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MEASUREMENT OF A QUANTUM PARTICLE POSITION AT TWO DISTANT LOCATIONS: A MODEL

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Abstract.

A simplified one-dimensional model of measurement of the position of a quantum particle by two distant detectors is considered. Detectors are modelled by quantum particles bounded in potential wells with just two bound states, prepared in the excited states. Their de-excitation due to the short range interaction with the measured particle is the signal for the detection. In the approximations of short time or small coupling between the particle and the measuring apparatuses, the simultaneous detection of the particle by both detectors is suppressed. The results extend to other models with two-level detectors.

Keywords: Position measurement, EPR paradox, quantum measurement.

1. Introduction

The theory of measurement is one of the topics in quantum mechanics from its early days [1]. The development can be found, for example, in reviews [2], [3], [4], [5], [6]. Recently, a comparison of an experimental measurement on a trapped 3-level ion by an interaction with a photon environment with the ideal quantum theoretical measurement was done in [7]. Measurements in quantum field theory with non-relativistic quantum-mechanical detectors coupled to the quantum field were studied in [8].

The famous Einstein-Podolsky-Rosen (EPR) [9] paradox concerns the relation of the measurement results performed on distant parts of a quantum system. Later, the subject evolved to a whole branch of the quantum physics, studies of the quantum entanglements and their applications in quantum informatics (see, e.g., Chapter 16 in [10] for the overview, also extracted in [11], review [12] with an extensive list of references, Chapters 4 and 6 in [13], Chapter 9 in [14], or [15]). For a brief review of the physical background up to 2005 see [16]. Consequences for the black-hole physics are discussed in [17]. Effects in several microscopic and statistical systems were recently studied in [18], [19], [20], [21], [22]. Further possible applications in quantum information are proposed in [23], [24]. The collapse of quantum state due to the measurement was, again, considered in [25], the entanglement in a system chaotic in the classical limit was studied in [26].

Although mostly entanglement of spin states of the two particles is considered, the general idea concerns any simultaneous measurements at distant locations with correlated results. The EPR paradox in the case of a particle position measurement means a simultaneous measurement of the particle presence at two distant places. As the particle can be detected at one place only, a sort of superluminal interaction between the two places seems to be necessary. Another possible interpretation could be that the simultaneous detection at the two places is excluded (or perhaps only strongly suppressed) by the quantum evolution of the *complete* system, consisting of the measured particle and both detectors (and perhaps even their environment), starting from a given initial state. We try to support such an idea by a simple, perhaps not very realistic, model. We obtain suppression of the simultaneous detection at the two places in an approximation but not exactly.

The description of the quantum measurement as the unitary evolution of the complete system was considered already by von Neumann in his classical book [1] (called as process 2 in Section V.1). There exist a number of attempts to model macroscopic measuring apparatuses as quantum objects, let us mention only a sample here. In [1] (Sect. VI.3), [27] (Sect. 139 and Appendix XIV), [28, 29] and essentially also in [30] (Chapter 22), the apparatus is modelled by one very massive body. More close to the quantum description of a macroscopic system are the models of the measuring apparatus as a many-body system, composed of infinite or very large finite number of particles, typically a spin chain - see [31], [32], [33] (Chapter 7).

We consider a rather simple model described in the next section. It consists of a particle interacting with two mutually distant detectors. However, we do not attempt to describe the detector as a macroscopic apparatus. We model it by a single quantum particle bounded in a potential well only.

2. The model

We consider one light particle and two detectors on a line. Each detector consists of a very massive particle bounded in the square potential well of a width $2a$ and depth V_0 tuned to the existence of just two bound states of energies $E_1 > E_0$. The detector is prepared for the measurement in the excited state E_1. The detection of the light particle at the detector is modelled by the passing of the detector to the ground state E_0 due to the interaction with the light particle. This is the determination of the particle position before the measurement as it gains energy $E_1 - E_0$ and is kicked away. The two detectors are located with centres of their potential wells at points $-R$, R with $R \gg a$ (see Figure 1).

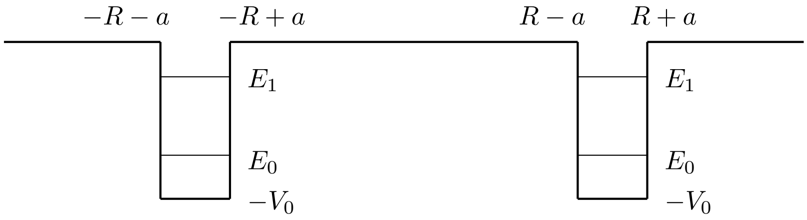

Figure 1. Potential wells of the two detectors. Remember that they bind two different massive particles (it is not a double-well potential but two different one-well potentials).

Let us write the Hamiltonian of the whole three-particle system. Let us denote the potential well

$$V(x) = -V_0\chi_{(-a,a)}(x) = \begin{cases} -V_0 & \text{for} \quad -\mathrm{a} < \mathrm{x} < \mathrm{a} \\ 0 & \text{for} \quad \mathrm{x} \leq -\mathrm{a} \quad \text{or} \quad \mathrm{x} \geq \mathrm{a} \end{cases} , \tag{1}$$

where $\chi_{(-a,a)}$ is the characteristic function of the interval $(-a, a)$ for definiteness although the detailed shape of the potential V is not really used. Denoting $r \in \mathbb{R}$ the coordinate of the measured light particle (mass m), $x \in \mathbb{R}$ the coordinate of the heavy particle in the left detector (mass M) and $y \in \mathbb{R}$ that in the right detector (mass M), the free Hamiltonian (without an interaction of the measured particle and detectors) reads

$$H_0 = -\frac{\hbar^2}{2m}\partial_r^2 - \frac{\hbar^2}{2M}\partial_x^2 + V(x+R) - \frac{\hbar^2}{2M}\partial_y^2 + V(y-R) \,, \tag{2}$$

understood as a self-adjoint operator in the state space $L^2(\mathbb{R}^3)$ with the domain $H^2(\mathbb{R}^3)$. Let us assume that the measured particle and the particle of the detector interact when their distance is shorter than a with the model interaction Hamiltonian (g is an interaction constant)

$$H_I = g\chi_{(-a,a)}(r-x) + g\chi_{(-a,a)}(r-y) \,. \tag{3}$$

For simplicity, the interaction potential between the measured particle and the heavy particle in each detector is chosen, again, in the form of a square well (or barrier). Its width $2a$ is chosen the same as for the potentials binding heavy particles in the detectors, which can be interpreted as the detector size (in the order of magnitude, at the least). The complete Hamiltonian

$$H = H_0 + H_I \,. \tag{4}$$

We assume that the detector with Hamiltonian

$$-\frac{\hbar^2}{2M}\partial_x^2 + V(x) \tag{5}$$

has just two bound states ψ_0, ψ_1 with energies $E_0 < E_1$. The detectors are prepared for measurement in the excited states ψ_1, and the measured particle in a state φ_0 so the initial wave function of the complete system is

$$\Psi(0, r, x, y) = \varphi_0(r)\psi_1(x+R)\psi_1(y-R) \,. \tag{6}$$

Let us assume that the energy of the light particle is insufficient to release the detector particles from their potential wells and that their states from the continuous spectrum can be neglected in the time evolution. In other words, we assume the wave function of the complete system in the form

$$\begin{aligned}\Psi(t, r, x, y) = {} & c_{00}(t, r)\psi_0(x+R)\psi_0(y-R) + c_{01}(t, r)\psi_0(x+R)\psi_1(y-R) \\ & + c_{10}(t, r)\psi_1(x+R)\psi_0(y-R) + c_{11}(t, r)\psi_1(x+R)\psi_1(y-R) \,. \end{aligned} \tag{7}$$

Inserting into the Schrödinger equation,

$$i\hbar\partial_t\Psi = H\Psi\ , \tag{8}$$

and projecting onto $\psi_j(x+R)\psi_k(y-R)$ $(j,k=0,1)$, we obtain

$$\begin{aligned} i\hbar\partial_t c_{jk}(t,r) &= -\frac{\hbar^2}{2m}\partial_r^2 c_{jk}(t,r) + (E_j+E_k)c_{jk}(t,r) \\ &+g\sum_{l=0}^{1}\left(f_{jl}(r+R)c_{lk}(t,r)+f_{kl}(r-R)c_{jl}(t,r)\right)\quad , \end{aligned} \tag{9}$$

where

$$f_{jk}(z) = \int_{z-a}^{z+a}\overline{\psi_j}(x)\psi_k(x)\,dx \tag{10}$$

come from the matrix elements of interaction term H_I (3) and the finite integration range here is a consequence of the finite range of interaction. As eigenfunctions ψ_j are exponentially decaying for $|x|>a$, the same is true for f_{jk} at $|z|>2a$.

In the matrix notation

$$C(t,r) = \begin{pmatrix} c_{00}(t,r) & c_{01}(t,r) \\ c_{10}(t,r) & c_{11}(t,r)\end{pmatrix}\quad ,\quad E = \begin{pmatrix} E_0 & 0 \\ 0 & E_1\end{pmatrix}\quad , \tag{11}$$

$$F(r) = \begin{pmatrix} f_{00}(r+R) & f_{01}(r+R) \\ f_{10}(r+R) & f_{11}(r+R)\end{pmatrix}\quad , \tag{12}$$

$$G(r) = \begin{pmatrix} f_{00}(r-R) & f_{10}(r-R) \\ f_{01}(r-R) & f_{11}(r-R)\end{pmatrix}\quad , \tag{13}$$

the equations (9) read

$$i\hbar\partial_t C(t,r) = -\frac{\hbar^2}{2m}\partial_r^2 C(t,r) + (E+gF(r))C(t,r) + C(t,r)(E+gG(r))\ . \tag{14}$$

The initial condition (6) gives

$$C(0,r) = \begin{pmatrix} 0 & 0 \\ 0 & \varphi_0(r)\end{pmatrix}\ . \tag{15}$$

Matrices $F(r)$ and $G(r)$ are Hermitian and bounded in $r\in\mathbb{R}$ according to their construction. So it is easily seen that the operator on the right-hand side of (14) is self-adjoint in $L^2(\mathbb{R},\mathbb{C}^4)$ with the domain $H^2(\mathbb{R},\mathbb{C}^4)$ and the solution $C(t,r)$ exists there for every $C(0,\cdot)\in H^2(\mathbb{R},\mathbb{C}^4)$.

The probability of finding the left detector in the state ψ_j and the right detector in the state ψ_k at the time t is calculated by (7)

$$P_{jk}(t) = \int_{\mathbb{R}}|c_{jk}(t,r)|^2\,dr\quad , \tag{16}$$

i.e., P_{11} is the probability that no detector reacts ($P_{11}(0)=1$ as detectors are initially prepared in the excited state ψ_1), P_{01} the probability of detection by the left detector only, P_{10} the probability of detection be the right detector only, and P_{00} the probability of simultaneous detection by both detectors.

3. Short-time evolution

In the approximation

$$C(t,r) = C(0,r) + t\partial_t C(0,r) + O(t^2)\ , \tag{17}$$

$$C(t,r) = \begin{pmatrix} 0 & -i\frac{gt}{\hbar}F_{01}(r)\varphi_0(r) \\ -i\frac{gt}{\hbar}G_{10}(r)\varphi_0(r) & i\frac{\hbar t}{2m}\partial_r^2\varphi_0(r) + (1-i\frac{t}{\hbar}(2E_1+gF_{11}(r)+gG_{11}(r)))\varphi_0(r)\end{pmatrix}, \tag{18}$$

assuming at the least that φ_0 belongs to the domain of ∂_r^2. Therefore, the probability $P_{00}(t)$ of the simultaneous detection by the both detectors, indicated by their de-excitations to the ground states, remains zero in the first nontrivial approximation as $t \to 0$,

$$P_{00}(t) = \int_{\mathbb{R}} |c_{00}(t,r)|^2 \, dr = O(t^4) \ , \tag{19}$$

in comparison with the probabilities of detection by just one detector $P_{01}(t)$, $P_{10}(t)$, which are of the order of $O(t^2)$. This is seen from (18), looking by which powers of t the expansions of $c_{jk}(t,r)$ start.

This result is an indication that quantum mechanics inherently prefers the detection of a particle at one place only. However, one should not take it more seriously than the model can approximate macroscopic measuring apparatuses. For the larger times, $P_{00}(t)$ surely is not zero, which also corresponds to the physical picture that the particle interacts with one detector and is bounced to the other.

4. Weak coupling approximation

In this section, we derive a result similar to the one of the previous section but in the first perturbation approximation according to the coupling constant g between the light particle and the measuring instruments. The procedure follows the standard approach of passing to the interaction picture and then iteration of the integral equivalent of the Schrödinger equation.

Let us start from equation (14) and denote as H_m the free Hamiltonian of the light particle, i.e.,

$$H_m = -\frac{\hbar^2}{2m}\partial_r^2 \quad , \tag{20}$$

where the right hand side is understood as the corresponding self-adjoint operator in $L^2(\mathbb{R})$. We transform

$$C(t,r) = \left(e^{-\frac{i}{\hbar}Et} e^{-\frac{i}{\hbar}H_m t} C_1(t,\cdot) e^{-\frac{i}{\hbar}Et}\right)(r) \quad . \tag{21}$$

Here $C_1(t,\cdot)$ denotes the matrix function of r with the value $C_1(t,r)$ at the point r (t is just a parameter). The operator $e^{-\frac{i}{\hbar}H_m t}$ transforms it to another function $C_2(t,\cdot)$, which is then multiplied by two exponentials depending on E and t only. Finally, the value of the resulting function at the point r, i.e., $e^{-\frac{i}{\hbar}Et} C_2(t,r) e^{-\frac{i}{\hbar}Et}$, is taken. A similar notation is used in a few following formulas.

Inserting (21) into (14), we obtain

$$\partial_t C_1(t,r) = -\frac{ig}{\hbar}(A(t)C_1(t,\cdot))(r) \quad , \tag{22}$$

where the operator $A(t)$ acts as

$$A(t)T = e^{\frac{i}{\hbar}H_m t}\left(\tilde{F}(t)(e^{-\frac{i}{\hbar}H_m t}T) + (e^{-\frac{i}{\hbar}H_m t}T)\tilde{G}(t)\right) \tag{23}$$

on a matrix function $T \in L^2(\mathbb{R},\mathbb{C}^4)$ and $\tilde{F}(t)$, $\tilde{G}(t)$ are multiplications by the matrix functions

$$\tilde{F}(t,r) = e^{\frac{i}{\hbar}Et} F(r) e^{-\frac{i}{\hbar}Et} \quad , \quad \tilde{G}(t,r) = e^{-\frac{i}{\hbar}Et} G(r) e^{\frac{i}{\hbar}Et} \ . \tag{24}$$

The integral form of (22) reads

$$C_1(t,r) = C_1(0,r) - \frac{ig}{\hbar}\int_0^t \left(A(\tau)C_1(\tau,\cdot)\right)(r)\, d\tau \ . \tag{25}$$

Let us iterate this equation starting with $C_1(0,r)$ as the first approximation. Since the operator $A(\tau)$ is uniformly bounded in τ, the resulting perturbation series is convergent to the unique solution.

Keeping only terms linear in g,

$$C_1(t,r) = C_1(0,r) - \frac{ig}{\hbar}\int_0^t \left(A(\tau)C_1(0,\cdot)\right)(r)\, d\tau + O(g^2) \ . \tag{26}$$

The initial value $C_1(0,r) = C(0,r)$ is given in (15) and (26) reads as

$$C_1(t,r) = \begin{pmatrix} 0 & gu_1(t,r) \\ gu_2(t,r) & \varphi_0(r) + gu_3(t,r) \end{pmatrix} + O(g^2) \ , \tag{27}$$

where

$$u_1(t,r) \;=\; -\frac{i}{\hbar}\int_0^t \left(e^{\frac{i}{\hbar}(E_0-E_1)\tau}e^{\frac{i}{\hbar}H_m\tau}f_{01}^{(+)}e^{-\frac{i}{\hbar}H_m\tau}\varphi_0\right)(r)\,d\tau \quad , \tag{28}$$

$$u_2(t,r) \;=\; -\frac{i}{\hbar}\int_0^t \left(e^{\frac{i}{\hbar}(E_0-E_1)\tau}e^{\frac{i}{\hbar}H_m\tau}f_{01}^{(-)}e^{-\frac{i}{\hbar}H_m\tau}\varphi_0\right)(r)\,d\tau \quad , \tag{29}$$

$$u_3(t,r) \;=\; -\frac{i}{\hbar}\int_0^t \left(e^{\frac{i}{\hbar}H_m\tau}(f_{11}^{(+)}+f_{11}^{(-)})e^{-\frac{i}{\hbar}H_m\tau}\varphi_0\right)(r)\,d\tau \quad , \tag{30}$$

denoting $f_{jk}^{(\pm)}(r) = f_{jk}(r \pm R)$ for $j,k = 0,1$. The free propagator $e^{-\frac{i}{\hbar}H_m\tau}$ can be expressed by the wellknown integral formula here (e.g., Eq. (IX.31) in [34]). Passing back to $C(t,r)$,

$$C(t,r) = \begin{pmatrix} 0 & ge^{-\frac{i}{\hbar}(E_0+E_1)t}e^{-\frac{i}{\hbar}H_m t}u_1 \\ ge^{-\frac{i}{\hbar}(E_0+E_1)t}e^{-\frac{i}{\hbar}H_m t}u_2 & e^{-2\frac{i}{\hbar}E_1 t}e^{-\frac{i}{\hbar}H_m t}(\varphi_0+gu_3) \end{pmatrix} + O(g^2), \tag{31}$$

and the probability of the simultaneous detection by both detectors remains zero in the first nontrivial approximation as $g \to 0$,

$$P_{00}(t) = O(g^4)\ , \tag{32}$$

in comparison with $P_{01}(t)$ and $P_{10}(t)$ which are of the order of $O(g^2)$.

5. Concluding remarks

The obtained results may be interpreted as a support of the idea that the detection of a particle at one place only is a consequence of the initial conditions for the complete system (particle plus detectors) without any superluminal interaction between two distant places. However, as they are obtained in the approximation of a short time or weak coupling only, it may be questioned whether they do not represent just a kind of initial state stability. A possible objection against this explanation is that C_{00} is more stable than C_{01} and C_{10}.

Another discussible point is the modelling of a macroscopic detector by a single quantum particle. We tacitly assume that the particles in detectors are heavy and thus have a large action. It was used only in heuristic justification of the assumption that their states remains in the two-dimensional subspace spanned by the vectors ψ_0 and ψ_1, in other words, the detectors are essentially two-level systems. The calculation for a more realistic models of the detectors and without approximations of small times or weak coupling would be very desirable but seems to be extremely difficult.

A very specific model was considered for the definiteness but the main results (19) and (32) hold for any detectors modelled as two-level systems and any interaction separated to the sum of two parts corresponding to the particle interaction with each detector instead of (3). The form of the initial particle state φ_0 is also not important here. The specific form of the model is not essentially used in the above calculations. However, the probability P_{00} of a simultaneous de-excitation of both detectors is already nonzero in the next iterations to (19) and (32). So the detection by both detectors is not excluded for large values of t or g in our model.

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References

[1] J. von Neumann. *Mathematical Foundations of Quantum Mechanics.* Princeton University Press, 2018.

[2] A. A. Clerk, M. H. Devoret, S. M. Girvin, et al. Introduction to quantum noise, measurement, and amplification. *Reviews of Modern Physics* **82**(2):1155–1208, 2010. https://doi.org/10.1103/RevModPhys.82.1155.

[3] A. E. Allahverdyan, R. Balian, T. M. Nieuwenhuizen. Understanding quantum measurement from the solution of dynamical models. *Physics Reports* **525**(1):1–166, 2013. https://doi.org/10.1016/j.physrep.2012.11.001.

[4] J. Zhang, Y. xi Liu, R.-B. Wu, et al. Quantum feedback: Theory, experiments, and applications. *Physics Reports* **679**:1–60, 2017. https://doi.org/10.1016/j.physrep.2017.02.003.

[5] B. d'Espagnat. *Conceptual Foundations of Quantum Mechanics.* CRC Press, Boca Raton, 2nd edn., 2019.

[6] C. Beck. *Local Quantum Measurement and Relativity.* Springer, Cham, 2021. https://doi.org/10.1007/978-3-030-67533-2.

[7] F. Pokorny, C. Zhang, G. Higgins, et al. Tracking the dynamics of an ideal quantum measurement. *Physical Review Letters* **124**(8-28):080401, 2020. https://doi.org/10.1103/PhysRevLett.124.080401.

[8] J. Polo-Gómez, L. J. Garay, E. Martín-Martínez. A detector-based measurement theory for quantum field theory. *Physical Review D* **105**:065003, 2022. https://doi.org/10.1103/PhysRevD.105.065003.

[9] A. Einstein, B. Podolsky, N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Physical Review* **47**:777–780, 1935. https://doi.org/10.1103/PhysRev.47.777.

[10] I. Bengtsson, K. Życzkowski. *Geometry of Quantum States: An Introduction to Quantum Entanglement.* Cambridge University Press, 2017. https://doi.org/10.1017/9781139207010.

[11] K. Zyczkowski, I. Bengtsson. An introduction to quantum entanglement: A geometric approach, 2006. https://doi.org/10.48550/ARXIV.QUANT-PH/0606228.

[12] N. Friis, G. Vitagliano, M. Malik, M. Huber. Entanglement certification from theory to experiment. *Nature Reviews Physics* **1**:72–87, 2019. https://doi.org/10.1038/s42254-018-0003-5.

[13] P. Meystre. *Quantum Optics: Taming the Quantum.* Springer, Cham, 2021. https://doi.org/10.1007/978-3-030-76183-7.

[14] D. D'Alessandro. *Introduction to Quantum Control and Dynamics.* CRC Press, Boca Raton, 2021. https://doi.org/10.1201/9781003051268.

[15] M. Fadel. *Many-particle entanglement, Einstein-Podolsky-Rosen steering and Bell correlations in Bose-Einstein condensates.* Springer, Cham, 2021. https://doi.org/10.1007/978-3-030-85472-0.

[16] M. Kupczynski. Seventy years of the EPR paradox. *AIP Conference Proceedings* **861**:516–523, 2006. https://doi.org/10.1063/1.2399618.

[17] L. Susskind. ER=EPR, GHZ, and the consistency of quantum measurements. *Fortschritte der Physik* **64**:72–83, 2016. https://doi.org/10.1002/prop.201500094.

[18] W. Zhang, M.-X. Dong, D.-S. Ding, et al. Einstein-Podolsky-Rosen entanglement between separated atomic ensembles. *Physical Review A* **100**(1):012347, 2019. https://doi.org/10.1103/PhysRevA.100.012347.

[19] M. Fadel, A. Usui, M. Huber, et al. Entanglement quantification in atomic ensembles. *Physical Review Letters* **127**(1):010401, 2021. https://doi.org/10.1103/PhysRevLett.127.010401.

[20] P. Marian, T. A. Marian. Einstein-Podolsky-Rosen uncertainty limits for bipartite multimode states. *Physical Review A* **103**(6):062224, 2021. https://doi.org/10.1103/PhysRevA.103.062224.

[21] W. Zhong, D. Zhao, G. Cheng, A. Chen. One-way Einstein–Podolsky–Rosen steering of macroscopic magnons with squeezed light. *Optics Communications* **497**:127138, 2021. https://doi.org/10.1016/j.optcom.2021.127138.

[22] K. Berrada, H. Eleuch. Einstein-Podolsky-Rosen steering and nonlocality in quantum dot systems. *Physica E: Low-dimensional Systems and Nanostructures* **126**:114412, 2021. https://doi.org/10.1016/j.physe.2020.114412.

[23] Y. Xiang, X. Su, L. Mišta, et al. Multipartite Einstein-Podolsky-Rosen steering sharing with separable states. *Physical Review A* **99**(1):010104, 2019. https://doi.org/10.1103/PhysRevA.99.010104.

[24] N. B. An. Quantum dialogue mediated by EPR-type entangled coherent states. *Quantum Information Processing* **20**:100, 2021. https://doi.org/10.1007/s11128-021-03007-1.

[25] O. V. Gritsenko. Quantum collapse as reduction from a continuum of conditional amplitudes in an entangled state to a single actualized amplitude in the collapsed state. *Physical Review A* **101**(1):012106, 2020. https://doi.org/10.1103/PhysRevA.101.012106.

[26] K. M. Frahm, D. L. Shepelyansky. Chaotic Einstein-Podolsky-Rosen pairs, measurements and time reversal. *The European Physical Journal D* **75**:277, 2021. https://doi.org/10.1140/epjd/s10053-021-00274-6.

[27] D. I. Blokhintsev. *Fundamentals of Quantum Mechanics.* Nauka, Moscow, 1976. In Russian.

[28] S. Machida, M. Namiki. Theory of measurement in quantum mechanics I. *Progress of Theoretical Physics* **63**(5):1457–1473, 1980. https://doi.org/10.1143/PTP.63.1457.

[29] S. Machida, M. Namiki. Theory of measurement in quantum mechanics II. *Progress of Theoretical Physics* **63**(6):1833–1847, 1980. https://doi.org/10.1143/PTP.63.1833.

[30] D. Bohm. *Quantum Theory.* Prentice-Hall, New York, 1952.

[31] K. Hepp. Quantum theory of mesurement and macroscopic observables. *Helvetica Physica Acta* **45**:237–248, 1972. https://doi.org/10.5169/seals-114381.

[32] M. Cini, M. De Maria, G. Mattioli, F. Nicolò. Wave packet reduction in quantum mechanics: A model of a measuring apparatus. *Foundations of Physics* **9**:479–500, 1979. https://doi.org/10.1007/BF00708363.

[33] P. Bóna. *Classical Systems in Quantum Mechanics.* Springer, Cham, 2020. https://doi.org/10.1007/978-3-030-45070-0.

[34] M. Reed, B. Simon. *Methods of Modern Mathematical Physics. II. Fourier Analysis. Self-Adjointness.* Academic Press, San Diego, 1975.