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FRACTIONAL CALCULUS AND LAMBERT FUNCTION I. LIOUVILLE–WEYL FRACTIONAL INTEGRAL

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Abstract. The interconnection between the Liouville–Weyl fractional integral and the Lambert function is studied. The class of modified Abel equations of the first kind is solved. A new integral formula for the Gamma function and possibly new transform pairs for the Laplace and Mellin transform have been found.

Keywords: variable order fractional integral, Liouville–Weyl fractional integral, Lambert function, Gamma function, Bickley function, McDonald function, exponential integral, entire functions, completely monotone functions, Laplace transform, Mellin transform, Euler integral transform, Volterra integral equations.

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1. Introduction

For a study of the interconnection between fractional integrals [1] and the Lambert function [2], we start with the following variant of the fractional integral, known as the Liouville, Liouville–Weyl or Weyl fractional integral [1]:

$$(I_-^\nu F)(y) = \frac{1}{\Gamma(\nu)}\int_y^\infty F(x)(x-y)^{\nu-1}\,\mathrm{d}x = \frac{1}{\Gamma(\nu)}\int_0^\infty F(u+y)u^{\nu-1}\,\mathrm{d}u, \tag{1.1}$$

where $\Re\nu > 0$ and $y > -\infty$. Parameter ν is known as the order of the fractional integral. The integral on the right of (1.1) is the Mellin transform of the shifted function $F(x)$. This means that the Mellin transform of function $F(\cdot)$ is its Liouville–Weyl fractional integral of the order ν at $y = 0$ times the Gamma function of argument ν. For our further investigation, we suppose that the integrals in (1.1) converge absolutely and that function $F(x)$ is an unilateral Laplace transform (picture, generating function) of the relevant original or determining function $f(t)$:

$$F(x) = \int_0^\infty \mathrm{e}^{-xt} f(t)\,\mathrm{d}t, \tag{1.2}$$

where $f(t)$ is a locally integrable function on the interval $[0,\infty)$, and the integral in (1.2) converges absolutely at some $x = c$. Then the integral in (1.2) converges absolutely and uniformly in the half-plane $\Re(x) > \Re(c)$ [3]. If $f(t)$ is a function of the exponential order α_0, i.e., $f(t) = \mathcal{O}(\exp\alpha_0 t)$ for $t = +\infty$ (see [21, Chap. 5]), then integral (1.2) converges absolutely at least in the region $\Re x > \alpha_0$ and uniformly for $\Re x > \alpha_1 > \alpha_0$. Function $F(x)$ is holomorphic in the region $\Re x > \alpha_0$. It can easily be shown [4] that

$$(I_-^\nu F)(y) = \int_0^\infty \mathrm{e}^{-yx} x^{-\nu} f(x)\,\mathrm{d}x, \quad \Re\nu \geq 0. \tag{1.3}$$

It should be noted that (1.3) is defined straightforwardly for $\Re\nu = 0$, whereas (1.1) is not. The integral in (1.3) can be regarded as a Laplace transform for variable y, or as a Mellin transform for variable $-\nu + 1$. It is often called the Laplace–Mellin transform [5]. Function $f(x)$ can be a generalized function, namely a Dirac delta function, see [3]. These generalized functions will be used as a tool only, and the results obtained here can be verified without making use of them.

Definition 1.1. A completely monotone function is defined as follows [6, Def. 1.3, p. 2]: A function $f : (0,\infty) \to \mathbb{R}$ is a completely monotone function if $f(\cdot)$ is a member of the class $\mathcal{C}^\infty$ and

$$(-1)^n f^{(n)}(\lambda) \geq 0, \quad n \in \mathbb{N}\cup\{0\},\ \lambda > 0. \tag{1.4}$$

Equation (1.3) has an interesting consequence based on the following theorems:

Theorem 1.2. *The Liouville–Weyl fractional integral of a completely monotone function is a completely monotone function.*

Proof. According to the Bernstein theorem [6, p. 3], the Laplace transform of the nonnegative function is completely monotone and, conversely, if function $F(x)$, defined by (1.2), is completely monotone, then function $f(t)$ is nonnegative for $t > 0$. Because function $f(x)$ is nonnegative by hypothesis, the Liouville–Weyl fractional integral defined by (1.3) is the Laplace transform of a nonnegative function. ∎

Theorem 1.3. *Suppose the following two conditions on $F(\cdot)$ are satisfied:*

(1.) *the Liouville–Weyl fractional integral of function $F(\cdot)$ is a completely monotone function;*

(2.) *function $F(\cdot)$ is a Laplace transform of some function $f(\cdot)$.*

Then function $F(\cdot)$ is completely monotone.

Proof. In consequence of hypothesis 2., the Liouville–Weyl fractional integral is represented by (1.3). According to hypothesis 1. and the Bernstein theorem, function $x^{-n} f(x)$ is nonnegative for $\Re x > 0$, $Re\nu > 0$. This implies that function $f(t)$ in (1.2) is also nonnegative. This means that function $F(\cdot)$ is completely monotone according to the Bernstein theorem. ∎

Completely monotone functions play a substantial role in probability theory, measure theory, etc. They can also be found in technological practice - we mention time-dependent shear, bulk and the Young moduli of linear viscoelastic materials, because they are Laplace transforms of the relevant nonnegative relaxation spectra [7].

2. Diagonal fractional integrals

In the case that order ν of the fractional integral in (1.1) is a function of variable y, i.e., $\nu = \nu(y)$, the integral in (1.1) is called a variable order fractional integral [8]. Variable order fractional integrals and derivatives have been studied recently due to their applications in many branches of science where the memory effect plays an essential role [9]. Our interest will be concentrated on the simplest case $\nu(y) = y$.

Definition 2.1. The fractional integral of variable order

$$(I_-^y F)(y) = \frac{1}{\Gamma(y)} \int_y^\infty F(x)(x-y)^{y-1}\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-yt} t^{-y} f(t)\,\mathrm{d}t, \quad y > 0, \tag{2.1}$$

will be called the right Liouville–Weyl diagonal fractional integral or simply the diagonal integral of the function $F(x)$, and the integral on the right hand side of (2.1) will be called the anti-diagonal Laplace–Mellin transform of function $f(t)$.

The following two theorems are seminal for this paper.

Theorem 2.2. *The right-most integral in* (2.1) *can be written as*

$$\int_0^\infty \mathrm{e}^{-yt} t^{-y} f(t)\,\mathrm{d}t = \int_{-\infty}^\infty \mathrm{e}^{-yu} f\big(W_0(\mathrm{e}^u)\big) \frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u, \tag{2.2}$$

where function $W_0(\cdot)$ is the 0th branch of the Lambert function [2].

Proof. We compute

$$\int_0^\infty \mathrm{e}^{-yt} t^{-y} f(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-y(t+\ln t)} f(t)\,\mathrm{d}t = \int_{-\infty}^\infty \mathrm{e}^{-yu} f\big(W_0(\mathrm{e}^u)\big) \frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u,$$

where the substitution $t + \ln t = u$, i.e., $t = W_0(\mathrm{e}^u)$ and $\mathrm{d}t = W_0(\mathrm{e}^u)/(1+W_0(\mathrm{e}^u))$, was performed. ∎

The Lambert function is defined as a solution of the functional equation $z = W(z)\mathrm{e}^{W(z)}$ for any complex z [2]. Theorem 2.2 means that the Liouville–Weyl diagonal fractional integral of the unilateral Laplace transform can be interpreted as a two-sided Laplace transform with the region of convergence $\beta_0 < y < \beta_1$.

Alternatively, we can express the anti-diagonal Laplace–Mellin transform as the Mellin transform.

Theorem 2.3. *The rightmost integral in* (2.1) *can be written as*

$$\int_0^\infty \mathrm{e}^{-yt} t^{-y} f(t)\,\mathrm{d}t = \int_0^\infty u^{-y-1} f\big(W_0(u)\big) \frac{W_0(u)}{1+W_0(u)}\,\mathrm{d}u\,. \tag{2.3}$$

Proof. We compute

$$\int_0^\infty \mathrm{e}^{-yt} t^{-y} f(t)\,\mathrm{d}t = \int_0^\infty (\mathrm{e}^t t)^{-y} f(t)\,\mathrm{d}t = \int_0^\infty u^{-y-1} f\big(W_0(u)\big) \frac{W_0(u)}{1+W_0(u)}\,\mathrm{d}u$$

where the substitution $\mathrm{e}^t t = u$, i.e., $t = W_0(u)$ and $\mathrm{d}t = \frac{W_0(u)}{u(1+W_0(u))}\,\mathrm{d}u$, was performed. ∎

The integral on the right of (2.3) is a Mellin transform in variable $-y$. In this paper, we will omit the words "in variable ... " for the sake of brevity. In the case of the Mellin transform in variable y we emphasize this circumstance as the "Mellin transform in standard notation" if necessary to avoid confusion.

This means that the Liouville–Weyl diagonal fractional integral can be represented as a Mellin transform with the region of convergence $-\beta_1 < -y < -\beta_0$ or $\beta_0 < y < \beta_1$. The following corollaries easily follow from Theorems 2.2 and 2.3:

Corollary 2.4. *Let the Laplace transform of function $h(t)$ exist for $0 \le a \le t < b \le \infty$, where $a = W_0(\mathrm{e}^A)$ and $b = W_0(\mathrm{e}^B)$, i.e., $A = a + \ln a$ and $B = b + \ln b$. Then for the Laplace transform of the compound function $h\big(W_0(\mathrm{e}^u)\big)$ it holds*

$$\int_A^B \mathrm{e}^{-yu} h\big(W_0(\mathrm{e}^u)\big)\,\mathrm{d}u = \int_a^b \mathrm{e}^{-yt} t^{-y}(1+t)h(t)/t\,\mathrm{d}t \tag{2.4}$$

whenever both integrals exist. If $-\infty < A < B < +\infty$, the Laplace transform on the left side represents the entire function. In that case, we can substitute the value $y = 0$ (and others of course), whenever we need it.

Corollary 2.5. *Let the Mellin transform of function $h(t)$ exist for $0 \le a < t < b \le \infty$, $a = W_0(A)$ and $b = W_0(B)$, i.e., $A = a\mathrm{e}^a$ and $B = b\mathrm{e}^b$. Then for the Mellin transform in the standard form of the compound function $h\big(W_0(u)\big)$ it holds*

$$\int_A^B u^{y-1} h\big(W_0(u)\big) du = \int_a^b t^{y-1}\mathrm{e}^{yt}(1+t)h(t)\,\mathrm{d}t \tag{2.5}$$

whenever both integrals exist.

Proof. Instead of $f(t)$, write $(1+t)h(t)/t$ in (2.2)–(2.3). In (2.3) change $-y$ to y. In both equations recount the limits of integration. ∎

The integrals on the right sides of (2.4)–(2.5) can often be calculated more easily than the integrals on the left side.

Example 2.6. We have

$$\int_0^{\mathrm{e}} \sin W_0(u)\,\mathrm{d}u = \int_0^1 \mathrm{e}^t(1+t)\sin t\,\mathrm{d}t = \frac{\mathrm{e}}{2}(2\sin 1 - \cos 1) \approx 1.55301.$$

3. Inversion of the diagonal fractional integral

A natural question is to find function $F(x)$ knowing its diagonal fractional integral $G(y)$:

$$\frac{1}{\Gamma(y)} \int_y^\infty F(x)(x-y)^{y-1}\,\mathrm{d}x = G(y). \tag{3.1}$$

This equation is a Volterra integral equation of the first kind and of the Abel type, but in contrast to (1.1), Equation (3.1) is not a convolution. If $F(x)$ is a unilateral Laplace transform of function $f(t)$, see (1.2), Equation (3.1) can easily be solved for $F(x)$ via its original function $f(t)$. Because

$$G(y) = \int_{-\infty}^\infty \mathrm{e}^{-yu} g(u) du = (I_-^y F)(y) = \int_{-\infty}^\infty \mathrm{e}^{-yu} f\big(W_0(\mathrm{e}^u)\big)\frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u, \tag{3.2}$$

the following theorem holds:

Theorem 3.1. *The solution of* (3.2) *is given by the relation*

$$f(t) = (1+t)g(t+\ln t)/t, \quad t \in (0,\infty). \tag{3.3}$$

Proof. Both integrands must be identical with an identical strip of convergence [10]. After substituting $W_0(\mathrm{e}^u) = t$, i.e., $u = t + \ln t$ into (3.2) we obtain (3.3). ∎

A similar argument holds for the "Mellin variant" (2.5). In this case

$$G(y) = \int_0^\infty u^{-y-1} g(u)\,\mathrm{d}u = (I_-^y F)(y) = \int_0^\infty u^{-y-1} f\big(W_0(u)\big)\frac{W_0(u)}{1+W_0(u)}\,\mathrm{d}u\,. \tag{3.4}$$

Theorem 3.2. *The solution of* (3.4) *is given by the relation*

$$f(t) = g(t\mathrm{e}^t)\frac{1+t}{t}, \quad t \in (0,\infty). \tag{3.5}$$

Proof. As in the case of Theorem 3.1, the two integrands must be identical with an identical strip of convergence. After substituting $W_0(u) = t$, i.e., $u = t\mathrm{e}^t$ in (3.4) we obtain (3.5). ∎

3.1. Examples

3.1.1. Example of the Laplace variant (cf. (3.2)–(3.3))

Let $G(y) = 1/(y-1)$, $y > 1$, i.e., $g(u) = \mathrm{e}^u$ for $u \geq 0$ and $g(u) = 0$ for $u < 0$. Then

$$f(t) = \begin{cases} \mathrm{e}^t(t+1) & \text{for } t \geq W_0(1), \\ 0 & \text{for } t < W_0(1), \end{cases}$$

$$F(x) = \int_{W_0(1)}^{\infty} \mathrm{e}^{-xt} f(t)\,\mathrm{d}t = W_0(1)^{x-1} \frac{xW_0(1) + x - W_0(1)}{(x-1)^2}, \quad x > 1,$$

and

$$(I_-^y F)(y) = \frac{1}{\Gamma(y)} \int_y^{\infty} F(x)(x-y)^{y-1}\,\mathrm{d}x = \int_{W_0(1)}^{\infty} \mathrm{e}^{-yt} t^{-y} f(t)\,\mathrm{d}t = \frac{1}{y-1}, \quad y > 1.$$

3.1.2. Example of the Mellin variant (cf. (3.4)–(3.5))

Let $G(y) = -\Gamma(-y) y^{y-1}$, $y \in (0,1)$, i.e., $g(u) = W_0(u)$ for $u \geq 0$ and $g(u) = 0$ for $u < 0$. Then

$$f(t) = \begin{cases} (1+t)g(t\mathrm{e}^t)/t = 1+t & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases}$$

beceause $W_0(t\mathrm{e}^t) = t$. The Laplace transform of $f(t)$ is

$$F(x) = \int_0^{\infty} \mathrm{e}^{-xt} f(t)\,\mathrm{d}t = \frac{1+x}{x^2}.$$

Finally we obtain

$$(I_-^y F)(y) = \frac{1}{\Gamma(y)} \int_y^{\infty} F(x)(x-y)^{y-1}\,\mathrm{d}x = \int_0^{\infty} \mathrm{e}^{-yt} t^{-y} f(t)\,\mathrm{d}t = \pi y^{y-2} \frac{\csc \pi y}{\Gamma(y)}, \quad y \in (0,1),$$

see [12, Entry 8, p. 201], where $\csc \pi y = 1/\sin \pi y$. This means that

$$G(y) = \int_0^{\infty} u^{-y-1} W_0(u)\,\mathrm{d}u = \pi y^{y-2} \frac{\csc \pi y}{\Gamma(y)}, \quad y \in (0,1).$$

In standard notation of the Mellin transform we have the known transform pair [2]

$$\int_0^{\infty} u^{y-1} W_0(u)\,\mathrm{d}u = \pi(-y)^{-y-2} \frac{\csc(-\pi y)}{\Gamma(-y)} = (-y)^{-y} \frac{\Gamma(y)}{y}, \quad y \in (-1,0).$$

4. Fixed point

The following theorems assert that the unilateral Laplace transform can be represented by the Liouville–Weyl diagonal fractional integral of another Laplace transform.

Theorem 4.1. *Every unilateral Laplace transform*

$$F_0(y) = \int_a^{\infty} \mathrm{e}^{-yt} f(t)\,\mathrm{d}t, \quad a \in [0,\infty)$$

of some function $f(t)$ of exponential order α_0, i.e., $f(t) = \mathcal{O}(\exp \alpha_0 t)$ for $t \to +\infty$, is equivalent to the Liouville–Weyl diagonal fractional integral of the function

$$F_1(x) = \int_{A_1}^{\infty} \mathrm{e}^{-xt} f(t + \ln t) \frac{1+t}{t}\,\mathrm{d}t, \quad x \geq \alpha_0, \tag{4.1}$$

where $A_1 = A(a) = W_0(\mathrm{e}^a)$.

Proof. We rewrite the Laplace transform from the hypothesis in the form

$$\int_a^{\infty} \mathrm{e}^{-yt} f(t)\,\mathrm{d}t = \int_a^{\infty} \mathrm{e}^{-yt} h_1\big(W_0(\mathrm{e}^t)\big) \frac{W_0(\mathrm{e}^t)}{1 + W_0(\mathrm{e}^t)}\,\mathrm{d}t\,.$$

Then $h_1(W_0(\mathrm{e}^t)) = f(t)(1+W_0(\mathrm{e}^t))/W_0(\mathrm{e}^t)$, $t > 0$, according to the uniqueness (Lerch) theorem for the unilateral Laplace transform [3, p. 120]. After substituting $W_0(\mathrm{e}^t) = u$, i.e., $t = u + \ln u, u \geq W_0(\mathrm{e}^a) = A(a) \equiv A_1 > 0$, we obtain

$$\int_a^\infty \mathrm{e}^{-yt} f(t)dt = \int_{A_1}^\infty \mathrm{e}^{-y(u+\ln u)} h_1(u)\,\mathrm{d}u = \int_{A_1}^\infty \mathrm{e}^{-yu} u^{-y} h_1(u)\,\mathrm{d}u\,. \tag{4.2}$$

The function $h_1(u) = f(u+\ln u) + f(u+\ln u)/u$, $u \geq W_0(\mathrm{e}^a)$ is also of exponential order α_0, because $f(u+\ln u)$ is of exponential order α_0 and the function $f(u+\ln u)/u$ is of the order $\alpha < \alpha_0$. This means that the Laplace transform

$$F_1(x) = \int_{A_1}^\infty \mathrm{e}^{-xt} h_1(t)\,\mathrm{d}t = \int_{A_1}^\infty \mathrm{e}^{-xt} f(t+\ln t)\frac{1+t}{t}\,\mathrm{d}t = \int_x^\infty \Phi(u)\,\mathrm{d}u + \Phi(x), \quad x \geq \alpha_0,$$

where

$$\Phi(x) = \int_{A_1}^\infty \mathrm{e}^{-xt} f(t+\ln t)\,\mathrm{d}t, \quad x \geq \alpha_0$$

exists, and that the Liouville–Weyl diagonal fractional integral

$$(I_-^y F_1)(y) = \frac{1}{\Gamma(y)}\int_y^\infty F_1(x)(x-y)^{y-1}\,\mathrm{d}x = \int_{A_1}^\infty \mathrm{e}^{-yu}u^{-y}h_1(u)\,\mathrm{d}u = \int_a^\infty \mathrm{e}^{-yt}f(t)\,\mathrm{d}t = F_0(y), \quad y > \alpha_0$$

also exists. ■

The same procedure as above can be applied to function $F_1(y)$, and we obtain

$$F_2(x) = \int_{A_2}^\infty \mathrm{e}^{-yt} h_2(t)\,\mathrm{d}t = \int_{A_2}^\infty \mathrm{e}^{-yt} h_1(t+\ln t)\frac{1+t}{t}\,\mathrm{d}t,$$
$$(I_-^y F_2)(y) = \frac{1}{\Gamma(y)}\int_y^\infty F_2(x)(x-y)^{y-1}\,\mathrm{d}x = \int_{A_2}^\infty \mathrm{e}^{-yu}u^{-y}h_2(u)\,\mathrm{d}u = \int_{A_1}^\infty \mathrm{e}^{-yt}h_1(t)\,\mathrm{d}t = F_1(y), \quad y > \alpha_0,$$

where $A_2 = A(A(a)) = W_0(\exp(W_0(\mathrm{e}^a))$ is a second iterate of the function $A(a)$, and

$$h_2(u) = h_1(u+\ln u)\frac{1+u}{u} = f\big(u+\ln u+\ln(u+\ln u)\big)\frac{1+u+\ln u}{u+\ln u}\frac{1+u}{u}, \quad u \geq A_2.$$

This procedure can be repeated without restraint. To understand its behaviour, we first observe that point $a = 1$ is a fixed point of the function $A(a) = W_0(\mathrm{e}^a)$, because $A(a)$ is a continuous function, $A(a) > a$ for $a < 1$, $A(a) < a$ for $a > 1$ and $A(1) = 1$. This fixed point is attractive because of $A'(1) = 1/2 < 1$. For this reason, $\lim_{n\to\infty} A^n(a) = 1$ for every $a \in [0,\infty)$, where $A^n(a)$ means the nth iterate of function $A(a)$. The interval $[0,\infty)$ is the domain of attraction of this fixed point.

For $F_n(x)$, we obtain

$$F_n(x) = \int_{A_n}^\infty \mathrm{e}^{-yt} h_n(t)\,\mathrm{d}t,$$

where $A_n = A^n(a)$,

$$h_n(t) = f\big((t+\ln t)^n\big)\prod_{k=0}^{n-1}\frac{1+(t+\ln t)^k}{(t+\ln t)^k}, \quad t \geq A_n,\, n > 0,$$

and $A^n(a)$ and $(t+\ln t)^n$ are the nth iterate (not power) of the function $A(a)$ and $t+\ln t$, respectively, $A^0(a) = a$, $(t+\ln t)^0 = \mathrm{t}$. The procedure just described can be reversed:

Theorem 4.2. *Let $H(y) = \int_b^\infty \mathrm{e}^{-yt}g(t)\,\mathrm{d}t$, $0 \leq b < \infty$, be a unilateral Laplace transform, where $g(t)$ is a locally integrable function on the interval $[b,\infty)$ and of exponential order α_0. Then the function*

$$H_1(y) = \int_B^\infty \mathrm{e}^{-yu} f(u)\,\mathrm{d}u = \int_B^\infty \mathrm{e}^{-yu} g\big(W_0(\mathrm{e}^u)\big)\frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u, \tag{4.3}$$

where $B = B(b) = b+\ln b$, is a Liouville–Weyl diagonal fractional integral of $H(y)$.

Proof. The proof is a direct consequence of Theorem 2.2. ■

This procedure can also be repeated. The point $b = 1$ is a fixed point of the function $B(b) = b+\ln b$, because $B(b)$ is a continuous function, $B(b) < b$ for $b < 1$, $B(b) > b$ for $b > 1$ and $B(1) = 1$. This fixed point is repulsive because of $B'(1) = 2 > 1$, so $\lim_{n\to\infty} B^n(b) = +\infty$ for $b > 1$, $\lim_{n\to\infty} B^n(b) = 1$ for $b = 1$ and for $b < 1$ the process ended at n for which $B^{n+1}(b) < 0$.

5. Complex domain

As yet there has been no need to reason about complex values of variable y in the definition of the diagonal integral and in the anti-diagonal Laplace–Mellin transform in (2.1), because this paper (with minimum exceptions — the Gamma function and Examples 6.12 and 6.13) deals with fractional integrals of generally complex order but on the real axis. Now we intend to infer an analog of the standard Bromwich inversion formula of the Laplace transform for the anti-diagonal Laplace–Mellin transform, and make a mention of the entire functions.

Theorem 5.1. *If the integral in* (3.2) *converges absolutely for function $g(u)$ in the region of convergence $-\infty \le \beta_0 < \Re y < \beta_1 \le +\infty$, giving the holomorphic function $G(y)$ there, the following holds:*

$$f(t) = \frac{1+t}{2\mathrm{i}\pi t}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{ty} t^y G(y)\,\mathrm{d}y, \quad t > 0,\ f(0) = \lim_{t\to 0+} f(t), \tag{5.1}$$

where $\beta_0 < c < \beta_1$ and the integral in (5.1) *is taken as the Cauchy principal value (see [3, § 5.8]).*

Moreover, (5.1) *is also true if "Equation* (3.2)*" in the hypothesis is changed to "Equation* (3.4)*".*

Proof. We start with the standard Bromwich inversion for the bilateral Laplace transform in (3.2) and perform the substitution $u = t + \ln t$. We obtain

$$g(t+\ln t) = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{ty} t^y G(y)\,\mathrm{d}y, \quad t > 0 \tag{5.2}$$

and according to (3.3) we obtain (5.1). If we apply the Bromwich inversion to the Mellin transform in (3.4) and perform the substitution $u = t\mathrm{e}^t$ we obtain

$$g(t\mathrm{e}^t) = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{ty} t^y G(y)\,\mathrm{d}y, \quad t > 0 \tag{5.3}$$

and according to (3.5) we also obtain (5.1). ∎

If $G(\cdot)$ is the Mellin transform of function $g(\cdot)$ in standard notation, formula (5.3) obtains the form

$$g(t\mathrm{e}^t) = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{-ty} t^{-y} G(y)\,\mathrm{d}y, \quad t > 0. \tag{5.4}$$

Corollary 5.2. *Let function $g(t)$ have the (generally bilateral) Laplace transform $G(x)$, then*

$$\int_0^\infty \mathrm{e}^{-xt} g(t+\ln t)\,\mathrm{d}t = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} (x-y)^{-y-1}\Gamma(1+y)G(y)\,\mathrm{d}y, \quad \Re x > c > -1,\ \Re x \ge \alpha_1 > \alpha_0, \tag{5.5}$$

where x lies in the interior of the region of holomorphy of the Laplace integral on the left side of (5.5).

Proof. Application of the Laplace transform to both sides of (5.2) with respect to the Fubini theorem gives (5.5). ∎

The formula for solving the integral equation (3.1) in the case that function $G(y)$ is a Laplace or Mellin transform of the pertinent determining functions $g(t)$ is given by the Laplace transform of (5.1):

Theorem 5.3. *Let function $G(y)$ on the right hand side of* (3.1) *be a Laplace or Mellin transform. Then solution $F(x)$ of integral equation* (3.1) *is*

$$\begin{aligned} F(x) = \int_0^\infty \mathrm{e}^{-xt} f(t)\,\mathrm{d}t &= \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} G(y)\int_0^\infty (1+t\mathrm{e}^{yt}t^y)\frac{\mathrm{e}^{-xt}}{t}\,\mathrm{d}t\,\mathrm{d}y \\ &= \frac{x}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} (x-y)^{-y-1}\Gamma(y)G(y)\,\mathrm{d}y, \quad \Re x > c,\ \Re c \ge \alpha_1 > \alpha_0, \end{aligned} \tag{5.6}$$

where the real constant c determines the Bromwich contour inside the region of holomorphy of function $G(y)$, providing $f(t)$ is of exponential order α_0, i.e., $f(t) = \mathcal{O}(\exp\alpha_0 t)$ as $t \to +\infty$.

Proof. We perform the Laplace transform of (5.1) and change the order of the integration according to the Fubini theorem. ∎

The importance of Theorem 5.3 in comparison with Theorem 3.1 is embodied in the fact that there is no need to know the Laplace inverse of function $G(y)$ and that solution $F(x)$ is obtained directly. Moreover numerical experiments indicate that (5.6) also holds for functions $G(y)$ that are either Laplace or Mellin transforms of generalized functions (const., y^n, e^{-ay}, $\zeta(1-y)$, $y^n\,\mathrm{e}^{-ay}$, $\tanh y$), or it is not known to the author if they are transforms at all ($\sin y$, $\ln y$, $1/\Gamma(y)$, $\tan y$). On the other hand, it should be emphasized that bare convergence of the integral in (5.6) does not guarantee the correct solution. An example is $G(y) = \exp y^2$, which is neither a Laplace nor a Mellin transform of any function. The resulting function $F(x)$ is not a solution of (3.1). Testing the solution is recommended.

Lemma 5.4. *If $F(x)$ is a finite Laplace transform of the original $f(t)$, i.e.,*

$$F(x) = \int_a^b \mathrm{e}^{-xt} f(t)\,\mathrm{d}t, \quad 0 \le a < b < \infty, \tag{5.7}$$

then

$$(I_-^y F)(y) = \int_a^b \mathrm{e}^{-xt} t^{-y} f(t)\,\mathrm{d}t = \int_{a+\ln a}^{b+\ln b} \mathrm{e}^{-yu} f\big(W_0(\mathrm{e}^u)\big)\frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u \tag{5.8}$$

and

$$(I_-^y F)(y) = \int_a^b \mathrm{e}^{-xt} t^{-y} f(t)\,\mathrm{d}t = \int_{a\exp a}^{b\exp b} u^{-y-1} f\big(W_0(u)\big)\frac{W_0(u)}{1+W_0(u)}\,\mathrm{d}u\,. \tag{5.9}$$

Proof. Substituting $W_0(\mathrm{e}^u) = t$, i.e., $u = t + \ln t$ in (5.6) and substituting $W_0(u) = t$, i.e., $u = t\mathrm{e}^t$ in left integral of (5.7), as in Theorems 3.1 and 3.2, respectively. ■

Theorem 5.5. *If $F(x)$ is a finite Laplace transform of the piecewise continuous original $f(t)$, i.e.,*

$$F(x) = \int_a^b \mathrm{e}^{-xt} f(t)\,\mathrm{d}t, \quad 0 < a < b < \infty, \tag{5.10}$$

then the Liouville–Weyl diagonal fractional integral

$$(I_-^y F)(y) = \int_a^b \mathrm{e}^{-xt} t^{-y} f(t)\,\mathrm{d}t, \quad y \in \mathbb{C} \tag{5.11}$$

is an entire function.

Proof. It is known that the finite Laplace transform of an almost piecewise continuous function is an entire function [10]. If $0 < a < b < \infty$, then the integral on the right of (5.8) is a finite Laplace transform of the almost piecewise continuous function, i.e., $(I_-^y F)(y)$, $y \in \mathbb{C}$, is an entire function. ■

Remark 5.6. The sharp inequality $0 < a$ in the hypothesis of Theorem 5.5 is essential. If $a = 0$, function $F(x)$ is entire, but its diagonal fractional integral need not be entire, because (5.8) is not a finite Laplace transform in that case.

Remark 5.7. From (5.11) it results that the question of the path of integration in the complex domain, not only for the diagonal integral but also for the general Liouville–Weyl fractional integral $(I_-^\nu F)(y)$, $\nu, y \in \mathbb{C}$, is irrelevant in the case that function $F(\cdot)$ is a Laplace transform. However, the specific path of integration often helps in the case when $\Gamma(\nu)(I_-^\nu F)(y)$, $\nu, y \in \mathbb{C}$ is used as the Euler transform for solving differential equations [17].

6. Applications

Many special functions have a connection with the Liouville–Weyl fractional integral according to (1.1)–(1.3). We will concentrate on the diagonal restriction $\nu = y$.

6.1. Gamma function

It is known [15] that there does not exist a pair $(F(x), \nu)$ such that the fractional integral (1.1) is equal to a constant. But there exists a function $F(x)$ the diagonal integral of which is equal to 1.

Lemma 6.1. *Let $G(y) = 1$, $g(u) = \delta(u)$, $u \in (-\infty, \infty)$, where $\delta(u)$ is the Dirac delta function. Then*

$$f(t) = (t+1)\delta(t + \ln t)/t = \delta\big(t - W_0(1)\big), \quad t > 0,$$

and for the Laplace transform we have

$$F(x) = \int_0^\infty \mathrm{e}^{-xt} f(t)\,\mathrm{d}t = W_0(1)^x, \quad x \in \mathbb{R}.$$

Proof. We start with the relation $\delta(t+\ln t) = \frac{\delta(t-W_0(1))}{1+1/W_0(1)}$ because the root of the equation $t+\ln t = 0$ is equal to $W_0(1)$. Then the Laplace transform of the function $f(t)$ is equal to $\exp(-xW_0(1)) = W_0(1)^x$ because $W_0(1)\exp W_0(1) = 1$. ∎

Lemma 6.2. *We have*

$$\Gamma(y) = \int_y^\infty W_0(1)^x (x-y)^{y-1}\,\mathrm{d}x, \quad y>0. \tag{6.1}$$

Proof. We have

$$(I_-^y F)(y) = \frac{1}{\Gamma(y)}\int_y^\infty W_0(1)^x (x-y)^{y-1}\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-yt} t^{-y} \delta\big(t - W_0(1)\big)\,\mathrm{d}t = 1,$$

because $W_0(1)\exp W_0(1) = 1$. Finally we get (6.1). ∎

After substituting $u = x-y$ in (6.1) we obtain its Mellin transform form

$$\Gamma(y) = W_0(1)^y \int_0^\infty W_0(1)^u u^{y-1}\,\mathrm{d}u, \tag{6.2}$$

which holds not only for $y>0$ but also for $\Re y > 0$, see [16, Entry 3.1, p. 25]. The usual Mellin transform formula for the Gamma function can be obtained from (6.1):

$$\Gamma(y) = \int_y^\infty W_0(1)(x-y)^{y-1}\,\mathrm{d}x = \int_0^\infty W_0(1)^{u/W_0(1)} u^{y-1}\,\mathrm{d}u = \int_0^\infty \mathrm{e}^{-u} u^{y-1}\,\mathrm{d}u,$$

using the linear substitution $x = y + u/W_0(1)$ and the relation $\ln(W_0(1))/W_0(1) = -1$. This proof of (6.1) is independent from generalized functions [18]. Equation (6.1) can be generalized for a complex argument:

Theorem 6.3. *The function $\Gamma(z)$, $\Re z > 0$ is an Euler integral transform [17, p. 258] with parameter z of the function $W_0(1)^u$ along path L, which is a half straight line starting at the point $z = a + \mathrm{i}c$ and is parallel to the x axis:*

$$\Gamma(z) = \int_L W_0(1)^u (u-z)^{z-1}\,\mathrm{d}u, \quad \Re z > 0. \tag{6.3}$$

Proof. Let $u = x + \mathrm{i}c$, $c \in (-\infty, \infty)$. Then

$$\begin{aligned}\int_L W_0(1)^u (u-z)^{z-1}\,\mathrm{d}u &= \int_a^\infty W_0(1)^{x+\mathrm{i}c} (x-a)^{a+\mathrm{i}c-1}\,\mathrm{d}x = \int_0^\infty W_0(1)^{y+a+\mathrm{i}c} y^{a+\mathrm{i}c-1}\,\mathrm{d}y \\ &= W_0(1)^{a+\mathrm{i}c}\int_0^\infty W_0(1)^y y^{a+\mathrm{i}c-1}\,\mathrm{d}y = \Gamma(a+\mathrm{i}c) = \Gamma(z), \end{aligned} \tag{6.4}$$

compare (6.2) or [19, Entry 3.3, p. 21]. ∎

Remark 6.4. Constant $W_0(1) = 0.56714\cdots$ is known as the Omega constant, see `http://oeis.org/A030178`.

6.2. Diagonal restriction of some special functions

Many standard special functions may be represented as the Laplace–Mellin transform in (1.3). As examples, we mention the incomplete Gamma function, the Exponential integral and the Macdonald function. The same representation occurs for a function specific for certain fields of physics. We mention here only the Bickley function, known in neutron physics [11]. Other applications, e.g., as thermonuclear reaction rate integrals [20, p. 371] are left for the reader to investigate.

6.2.1. Incomplete Gamma function

We start with the formula

$$\Gamma(\nu, y) = \frac{\mathrm{e}^{-y}}{\Gamma(1-\nu)}\int_0^\infty \mathrm{e}^{yt}\frac{t^{-\nu}}{t+1}\,\mathrm{d}t = \frac{\mathrm{e}^{-y}}{\Gamma(1-\nu)\Gamma(\nu)}\int_y^\infty \mathrm{e}^x \Gamma(0,x)(x-y)^{\nu-1}\,\mathrm{d}t, \quad y>0,\, \nu<1, \tag{6.5}$$

see [13, p. 87]. Its diagonal restriction $\nu = y$ gives

$$\begin{aligned}\Gamma(y,y) = \frac{\mathrm{e}^{-y}}{\Gamma(1-y)}\int_0^\infty \mathrm{e}^{-yt}\frac{t^{-y}}{t+1}\,\mathrm{d}t &= \frac{\mathrm{e}^{-y}}{\Gamma(y-1)}\int_{-\infty}^\infty \mathrm{e}^{yu}\frac{W_0(\mathrm{e}^u)}{(1+W_0(\mathrm{e}^u))^2}\,\mathrm{d}u \\ &= \frac{\mathrm{e}^{-y}\sin\pi y}{\pi}\int_y^\infty \mathrm{e}^x\Gamma(0,x)(x-y)^{y-1}\,\mathrm{d}x, \quad y\in(0,1). \end{aligned} \tag{6.6}$$

On the other side, integral (6.1) can be split into two parts

$$\Gamma(y) = \int_y^{y(1+1/W_0(1))} W_0(1)^x (x-y)^{y-1}\,\mathrm{d}x + \int_{y(1+1/W_0(1))}^{\infty} W_0(1)^x (x-y)^{y-1}\,\mathrm{d}x = \gamma(y,y) + \Gamma(y,y), \quad y>0, \tag{6.7}$$

see [16, Entries 3.2 and 3.3, p. 25] and (1.3), i.e., another representation of the diagonal restriction of both incomplete Gamma functions. The second integral means that $\Gamma(y,y)$ is equal to the $\Gamma(y)$ times Liouville–Weyl fractional integral of the order y at the point $y(1+1/W_0(1))$ of the function $W_0(1)^x$.

6.2.2. Exponential integral

The general exponential integral function is defined as [13, p. 132]:

$$E_\nu(y) = \int_1^\infty \mathrm{e}^{-yt} t^{-\nu}\,\mathrm{d}t = y^{\nu-1}\Gamma(1-\nu, y), \quad \Re y > 0,\ \nu \in \mathbb{C}. \tag{6.8}$$

This formula can be written in the form of the Liouville–Weyl fractional integral

$$E_\nu(y) = \int_0^\infty \mathrm{e}^{-yt} t^{-\nu} H(t-1)\,\mathrm{d}t = \frac{1}{\Gamma(\nu)} \int_y^\infty \frac{\mathrm{e}^{-x}}{x}(x-y)^{\nu-1}\,\mathrm{d}x, \quad y>0,\ \nu>0.$$

The integral on the right is the Liouville–Weyl fractional integral of the Laplace transform of the shifted Heaviside function $H(t-1)$.

Theorem 6.5. *The general exponential integral $E_\nu(y)$ is a completely monotone function in variable $y \in \mathbb{R}^+$ for fixed ν and a completely monotone function in parameter $\nu \in \mathbb{R}^+$ for fixed $y \in \mathbb{R}^+$.*

Proof. The integral on the left of (6.8) is the Laplace transform of the non-negative function in the interval $t \in [0,\infty)$ and $\nu > 0$. This means, according to Theorem 1.3, that $E_\nu(y)$ is a completely monotone function in variable y. After substituting $t = \mathrm{e}^x$ into the integral in (6.8), we obtain

$$E_\nu(y) = \int_0^\infty \mathrm{e}^{x - y\mathrm{e}^x} \mathrm{e}^{-\nu x}\,\mathrm{d}x\,.$$

This integral is the Laplace transform of a positive function, and according to Theorem 1.3 $E_\nu(y)$ is a completely monotone function in parameter ν. ∎

The diagonal restriction of $E_\nu(y)$ is

$$E_y(y) = \int_1^\infty \mathrm{e}^{-yt} t^{-y}\,\mathrm{d}t = \int_1^\infty \mathrm{e}^{-yu} \frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u = y^{y-1}\Gamma(1-y,y). \tag{6.9}$$

It is evident that $E_y(y)$ is a completely monotone function for $y \in \mathbb{R}^+$, because it is the Laplace transform of the nonnegative function $W_0(\mathrm{e}^u)/(1+W_0(\mathrm{e}^u))$.

6.2.3. Modified Bessel function of the second kind (Macdonald function) $K_\nu(x)$

We start with the relation

$$\int_0^\infty \mathrm{e}^{-yt} t^{-\nu-1} \mathrm{e}^{-1/t}\,\mathrm{d}t = 2y^{\nu/2} K_\nu(2\sqrt{y}), \quad y>0,\ \nu \geq 0. \tag{6.10}$$

According to (2.2) we have for $\nu = y$

$$2y^{y/2} K_y(2\sqrt{y}) = \int_{-\infty}^\infty \mathrm{e}^{-yu} \frac{\exp(-1/W_0(\mathrm{e}^u))}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u, \quad y>0, \tag{6.11}$$

or equivalently

$$K_y(2\sqrt{y}) = \frac{y^{-y/2}}{\Gamma(y)} \int_y^\infty K_0(2\sqrt{x})(x-y)^{y-1}\,\mathrm{d}x, y>0, \tag{6.12}$$

because

$$\int_0^\infty \mathrm{e}^{-yt} \frac{\mathrm{e}^{-1/t}}{t}\,\mathrm{d}t = 2K_0(2\sqrt{y}), \quad y>0.$$

In classical fractional calculus, we have from (6.10) the relation

$$K_\nu(2\sqrt{y}) = \frac{y^{-\nu/2}}{\Gamma(\nu)} \int_y^\infty K_0(2\sqrt{x})(x-y)^{\nu-1}\,\mathrm{d}x, \quad y>0,\ \nu>0, \tag{6.13}$$

i.e., the modified Bessel function of the second kind with index ν is given by the Liouville–Weyl fractional integral of order ν of the modified Bessel function of the second kind with zero index. Particularly if $\nu = 1$ we obtain

$$K_1(2\sqrt{y}) = y^{-1/2}\int_y^\infty K_0(2\sqrt{x})\,\mathrm{d}x, \quad y > 0.$$

Due to the semigroup property of the fractional integrals (additive index law) [1]

$$(I_-^{\nu+\mu}F)(y) = \big(I_-^\nu(I_-^\mu F)\big)(y), \quad \nu > 0,\, \mu > 0,$$

providing that $F(x) \in L^1_{\mathrm{loc}}(0,\infty)$ (space of locally integrable functions), we can rewrite (6.13) easily in the form

$$\begin{aligned} K_{\nu+\mu}(2\sqrt{y}) &= \frac{y^{-(\nu+\mu)/2}}{\Gamma(\nu)}\int_y^\infty x^{\mu/2}K_\mu(2\sqrt{x})(x-y)^{\nu-1}\,\mathrm{d}x \\ &= \frac{y^{-(\nu+\mu)/2}}{\Gamma(\mu)}\int_y^\infty x^{\nu/2}K_\nu(2\sqrt{x})(x-y)^{\mu-1}\,\mathrm{d}x, \quad y > 0,\, \nu > 0. \end{aligned} \tag{6.14}$$

6.2.4. Bickley function

The Bickley function of order ν is defined by the fractional integral [11, Entry 10.43.11, p. 259]

$$Ki_\nu(y) = \frac{1}{\Gamma(\nu)}\int_y^\infty Ki_0(x)(x-y)^{\nu-1}\,\mathrm{d}x, \quad y > 0,\, \nu > 0, \tag{6.15}$$

where $Ki_0(x) = K_0(x)$. Because

$$K_0(y) = \int_1^\infty \frac{\mathrm{e}^{-yu}}{\sqrt{u^2-1}}\,\mathrm{d}u, \quad \Re y > 0,$$

it holds that the Bickley function is given by the Laplace–Mellin transform

$$Ki_\nu(y) = \int_1^\infty \frac{\mathrm{e}^{-yu}u^{-\nu}}{\sqrt{u^2-1}}\,\mathrm{d}u, \quad \Re y > 0,\, \nu > 0. \tag{6.16}$$

From (6.16), it is evident that the Bickley function is completely monotone in y for fixed ν and completely monotone in ν for fixed y.

Diagonal restriction of (6.16) in the form of the pure Laplace or Mellin transform can be obtained very simply, and is not introduced here. Instead the Mellin transform pair in standard notation for the Bickley function is elicited.

Theorem 6.6. *The Mellin transform of the Bickley function is given by the formula*

$$\int_0^\infty y^{p-1}Ki_\nu(y)\,\mathrm{d}y = \sqrt{\frac{\pi}{4}}\frac{\Gamma(p)\Gamma(\frac{p}{2}+\frac{\nu}{2})}{\Gamma(\frac{p}{2}+\frac{\nu}{2}+\frac{1}{2})}, \quad \Re p > 0,\, \nu > 0. \tag{6.17}$$

Proof. We have

$$\begin{aligned} \int_0^\infty y^{p-1}Ki_\nu(y)\,\mathrm{d}y &= \int_1^\infty \frac{u^{-\nu}}{\sqrt{u^2-1}}\int_0^\infty y^{p-1}\mathrm{e}^{-yu}\,\mathrm{d}y\,\mathrm{d}u \\ &= \Gamma(p)\int_1^\infty \frac{u^{-\nu-p}}{\sqrt{u^2-1}}\,\mathrm{d}u = \sqrt{\frac{\pi}{4}}\frac{\Gamma(p)\Gamma(\frac{p}{2}+\frac{\nu}{2})}{\Gamma(\frac{p}{2}+\frac{\nu}{2}+\frac{1}{2})}, \quad \Re p > 0,\, \nu > 0, \end{aligned}$$

where (6.16) and the Fubini theorem have been applied. ■

The fact that the region of holomorphy of the Mellin transform (6.17) is unbounded from above unveils that

$$M_n = \int_0^\infty y^n Ki_\nu(y)\,\mathrm{d}y = \sqrt{\frac{\pi}{4}}\frac{\Gamma(n+1)\Gamma(\frac{n}{2}+\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+\frac{\nu}{2}+1)}, \quad \nu > 0,\, n = 0,1,2,\ldots$$

is the n-th Stieltjes moment of the Bickley function $Ki_\nu(y)$.

6.3. Integral transform pairs containing the Lambert function

We use (2.4) from Corollary 2.4 and (2.5) from Corollary 2.5.

Example 6.7. Let $h(t) = t$. Then

$$\int_1^\infty \mathrm{e}^{-yu} W_0(\mathrm{e}^u)\,\mathrm{d}u = \int_{\mathrm{e}}^\infty u^{-y-1} W_0(u)\,\mathrm{d}u = \int_1^\infty \mathrm{e}^{-yt} t^{-y}(1+t)\,\mathrm{d}t = \frac{E_y(y)+\mathrm{e}^{-y}}{y}, \quad \Re y > 0, \tag{6.18}$$

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-yu} W_0(\mathrm{e}^u)\,\mathrm{d}u = \int_1^\infty u^{-y-1} W_0(u)\,\mathrm{d}u &= \int_{W_0(1)}^\infty \mathrm{e}^{-yt} t^{-y}(1+t)\,\mathrm{d}t \\ &= W_0(1)\frac{W_0(1)^{-y} E_y(yW_0(1)) + 1}{y}, \quad \Re y > 0,\end{aligned}$$

$$\int_{-\infty}^\infty \mathrm{e}^{-yu} W_0(\mathrm{e}^u)\,\mathrm{d}u = \int_0^\infty u^{-y-1} W_0(u)\,\mathrm{d}u = \int_0^\infty \mathrm{e}^{-yt} t^{-y}(1+t)\,\mathrm{d}t = y^{y-2}\Gamma(1-y), \quad 0 < \Re y < 1. \tag{6.19}$$

Inversion of the Mellin transform in (6.18) gives

$$W_0(u) = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} u^y \frac{E_y(y)+\mathrm{e}^{-y}}{y}\,\mathrm{d}y, \quad c > 0,\, u > \mathrm{e} \tag{6.20}$$

and inversion in (6.19) gives

$$W_0(u) = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} u^y y^{y-2}\Gamma(1-y)\,\mathrm{d}y, \quad c \in (0,1),\, u \geq 0. \tag{6.21}$$

This is a representation of $W_0(u)$ in the form of complex integrals. It should be noted, however, that the integral in (6.20) defines a function different from the function defined by the integral in (6.21). The two functions are equivalent for $u > \mathrm{e}$. But (6.21) is equal to $W_0(u)$ also for $0 < u < \mathrm{e}$, whereas (6.20) is equal to zero for $0 < u < \mathrm{e}$ and to $W_0(\mathrm{e})/2$ for $u = \mathrm{e}$. This is typical for integration along the Bromwich contour. Equation (6.18) is compelling also for another reason given by the following conjecture that links the Lambert function and the residue of another function:

Conjecture 6.8. Function $u^y(E_y(y)+\mathrm{e}^{-y})/y$ of complex variable $y \in \mathbb{C}$ has for fixed $u \in \mathbb{C}$ only one singular point $y = 0$ in which this function has non-removable point singularity. If this is true, then

$$W_0(u) = \operatorname{Res}\Big(u^y \frac{E_y(y)+\mathrm{e}^{-y}}{y}, y = 0\Big), \quad u \geq \mathrm{e}. \tag{6.22}$$

This formula is a consequence of the Cauchy residue theorem and the Jordan lemma because the Bromwich contour in (6.20) can be closed to the left for $u > \mathrm{e}$ and $c > 0$. Indeed, because $E_\nu(y) \propto \mathrm{e}^{-y}/y$, $y \to \infty$ independently on ν, it holds that $u^y(E_y(y)+\mathrm{e}^{-y})/y \propto \mathrm{e}^{-y}/y$, $y \to \infty$ and $|u^y(E_y(y)+\mathrm{e}^{-y})/y| \to 0$ for $|y| \to \infty$, $u > \mathrm{e}$, $\Re y < c$.

This conjecture is supported by numerical experiments in *Mathematica*®. Numerical calculation of the residue by closed contour integration in the complex plane by the built-in *NIntegrate* function gave excellent results.

The Laplace transform of (6.21) gives the following transform pair:

Theorem 6.9. *We have*

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-pu} W_0(u)\,\mathrm{d}u &= \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} p^{-1-y} y^{y-2}\Gamma(1-y)\Gamma(1+y)\,\mathrm{d}y \\ &= \frac{1}{2\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} p^{-1-y}\frac{y^{y-1}}{\sin \pi y}\,\mathrm{d}y, \quad \Re p > 0,\, c \in (0,1).\end{aligned} \tag{6.23}$$

Proof. Perform the Laplace transform of (6.21) and change the order of integration according to the Fubini theorem. Then the Laplace transform of the function $\phi(u) = u^y$ is $p^{-1-y}\Gamma(1+y)$. The integrand in the second integral on the right side is given by the reflection formula $\Gamma(1+y)\Gamma(1-y) = \pi y/\sin \pi y$. ∎

Both complex integrals on the right side of (6.23) are de facto Bromwich inversion integrals of the Mellin transform and converge for all complex p except $p = 0$, while the Laplace integral on the left side converges for $\Re p > 0$ only. We thus have the analytic continuation of the function defined by the Laplace integral on the left to the region $\mathbb{C} \setminus \{0\}$.

Example 6.10. Let $h(t) = t/(1+t)$. Then

$$\int_1^\infty \mathrm{e}^{-yu} \frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u = \int_\mathrm{e}^\infty u^{-y-1} \frac{W_0(u)}{1+W_0(u)}\,\mathrm{d}u = \int_1^\infty \mathrm{e}^{-yt} t^{-y}\,\mathrm{d}t = E_y(y) = y^{y-1}\Gamma(1-y,y), \quad \Re y > 0,$$
$$\int_0^\infty \mathrm{e}^{-yu} \frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u = \int_1^\infty u^{-y-1} \frac{W_0(u)}{1+W_0(u)}\,\mathrm{d}u = \int_{W_0(1)}^\infty \mathrm{e}^{-yt} t^{-y}\,\mathrm{d}t = y^{y-1}\Gamma\big(1-y, yW_0(1)\big), \quad \Re y > 0,$$
$$\int_{-\infty}^\infty \mathrm{e}^{-yu} \frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u = \int_0^\infty u^{-y-1} \frac{W_0(u)}{1+W_0(u)}\,\mathrm{d}u = \int_0^\infty \mathrm{e}^{-yt} t^{-y}\,\mathrm{d}t = -y^y\Gamma(-y), \quad 0 < \Re y < 1.$$

Example 6.11. Let $h(t) = 1/(1+t)$. Then

$$\int_1^\infty \frac{\mathrm{e}^{-yu}}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u = \int_\mathrm{e}^\infty \frac{u^{-y-1}}{1+W_0(u)}\,\mathrm{d}u = \int_1^\infty \mathrm{e}^{-yt} \frac{t^{-y}}{t}\,\mathrm{d}t = E_{1+y}(y) = y^y\Gamma(-y,y), \quad \Re y > 0,$$
$$\int_0^\infty \frac{\mathrm{e}^{-yu}}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u = \int_1^\infty \frac{u^{-y-1}}{1+W_0(u)}\,\mathrm{d}u = \int_{W_0(1)}^\infty \mathrm{e}^{-yt} \frac{t^{-y}}{t}\,\mathrm{d}t = y^y\Gamma\big(-y, yW_0(1)\big), \quad \Re y > 0.$$

Moreover function $1/(1+W_0(\mathrm{e}^u))$ is also a Laplace transform [14]:

$$\int_0^\infty \mathrm{e}^{-ux} x^{-x} \sin \pi x\, \Gamma(x)\,\mathrm{d}x = \frac{\pi}{1+W_0(\mathrm{e}^u)}, \quad u > -1.$$

This means that the following Stieltjes transform pair holds:

$$\int_0^\infty x^{-x}\mathrm{e}^{-x} \frac{\sin \pi x\, \Gamma(x)}{y+x}\,\mathrm{d}x = \pi \mathrm{e}^y y^y \Gamma(-y,y), \quad y \in \mathbb{C} \setminus (-\infty, 0], \tag{6.24}$$
$$\int_0^\infty x^{-x} \frac{\sin \pi x\, \Gamma(x)}{y+x}\,\mathrm{d}x = \pi y^y \Gamma(-y, yW_0(1)), \quad y \in \mathbb{C} \setminus (-\infty, 0]. \tag{6.25}$$

We now take the finite Laplace transform

$$Q_x(y) = \int_1^{x+\ln x} \frac{\mathrm{e}^{-yu}}{1+W_0(\mathrm{e}^u)} = \int_\mathrm{e}^{x\exp x} \frac{u^{-y-1}}{1+W_0(u)}\,\mathrm{d}u = \int_1^x \mathrm{e}^{-yt}\frac{t^{-y}}{t}\,\mathrm{d}t = y^y\big(\Gamma(-y,y) - \Gamma(-y,xy)\big), \quad x > 0.$$

Because the finite Laplace transform is an entire function in the transform variable y it holds that $Q_x(0) = \int_1^x \frac{1}{t}\,\mathrm{d}t = \ln x$, which means that

$$\lim_{y\to 0} y^y\big(\Gamma(-y,y) - \Gamma(-y,xy)\big) = \ln x, \quad x > 0, \tag{6.26}$$
$$\int_1^a \frac{1}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u = \ln W_0(\mathrm{e}^a) = a - W_0(\mathrm{e}^a). \tag{6.27}$$

Example 6.12. Let $h(t) = t/(1+t)^3$. Then

$$\int_{-\infty}^\infty \mathrm{e}^{-yu} \frac{W_0(\mathrm{e}^u)}{(1+W_0(\mathrm{e}^u))^3} = \int_0^\infty u^{-y-1} \frac{W_0(u)}{(1+W_0(u))^3}\,\mathrm{d}u$$
$$= \int_0^\infty \mathrm{e}^{-yt} \frac{t^{-y}}{(1+t)^2}\,\mathrm{d}t = y^y\Gamma(1-y), \quad 0 < \Re y < 1. \tag{6.28}$$

Examples 6.7, 6.10–6.12 have one common feature. This is the function y^y, first studied by Johann Bernoulli in the 17th century. It is known that this function is a bilateral Laplace transform of the Landau probability density function that describes the energy loss of a fast charged particle by ionization while passing through a thin layer of a matter [14]:

$$y^y = \int_{-\infty}^\infty \mathrm{e}^{-yu} L(u)\,\mathrm{d}u, \quad \Re y > 0,$$

where

$$L(u) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-ux} x^{-x} \sin \pi x\,\mathrm{d}x, \quad u > -\infty$$

is the Landau p.d.f. For example, the left hand side $y^y\Gamma(1-y)$, $0 < \Re y < 1$, of (6.28) is a bilateral Laplace transform of the convolution

$$\int_{-\infty}^\infty \exp(-\mathrm{e}^{x-t})\mathrm{e}^{x-t} L(t)\,\mathrm{d}t,$$

and from this and from (6.28) it follows that

$$\int_{-\infty}^{\infty} \exp(-\mathrm{e}^{x-t})\mathrm{e}^{x-t} L(t)\,\mathrm{d}t = \frac{W_0(\mathrm{e}^x)}{(1+W_0(\mathrm{e}^x))^3}, \quad x \in \mathbb{R}.$$

Example 6.13. Let $h(t) = t^2\mathrm{e}^t/(1+t)$, $t > \ln 2$. This example originates from the theory of the distribution of prime numbers [21] and consists in calculating the integral

$$\int_{\ln 2}^{\infty} t^{-(y-1)}\mathrm{e}^{-(y-1)t}\,\mathrm{d}t = \int_{2\ln 2}^{\infty} u^{-y-1} \exp W_0(u) \frac{\big(W_0(u)\big)^2}{1+W_0(u)}\,\mathrm{d}u = (y-1)^{y-2}\Gamma\big(2-y,(y-1)\ln 2\big), \quad \Re y > 1. \tag{6.29}$$

Example 6.14. Let $h(t) = \sin t$. The task is to calculate

$$\int_{-\infty}^{\infty} \mathrm{e}^{-yu} \sin W_0(\mathrm{e}^u)\,\mathrm{d}u = \int_0^{\infty} u^{-y-1} \sin W_0(\mathrm{e}^u)\,\mathrm{d}u\,.$$

According to Corollary 2.4, these integrals are equal to

$$\begin{aligned}\int_0^{\infty} \mathrm{e}^{-yt} t^{-y} \frac{(1+t)\sin t}{t}\,\mathrm{d}t &= -\frac{1}{2}\big((y-\mathrm{i})^{y-1} + (y+\mathrm{i})^{y-1}\big)\Gamma(-y) \\ &= -(y^2+1)^{(y-1)/2}\Gamma(-y)\cos\big((y-1)\arctan\tfrac{1}{y}\big), \quad 0 < \Re y < 1,\end{aligned}$$

where $\mathrm{i} = \sqrt{-1}$.

7. Eigenproblem

The next topic in this paper is the problem of real eigenvalues and eigenvectors in either form:

$$\frac{1}{\Gamma(y)} \int_y^{\infty} F(x)(x-y)^{y-1}\,\mathrm{d}x = \lambda F(y), \tag{7.1}$$

$$\frac{\mu}{\Gamma(y)} \int_y^{\infty} F(x)(x-y)^{y-1}\,\mathrm{d}x = F(y). \tag{7.2}$$

It can easily be shown that $F(y) = \mathrm{e}^{-y}$ is an eigenfunction of the Liouville–Weyl diagonal fractional integral for the eigenvalue $\mu = \lambda = 1$. The determining function (original) $f(t)$ of $F(y)$ is the Dirac delta-function $\delta(t-1)$. There exists at least one other real eigenvalue.

Theorem 7.1. *We have that $\lambda = 1/2$ is the eigenvalue of* (7.1).

Proof. Equation (7.1) is equivalent to

$$\int_1^{\infty} \mathrm{e}^{-yu} f\big(W_0(\mathrm{e}^u)\big) \frac{W_0(\mathrm{e}^u)}{1+W_0(\mathrm{e}^u)}\,\mathrm{d}u = \lambda \int_1^{\infty} \mathrm{e}^{-yu} f(u)\,\mathrm{d}u, \tag{7.3}$$

where the lower limit of the integrals must be equal to 1 because this value satisfies the requirement $x = x + \ln x$, and the Laplace transform on the right must be unilateral, see (1.2). In this case, the uniqueness (Lerch) theorem [10] can be applied and the functional equation

$$f\big(W_0(\mathrm{e}^u)\big) = \lambda f(u) \frac{1+W_0(\mathrm{e}^u)}{W_0(\mathrm{e}^u)}, \quad 1 \le u < \infty, \tag{7.4}$$

is to be solved. According to [22, Eqs. (2.3.6)–(2.3.7), p. 63] we can deduce that $\lambda = 1/2$ (see later) is a necessary condition for $f(u)$ being a monotonic solution of (7.3):

$$f(u) = f(u_0) \prod_{n=0}^{\infty} \frac{p(q^n(u_0))}{p(q^n(u))}, \tag{7.5}$$

where $q(u) = W_0(\mathrm{e}^u)$, $q^n(u)$ is the n-th iterate (not power) of the function $q(u)$, $p(u) = \lambda(1+q(u))/q(u)$ and $1 \le u_0 < \infty$ is fixed. This implies that the eigenvalue $\lambda = 1/2$ is degenerated. The fact that $\lim_{n\to\infty} q^n(u) = 1$, $\lim_{n\to\infty} p(q^n(u)) = 2\lambda$ for $1 \le u < \infty$ implies that λ must be equal to 1/2, because a necessary condition for (7.5) is $\lim_{n\to\infty} p(q^n(u)) = 2\lambda$ for $1 \le u < \infty$ (see [22, p. 63]). ∎

Function $f(u)$ in (7.5) was calculated numerically with the aid of the *Mathematica* function *Nest* for calculating the iterate of the function.

8. Generalization

The diagonal fractional integral is the simplest form of the variable order fractional integral

$$G(y) = (I_-^{\nu(y)}F)(y) = \frac{1}{\Gamma(\nu(y))}\int_y^\infty F(x)(x-y)^{\nu(y)-1}\,\mathrm{d}x = \frac{1}{\Gamma(\nu(y))}\int_0^\infty F(u+y)y^{\nu(y)-1}\,\mathrm{d}u, \quad \Re\nu(y) > 0. \tag{8.1}$$

Then integral (8.1) is equivalent to

$$G(y) = \int_0^\infty \mathrm{e}^{-yt}t^{-\nu(t)}f(t)\,\mathrm{d}t, \tag{8.2}$$

providing that $F(x)$ is a unilateral Laplace transform of function $f(t)$.

Example 8.1. Let $\nu(y) = \frac{1}{2}\sin y = S(y)$ and $f(t) = 1$. Then we obtain

$$\int_0^\infty \mathrm{e}^{-yt}t^{-S(y)}\,\mathrm{d}t = y^{S(y)-1}\Gamma\big(1 - S(y)\big), \quad y > 0, \tag{8.3}$$

which represents values of the fractional integral (8.1) of the function $F(x) = 1/x$ (i.e., $f(t) = 1$, $t > 0$) of the order $\nu(y) = \frac{1}{2}\sin y$ in regions where the sinusoid has non-negative values and the fractional derivative of the same order of the function $F(x)$, otherwise [23]. The connection to the Lambert function for a general variable order fractional integral is lost, but for the linear order $\nu(y) = a + by \geq 0$, equation (8.2) has the Laplace transform form

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-yt}t^{-1-by}f(t)\,\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-y(t+b\ln t)}t^{-a}f(t)\,\mathrm{d}t\\ &= b^{-a}\int_{-\infty}^\infty \mathrm{e}^{-yu}f\big(bW_0(\mathrm{e}^{u/b}/b)\big)\frac{(W_0(\mathrm{e}^{u/b}/b))^{1-a}}{1 + W_0(\mathrm{e}^{u/b}/b)}\,\mathrm{d}u, \quad y > 0. \end{aligned}\tag{8.4}$$

In the case that $a < 0$ and $b > 0$ there exists a critical point $c = -a/b$ such that for $y > c$, Equation (8.4) represents the Liouville–Weyl fractional integral and for $y < c$ the Liouville–Weyl fractional derivative [23]. The same critical point $c = -a/b$ exists for $a > 0$ and $b < 0$. In that case (8.4) represents fractional integral for $y < c$ and fractional derivative for $y > c$.

Example 8.2. Let $f(t) = \sin t$, $a = -3$, $b = 1$. Then $F(x) = 1/(1 + x^2)$ and $c = 3$. Then integral (8.4) gives:

$$\int_{-\infty}^\infty \mathrm{e}^{-yt}t^{3-y}\sin t\,\mathrm{d}t = -(y^2+1)^{(y-4)/2}\Gamma(4-y)\sin\big((y-4)\arctan\frac{1}{y}\big).$$

This gives for $y < 3$ the Liouville–Weyl fractional derivative and for $3 < y < 5$ the fractional integral of the function $1/(1 + x^2)$ both of the variable order $|-3 + y|$ at point y. It must be emphasized, however, that the case $b < 0$ essentially changes the situation. This is the topic of [23].

Remark 8.3. We call the transform $\int_0^\infty \mathrm{e}^{-yt}t^{-y}f(t)\,\mathrm{d}t$, $y > 0$, the anti-diagonal Laplace–Mellin transform because the term diagonal Laplace–Mellin transform is reserved for the transformation $\int_0^\infty \mathrm{e}^{-yt}t^{y}f(t)\,\mathrm{d}t$, $y > 0$, which is closely related to the fractional derivative [23].

9. Conclusion

This paper has focused on the simplest form of the variable order Liouville–Weyl fractional integral, where the order $\nu(y) = y$. The Liouville–Weyl fractional integral is not so often used in physical and technical applications as the Riemann–Liouville integral, but the Liouville–Weyl fractional integral fulfills the relation of (1.3). This makes it possible to take advantage of the Laplace transform or the Mellin transform, and to find a connection to the Lambert function. Numerical calculations have been performed with the aid of *Mathematica*® ver. 8.0.1.0. Analytical calculations were checked by the same version of *Mathematica*. In several cases, the formulas generated by *Mathematica* have been used instead of the equivalent formulas presented in [12, 16, 19].

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