{W(u_1^{(1)}, u_2^{(1)})}, \quad \widetilde{\psi}_{\mathcal{E}_2}^{(1)} \propto \frac{u_1^{(1)}}{W(u_1^{(1)}, u_2^{(1)})}, \tag{20}$$

due to a stronger divergent behaviour of the Wronskian when $|x| \to \infty$ than the solutions $u_1^{(1)}$,$u_2^{(1)}$, the Hamiltonian $\widetilde{H}^{(1)}$ contains two new bounded states $\widetilde{\psi}_{\mathcal{E}_1}^{(1)}$, and $\widetilde{\psi}_{\mathcal{E}_2}^{(1)}$, so its spectrum is $\mathrm{Sp}\{\widetilde{H}^{(1)}\} = \{E_{-2}, \mathcal{E}_1, E_n,\ n = 0,\ 1,\ 2, \dots\}$.

3.2. Second but equivalent SUSY transformation

We can obtain the same Hamiltonian $\widetilde{H}^{(1)}$, up to an additive constant, with a different second-order SUSY transformation. Let us choose the following seed solutions:

$$u_1^{(2)}(x) = \psi_1(x),$$
$$u_2^{(2)}(x) = e^{-\frac{x^2}{2}}[H_{\lambda_2}(x) + \gamma H_{\lambda_2}(-x)], \tag{21}$$

with the factorization energies $\mathcal{E}_3 = E_1$, and $\mathcal{E}_4 = \mathcal{E}_1 + 2$, respectively. Note that $\lambda_2 = \lambda_1 + 2$. Again, through the relations (7) and (8), we can define second-order differential operators $B^{(2)},\ B^{(2)+}$, which intertwine a Hamiltonian $\widetilde{H}^{(2)}$ with H as

$$\widetilde{H}^{(2)} B^{(2)+} = B^{(2)+} H. \tag{22}$$

The supersymmetric partner potential is

$$\widetilde{V}^{(2)} = \frac{1}{2}x^2 - \left[\ln W(u_1^{(2)}, u_2^{(2)})\right]''. \tag{23}$$

Since $u_2^{(2)}$ is an infinite series, $\widetilde{V}^{(2)}$ is a non-rational extension of V. The eigenfunctions of its Hamiltonian are

$$\widetilde{\psi}_n^{(2)} = \frac{B^{(2)+}\psi_n}{\sqrt{(E_n - \mathcal{E}_3)(E_n - \mathcal{E}_4)}}, \quad n = 0, 2, 3, \dots, \tag{24}$$

and the missing states

$$\widetilde{\psi}_{\mathcal{E}_3}^{(2)} \propto \frac{u_2^{(2)}}{W(u_1^{(2)}, u_2^{(2)})}, \quad \widetilde{\psi}_{\mathcal{E}_4}^{(2)} \propto \frac{u_1^{(2)}}{W(u_1^{(2)}, u_2^{(2)})}. \tag{25}$$

In this case, owing to the divergent asymptotic behaviour of the solution $u_2^{(2)}$ when $|x| \to \infty$, the missing state $\widetilde{\psi}_{\mathcal{E}_3}^{(2)}$ is not normalizable, and since $u_1^{(2)}$ converges, the state $\widetilde{\psi}_{\mathcal{E}_4}^{(2)}$ is square-integrable. Therefore, the energy spectrum of $\widetilde{H}^{(2)}$ is $\mathrm{Sp}(\widetilde{H}^{(2)}) = \{E_0, \mathcal{E}_3, E_2, \dots\}$.

It is important to notice that the seed functions $u_1^{(1)},\ u_2^{(1)}$ used to construct $\widetilde{H}^{(1)}$ are related to the seed solutions $u_1^{(2)},\ u_2^{(2)}$ involved in $\widetilde{H}^{(2)}$. The functions $u_1^{(1)}$ and $u_2^{(2)}$ satisfy $u_2^{(1)} = \sqrt{2\sqrt{\pi}} e^{x^2} u_1^{(2)}$, and $a^- a^- u_2^{(2)} = 2\lambda(\lambda - 1) u_1^{(1)}$, where a^- is the annihilation operator of the harmonic oscillator. Then, by a direct substitution, it can be shown that

$$\widetilde{H}^{(2)} = \widetilde{H}^{(1)} + 2.$$

Thus, $\widetilde{V}^{(1)}$ and $\widetilde{V}^{(2)}$ are equivalent non-rational extensions of the harmonic oscillator. Notice that due to this equivalence, the eigenfunctions obtained by both transformations are the same but with eigenvalues displaced. In the first extension, the ground state is the missing state $\widetilde{\psi}^{(1)}_{\mathcal{E}_2}$, which is also obtained by $B^{(2)+}\psi_0$. Moreover, the missing state $\widetilde{\psi}^{(1)}_{\mathcal{E}_1}$ corresponds to the missing state $\widetilde{\psi}^{(1)}_{\mathcal{E}_4}$. Finally, relations (19) and (24) are also equivalent as $\widetilde{\psi}^{(2)}_n \propto \widetilde{\psi}^{(1)}_{n-2}$, where $n = 2, 3, 4, \ldots$

3.3. Ladder operators

Since both Hamiltonians $\widetilde{H}^{(1)}$ and $\widetilde{H}^{(2)}$ are equivalent, we can simplify the notation by defining $\widetilde{H}^{(2)}$ as $\widetilde{H}$, its eigenfunctions simply by $\widetilde{\psi}$, the potential $\widetilde{V}^{(2)}$ as $\widetilde{V}$, and $\mathcal{E}_3$ as ϵ. Be aware that $1/2 < \epsilon < 5/2$.

Now, we can define the ladder operators for the SUSY extension $\widetilde{H}$ as the product of the intertwining operators related to the equivalent SUSY transformations as in [32], i.e.

$$\mathcal{L}^+ = B^{(1)+}B^{(2)}, \quad \mathcal{L}^- = B^{(2)+}B^{(1)}. \tag{26}$$

They satisfy the following commutation algebra

$$[\widetilde{H}, \mathcal{L}^{\pm}] = \pm 2\mathcal{L}^{\pm}, \tag{27}$$

and

$$\begin{aligned}[\mathcal{L}^-, \mathcal{L}^+] = {} & (\widetilde{H} + 2 - \mathcal{E}_1)(\widetilde{H} + 2 - \mathcal{E}_2) \\ & (\widetilde{H} + 2 - \mathcal{E}_3)(\widetilde{H} + 2 - \mathcal{E}_4) \\ & - (\widetilde{H} - \mathcal{E}_1)(\widetilde{H} - \mathcal{E}_2)(\widetilde{H} - \mathcal{E}_3)(\widetilde{H} - \mathcal{E}_4). \end{aligned} \tag{28}$$

From the relation (27) and the diagram in Figure 1, we can observe how these operators are two-step ladder operators. Furthermore, the commutation relation (28) indicates that these operators, together with $\widetilde{H}$, realize a polynomial Heisenberg algebra of third-order [33], with a generalized number operator:

$$\begin{aligned} N_4(\widetilde{H}) &= \mathcal{L}^+\mathcal{L}^- \\ &= (\widetilde{H} - \mathcal{E}_1)(\widetilde{H} - \mathcal{E}_2)(\widetilde{H} - \mathcal{E}_3)(\widetilde{H} - \mathcal{E}_4). \end{aligned} \tag{29}$$

The kernel of the annihilation operator $\mathcal{L}^-$ is composed by the functions

$$K_{\mathcal{L}^-} = \{\widetilde{\psi}_{E_0}, \widetilde{\psi}_{\epsilon}, \widetilde{\psi}_{E_3}, B^{(1)+}u_2^{(2)}\}. \tag{30}$$

The first three elements of the kernel are eigenfunctions of $\widetilde{H}$ and the last one is a non-normalizable solution of the corresponding Schrödinger equation. By applying iteratively the operator $\mathcal{L}^+$ onto these three eigenfunctions, we can construct a basis of three subspaces of the Hilbert space, the direct sum of the three Hilbert-subspaces compose the whole Hilbert space (see Figure 2). Notice that $\widetilde{\psi}_{\epsilon}$ is annihilated by $\mathcal{L}^+$, then the corresponding subspace will be one-dimensional whereas the other two are infinite-dimensional subspaces.

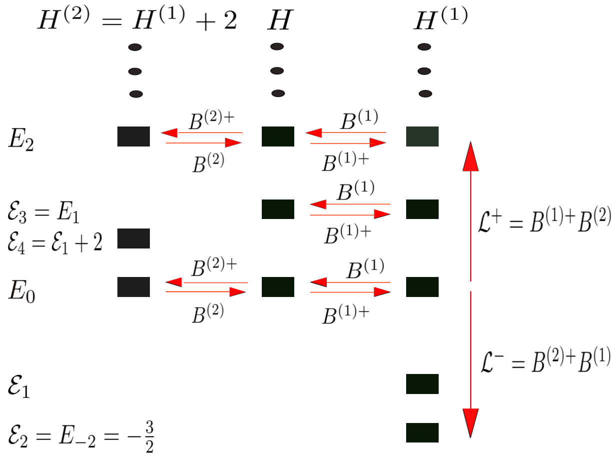

Figure 1. Diagram of the mechanism of the two-step ladder operators (26)

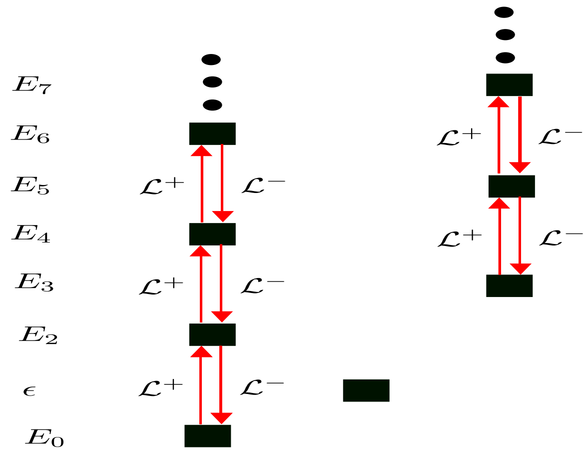

Figure 2. Three independent energy ladders that make up the spectrum of $\widetilde{H}$. This spectrum is composed by two infinite energy ladders and a single-element one.

4. Linearised coherent states and their properties

Once we have defined the ladder operators $\mathcal{L}^{\pm}$ in (26), and clarify how they divide the Hilbert space into two infinite subspaces (or energy ladders) plus a one-dimensional subspace, we proceed to linearise them. We focus on the two infinite subspaces since the construction of the coherent state of the third subspace is trivial. We define new ladder operators for each infinite subspace as

$$l_\nu^+ = \sigma_\nu(\widetilde{H})\mathcal{L}^+, \quad l_\nu^- = \sigma_\nu(\widetilde{H} + 2)\mathcal{L}^-, \tag{31}$$

where $\nu = 0,\ 3$ is the index of the subspace. When $\nu = 0$, we refer to the subspace span$\{\widetilde{\psi}_0,\ \widetilde{\psi}_2, \widetilde{\psi}_4,\ \ldots\}$ and, when $\nu = 3$, we refer to the subspace span$\{\widetilde{\psi}_3,\ \widetilde{\psi}_5,\ \widetilde{\psi}_7,\ \ldots\}$. The operators σ_ν are defined as

$$\begin{aligned} \sigma_0(\widetilde{H}) &= [(\widetilde{H} - \mathcal{E}_1)(\widetilde{H} - \mathcal{E}_3)(\widetilde{H} - \mathcal{E}_4)]^{-1/2}, \\ \sigma_3(\widetilde{H}) &= [(\widetilde{H} - \mathcal{E}_1)(\widetilde{H} - \mathcal{E}_2)(\widetilde{H} - \mathcal{E}_3)]^{-1/2}. \end{aligned} \tag{32}$$

From (27), and considering $\sigma_\nu(x)$ a regular function, we obtain the following useful relations.

$$\sigma_\nu(\widetilde{H})\mathcal{L}^+ = \mathcal{L}^+\sigma_\nu(\widetilde{H}+2), \quad \sigma_\nu(\widetilde{H})\mathcal{L}^- = \mathcal{L}^-\sigma_\nu(\widetilde{H}-2);$$
$$\mathcal{L}^+\sigma_\nu(\widetilde{H}) = \sigma(\widetilde{H}-2)\mathcal{L}^+, \quad \mathcal{L}^-\sigma_\nu(\widetilde{H}) = \sigma_\nu(\widetilde{H}+2)\mathcal{L}^-.$$

Using (29), it is direct to show that the operators $l_\nu^\pm$ fulfill the linear commutation relation

$$[l_\nu, l_\nu^+] = 2\mathbb{1}_{\mathbb{H}^\nu}, \tag{33}$$

where $\mathbb{1}_{\mathbb{H}^\nu}$ is the identity in the subspace $\mathbb{H}^\nu$. Therefore, on both Hilbert subspaces, the action of the linearised ladder operators is

$$l_\nu^-\widetilde{\psi}_{\nu+2n} = \sqrt{2n}\widetilde{\psi}_{\nu+2(n-1)},$$
$$l_\nu^+\widetilde{\psi}_{\nu+2n} = \sqrt{2(n+1)}\widetilde{\psi}_{\nu+2(n+1)}, \tag{34}$$

where $n = 0, 1, 2, \ldots$

At this stage, we can define the linearised coherent states as eigenstates of the linear annihilation operator,

$$l_\nu^- \left|z^\nu\right\rangle = z\left|z^\nu\right\rangle, \quad \nu = 0,\ 3, \tag{35}$$

where $z \in \mathbb{C}$. We can make the expansion

$$\left|z^\nu\right\rangle = \sum_{n=0}^{\infty} c_n \left|\nu+2n\right\rangle, \tag{36}$$

where $\widetilde{\psi}_{\nu+2n}(x) = \langle x|\nu+2n\rangle$ are the eigenfunctions of the SUSY Hamiltonian, and following the definition (35), we find that the explicit form of the normalised coherent states is

$$\left|z^\nu\right\rangle = e^{-\frac{|z|^2}{4}}\sum_{n=0}^{\infty}\frac{(z/\sqrt{2})^n}{\sqrt{n!}}\left|\nu+2n\right\rangle. \tag{37}$$

Notice that we obtained a similar expression of the standard coherent states but with the relevant difference that the expansion is in terms of eigenfunctions of the supersymmetric partner Hamiltonian $\widetilde{H}$ in the subspace ν.

4.1. Completeness relation

An important property that the constructed coherent states fulfill is that they form an over-complete set on Hilbert subspaces, i.e., they solve an identity expression [25]

$$\frac{1}{2\pi}\int_{\mathbb{C}}\left|z^\nu\right\rangle\left\langle z^\nu\right| d^2z = \mathbb{1}_{\mathbb{H}^\nu}. \tag{38}$$

4.2. Mean-energy values

The eigenvalue equation of the Hamiltonian $\widetilde{H}$ is given by

$$\widetilde{H}\left|\nu+2n\right\rangle = \left(\nu+\frac{1}{2}+2n\right)\left|\nu+2n\right\rangle, \tag{39}$$

which leads to the energy expectation

$$\left\langle z^\nu\right|\widetilde{H}\left|z^\nu\right\rangle = \nu+\frac{1}{2}+|z|^2. \tag{40}$$

We observe that we obtain the well-known quantity of energy-growth corresponding to the oscillator coherent states, this result is another direct consequence of the linear commutation relation between the linearised ladder operators.

4.3. Temporal stability

Another relevant property of the coherent states is that they must remain coherent as they evolve in time. By applying the time evolution operator $U(t)$, we obtain

$$U(t)\left|z^\nu\right\rangle = e^{-i\left(\nu+\frac{1}{2}\right)t}\left|z^\nu(t)\right\rangle,$$

i.e., our linearised coherent states fulfill this condition. The period of evolution of these states is $\tau = \pi$, the half of the harmonic oscillator coherent states ($T = 2\pi$). This means that in the phase-space, our states need just the half of the time to return to the same point with an acquired phase. This represents a first clear indication of non-classical behaviour.

4.4. Evolution of the probability densities

Let us analyse the time evolution of the probability densities. For the classical coherent states, this quantity is represented by a Gaussian wave packet oscillating around the minimum of the potential. In our case, we have:

$$\rho_z(z,x,t) = |\langle x|\,U(t)\left|z^\nu\right\rangle|^2$$
$$= |\sum_{n=0}^{\infty} e^{-\frac{|z|^2}{4}}\frac{(ze^{-i2t}/\sqrt{2})^n}{\sqrt{n!}}\widetilde{\psi}_{\nu+2n}(x)|^2. \tag{41}$$

In the Figure 3, we plot this evolution. We observe that each coherent state is composed by two wavepackets with a back-and-forth motion resembling a semi-classical behaviour, since each wavepacket looks like a harmonic-oscillator coherent state. The two wavepackets interfere with each other, and it is more noticeable when they collide around $x = 0$. A parity symmetry $x \to -x$, is only apparent and cannot be guaranteed for the SUSY extensions since the potential $\widetilde{V}$ is only symmetric around $x = 0$ when the parameter $\gamma = 0$ in the seed function $u_2^{(2)}$.

4.5. Wigner distributions

An efficient tool to determine the nature of quantum wave functions is the Wigner quasiprobability distribution in the phase space, defined by

$$W(x,p) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi^*\left(x-\frac{y}{2}\right)\psi\left(x+\frac{y}{2}\right)e^{ipy}dy. \tag{42}$$

In Figure 4, we show the corresponding Wigner functions of coherent states for both subspaces. We observe that the distributions possess regions with non-positive values, which is a clear indication of the non-classical behaviour or pure quantum nature of our linearised coherent states.

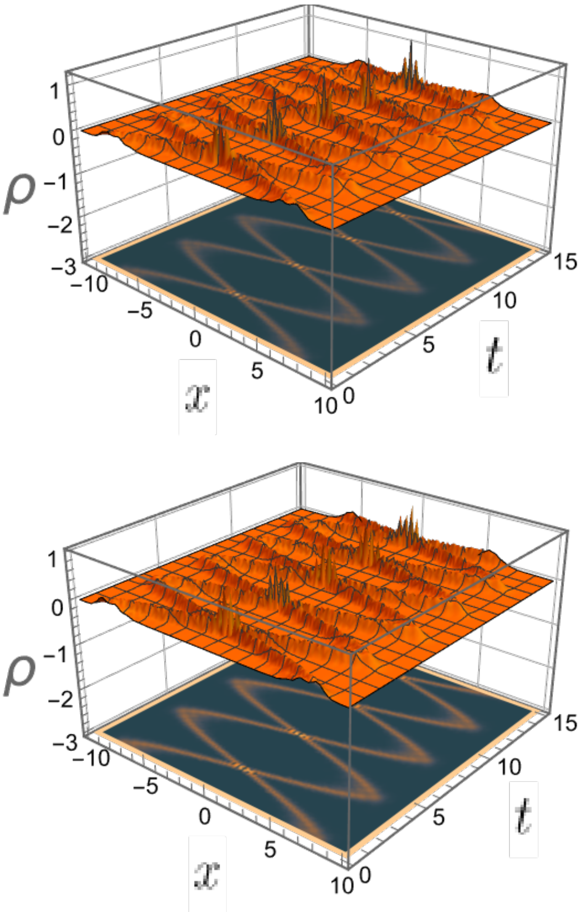

Figure 3. Time evolution of the probability densities (41) of the linearised coherent states (37) with $\epsilon = 2$, $\gamma = 2$, **Top:** $\nu = 0$, $z = 5$, and **Bottom:** $\nu = 3$, $z = 5$.

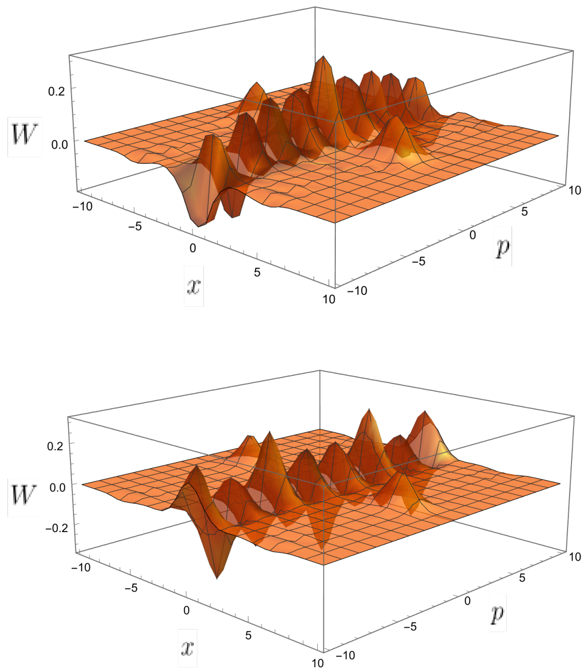

Figure 4. Wigner distributions of the linearised coherent states with $\epsilon = 2$, $\gamma = 2$, $z = 5$, **Top:** $\nu = 0$, and **Bottom:** $\nu = 3$.

4.6. Heisenberg uncertainty relation

First, we introduce two Hermitian quadrature operators

$$X_1 = \frac{l_\nu^+ + l_\nu^-}{2}, \quad X_2 = \frac{l_\nu^- - l_\nu^+}{2i}, \tag{43}$$

and the uncertainties

$$\sigma_{X_i}^2 = \langle X_i^2 \rangle_{z^\nu} - \langle X_i \rangle_{z^\nu}^2, \quad i = 1, 2. \tag{44}$$

Since the coherent states are eigenfunctions of l^-, it is found that these uncertainties follow the product

$$\sigma_{X_1}^2 \sigma_{X_2}^2 = \frac{1}{4}, \tag{45}$$

indicating that they saturate the Heisenberg inequality.

5. Conclusions

We have found a family of equivalent non-rational extensions of the harmonic oscillator potential generated through two different SUSY transformations involving general solutions of the stationary Schrödinger equation in terms of Hermite functions. These SUSY transformations consisted in moving the first-excited state to an arbitrary level between the ground and the second-excited states, and, on the other hand, adding two new levels below the ground state. We built fourth-order differential ladder operators as the product of the intertwining operators related to the equivalent SUSY transformations. Then, we linearised these ladder operators to have a linear commutation relation. In addition, we realized that these operators divide the entire Hilbert space of eigenfunctions into two infinite energy ladders or Hilbert-subspaces, and one single-element subspace. Then, we derived coherent states of the linearised annihilation operator as eigenstates. We uncovered that they are temporally stable cyclic states with a period $\tau = \pi$, and we showed as well that they form an overcomplete set in each subspace. Moreover, they present the same energy growth as the oscillator coherent states. For the time evolution of the probability densities, we obtained the structure of two oscillating wave-packets, each one with a period 2π, but the collective behaviour with a period τ. For the Wigner functions, we observed that they possess regions with non-positive values, unveiling the quantum nature of these states. Finally, by defining two Hermitian quadrature operators as in the harmonic oscillator, we got the linearised coherent states saturate the Heisenberg inequality. Therefore, as we already mentioned, we conclude that our states present both classical and quantum behaviour.

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