BIPERIODIC FIBONACCI WORD AND ITS FRACTAL CURVE

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ABSTRACT. In the present article, we study a word-combinatorial interpretation of the biperiodic Fibonacci sequence of integer numbers $(F_n^{(a,b)})$. This sequence is defined by the recurrence relation $aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}$ if n is even and $bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}$ if n is odd, where a and b are any real numbers. This sequence of integers is associated with a family of finite binary words, called finite biperiodic Fibonacci words. We study several properties, such as the number of occurrences of 0 and 1, and the concatenation of these words, among others. We also study the infinite biperiodic Fibonacci word, which is the limiting sequence of finite biperiodic Fibonacci words. It turns out that this family of infinite words are Sturmian words of the slope $[0, \overline{a, b}]$. Finally, we associate to this family of words a family of curves with interesting patterns.

KEYWORDS: Fibonacci word, biperiodic Fibonacci word, biperiodic Fibonacci curve.

1. INTRODUCTION

The Fibonacci numbers and their generalizations have many interesting properties and combinatorial interpretations, see, e.g., [13]. The Fibonacci numbers F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for all integer $n \geq 2$, and with initial values $F_0 = 1 = F_1$.

Many kinds of generalizations of the Fibonacci sequence have been presented in the literature. For example, Edson and Yayenie [12] introduced the biperiodic Fibonacci sequence $\left\{ F_n^{(a,b)} \right\}_{n \in \mathbb{N}}$. For any two nonzero real numbers a and b, this is defined recursively by

$$F_0^{(a,b)} = 0, \quad F_1^{(a,b)} = 1,$$
$$F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & \text{if } n \geq 2 \text{ is even,} \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & \text{if } n \geq 2 \text{ is odd.} \end{cases}$$

To avoid cumbersome notation, let us denote $F_n^{(a,b)}$ by q_n. The first few terms are

$$\{q_n\}_{n=0}^{\infty} = \{0, 1, a, ab+1, a^2b + 2a,$$
$$a^2b^2 + 3ab + 1, a^3b^2 + 4a^2b + 3a, \dots \}.$$

Note that if $a = b = 1$, then q_n is the nth Fibonacci number. A Binet-like formula to the biperiodic Fibonacci sequence is

$$q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \right) \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1}$$

where

$$\alpha = \frac{ab + \sqrt{(ab)^2 + 4ab}}{2}, \quad \beta = \frac{ab - \sqrt{(ab)^2 + 4ab}}{2},$$
$$\text{and} \quad \xi(n) := n - 2\left\lfloor \frac{n}{2} \right\rfloor.$$

On the other hand, there exists a well-known word-combinatorial interpretation of the Fibonacci sequence. Let f_n be a binary word defined inductively as follows

$$f_0 = 1, \quad f_1 = 0, \quad f_n = f_{n-1}f_{n-2}, \quad \text{for } n \geq 2.$$

It is clear that $|f_n| = F_n$, i.e., the length of the word f_n is the nth Fibonacci number. The words f_n are called *finite Fibonacci words*. The infinite Fibonacci word,

$$\boldsymbol{f} = 01001010010010100101001001010010101 \cdots$$

is defined by the limit sequence of the infinite sequence $\{f_n\}_{n \in \mathbb{N}}$. It is the archetype of a Sturmian word [2, 14], and one of the most studied examples in the combinatorial theory of infinite words; see, e.g., [3, 5, 6, 8, 10, 15, 17].

The word $\boldsymbol{f}$ can be associated with a curve from a drawing rule, which has geometric properties obtained from the combinatorial properties of $\boldsymbol{f}$ [4, 16]. The curve produced depends on the rules given. We read the symbols of the word in order and depending on what we read, we draw a line segment in a certain direction; this idea is the same as that used in L-Systems [18]. In this case, the drawing rule is called the "odd-even drawing rule" [16]. This is defined as shown in Table 1.

The *nth-curve of Fibonacci*, denoted by $\mathcal{F}_n$, is obtained by applying the odd-even drawing rule to the word f_n. The *Fibonacci word fractal* $\mathcal{F}$, is defined as $\mathcal{F} = \lim_{n \to \infty} \mathcal{F}_n$.

For example, in Figure 1, we show the curve $\mathcal{F}_{10}$ and $\mathcal{F}_{17}$. The graphics in the present article were generated using the Mathematica 9.0 software, [19, 21].

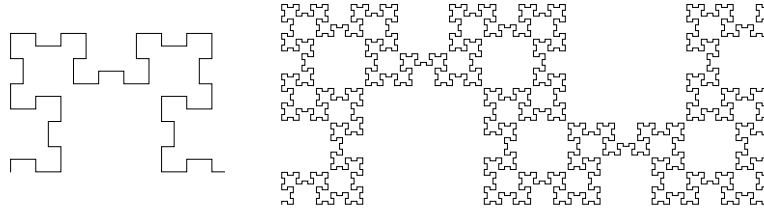$$f_{10} = 01001010010010100101001001010010 \\ 00101001010010010100101001001010 \\ 01001010010100100101001001.$$

FIGURE 1. Fibonacci curves $\mathcal{F}_{10}$ and $\mathcal{F}_{17}$ corresponding to the words f_{10} and f_{17}.

Symbol	Action
1	Draw a line forward.
0	Draw a line forward and if the symbol 0 is in an even position, then turn left and if 0 is in an odd position, then turn right.

TABLE 1. Odd-even drawing rule.

Ramírez et al. [22] introduced a generalization of the Fibonacci word, the i-Fibonacci word. Specifically, the (n, i)-*Fibonacci words* are words over {0,1} defined inductively as follows: $f_0^{[i]} = 0$, $f_1^{[i]} = 0^{i-1}1$, $f_n^{[i]} = f_{n-1}^{[i]} f_{n-2}^{[i]}$, for $n \geq 2$ and ≥ 1. The infinite word $\boldsymbol{f}^{[i]} := \lim_{n\to\infty} f_n^{[i]}$ is called the *i-Fibonacci word*. For $i = 2$, we have the classical Fibonacci word. Note that $|f_n^{[i]}| = F_n^{[i]}$, where $F_n^{[i]}$ is the integer sequence defined recursively by $F_0^{[i]} = 1, F_1^{[i]} = i, F_n^{[i]} = F_{n-1}^{[i]} + F_{n-2}^{[i]}$, for $n \geq 2$, and $i \geq 1$. For $i = 1, 2$ we have the Fibonacci numbers.

Ramírez and Rubiano [20] studied a similar binary word to $f_n^{[i]}$, which is denoted by $f_{k,n}$. The initial values are the same, i.e., $f_{k,0} = 0, f_{k,1} = 0^{k-1}1$. However, the recurrence relation is defined by $f_{k,n} = f_{k,n-1}^k f_{k,n-2}$, for $n \geq 2$, and $k \geq 1$. The infinite word $\boldsymbol{f}_k$ is the limit sequence of the infinite sequence $\{f_{k,n}\}_{n\in\mathbb{N}}$. For $k = 1$, we have the word $\overline{\boldsymbol{f}} = 1011010110110\ldots$. Here the overline is shorthand for the morphism that maps 0 to 1 and 1 to 0. It is clear that $|f_{k,n}| = F_{k,n+1}$, where $F_{k,n+1}$ is the integer sequence defined recursively by $F_{k,0} = 0, F_{k,1} = 1, F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, for $n \geq 1$.

By analogy with the Fibonacci word fractal, when we apply the odd-even drawing rule to the words $f_n^{[i]}$ and $f_{k,n}$, we obtain the nth word fractal $\mathcal{F}_n^{[i]}$ and $\mathcal{F}_{k,n}$, respectively. Moreover, we have the curves $\mathcal{F}^{[i]}$ and $\mathcal{F}_k$, which are defined as $\mathcal{F}^{[i]} = \lim_{n\to\infty} \mathcal{F}_n^{[i]}$ and $\mathcal{F}_k = \lim_{n\to\infty} \mathcal{F}_{k,n}$. In Table 2, we show some curves $\mathcal{F}_{16}^{[i]}$ and $\mathcal{F}_{k,n}$, and their associated words.

In this paper, we study a word-combinatorial interpretation of the biperiodic Fibonacci sequence [12]. This problem was recently proposed by Ramírez et al. [22]. We study a family of infinite words $\boldsymbol{f}_{(a,b)}$ that generalize the Fibonacci word and the word $\boldsymbol{f}_k$. Specifically, the nth *biperiodic Fibonacci words* are words over {0,1} defined inductively as follows

$$f_{(a,b,0)} = \epsilon, \quad f_{(a,b,1)} = 0, \quad f_{(a,b,2)} = 0^{a-1}1,$$

$$f_{(a,b,n)} = \begin{cases} f_{(a,b,n-1)}^a f_{(a,b,n-2)}, & \text{if } n \geq 3 \text{ is even,} \\ f_{(a,b,n-1)}^b f_{(a,b,n-2)}, & \text{if } n \geq 3 \text{ is odd,} \end{cases}$$

for all $a, b \geq 1$. It is clear that $|f_{(a,b,n)}| = F_n^{(a,b)} = q_n$. The infinite word

$$\boldsymbol{f}_{(a,b)} := \lim_{n\to\infty} f_{(a,b,n)},$$

is called the *biperiodic Fibonacci word*. In addition to this definition, we investigate some new combinatorial properties and we associate a family of curves with interesting geometric properties. These properties are obtained from the combinatorial properties of the word $\boldsymbol{f}_{(a,b)}$.

2. DEFINITIONS AND NOTATION

The terminology and notations are mainly those of Lothaire [14] and Allouche and Shallit [1]. Let Σ be a finite alphabet, whose elements are called *symbols*. A *word* over Σ is a finite sequence of symbols from Σ. The set of all words over Σ, i.e., the free monoid generated by Σ, is denoted by Σ^*. The identity element ϵ of Σ^* is called the *empty word*. For any word $w \in \Sigma^*$, $|w|$ denotes its *length*, i.e., the number of symbols occurring in w. The length of ϵ is taken to be equal to 0. If $a \in \Sigma$ and $w \in \Sigma^*$, then $|w|_a$ denotes the number of occurrences of a in w.

For two words $u = a_1 a_2 \cdots a_k$ and $v = b_1 b_2 \cdots b_s$ in Σ^* we denote by uv the *concatenation* of the two words, that is, $uv = a_1 a_2 \cdots a_k b_1 b_2 \cdots b_s$. If $v = \epsilon$, then $u\epsilon = \epsilon u = u$. Moreover, by u^n we denote the word $uu\cdots u$ (n times). A word v is a *factor* or *subword* of u if there exist $x, y \in \Sigma^*$ such that $u = xvy$. If $x = \epsilon$ ($y = \epsilon$), then v is called a *prefix* (*suffix*) of u.

The *reversal* of a word $u = a_1 a_2 \cdots a_n$ is the word $u^R = a_n \cdots a_2 a_1$ and $\epsilon^R = \epsilon$. A word u is a *palindrome* if $u^R = u$.

An *infinite word* over Σ is a map $\boldsymbol{u} : \mathbb{N} \to \Sigma$. It is written $\boldsymbol{u} = a_1 a_2 a_3 \ldots$. The set of all infinite words over Σ is denoted by Σ^ω.
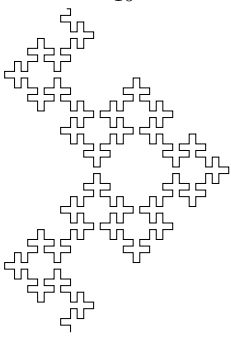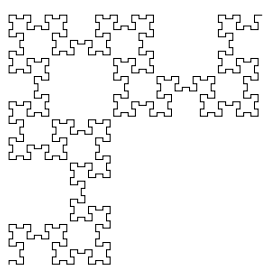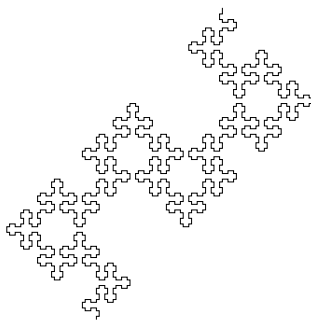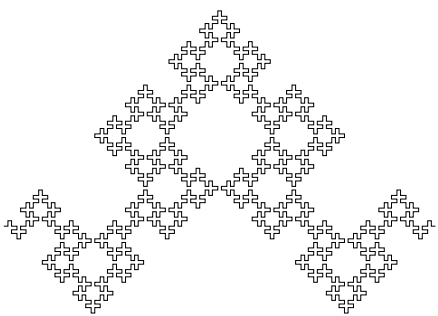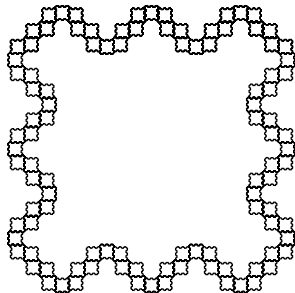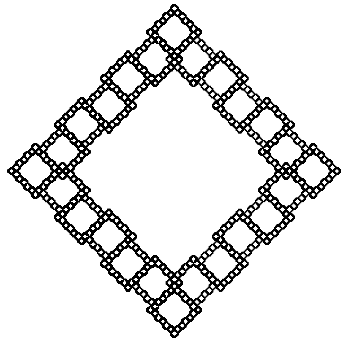
$\boldsymbol{f}^{[1]} = 1011010110110\cdots = \overline{\boldsymbol{f}}$ $\boldsymbol{f}^{[2]} = 0100101001001\cdots = \boldsymbol{f}$ $\boldsymbol{f}^{[3]} = 0010001001000\cdots$
$\mathcal{F}_{16}^{[1]}$ $\mathcal{F}_{16}^{[2]}$ $\mathcal{F}_{16}^{[3]}$

$\boldsymbol{f}_1 = 1011010110110\cdots = \overline{\boldsymbol{f}}$ $\boldsymbol{f}_5 = 0000100001000\cdots$ $\boldsymbol{f}_6 = 0000010000010\cdots$
$\mathcal{F}_{1,18}$ $\mathcal{F}_{5,6}$ $\mathcal{F}_{6,6}$

TABLE 2. Some curves $\mathcal{F}_{16}^{[i]}$ and $\mathcal{F}_{k,n}$.

Example 1. Let $\boldsymbol{p} = (p_n)_{n \geq 1} = 0110101000101\cdots$ be an infinite word, where $p_n = 1$ if n is a prime number and $p_n = 0$ otherwise. The word $\boldsymbol{p}$ is called the characteristic sequence of the prime numbers.

Let Σ and Δ be alphabets. A *morphism* is a map $h : \Sigma^* \to \Delta^*$ such that $h(xy) = h(x)h(y)$ for all $x, y \in \Sigma^*$. It is clear that $h(\epsilon) = \epsilon$. Furthermore, a morphism is completely determined by its action on single symbols. For instance, the Fibonacci word $\boldsymbol{f}$ satisfies $\lim_{n\to\infty} \sigma^n(1) = \boldsymbol{f}$, where $\sigma : \{0,1\} \to \{0,1\}^*$ is the morphism defined by $\sigma(0) = 01$ and $\sigma(1) = 0$. This morphism is called *the Fibonacci morphism*. The Fibonacci word $\boldsymbol{f}$ can be defined in several different ways, see, e.g., [3].

There is a special class of infinite words, with many remarkable properties, the so-called Sturmian words. These words admit several equivalent definitions (see, e.g. [1, 2, 14]). Let $\boldsymbol{w} \in \Sigma^\omega$. We define $P(\boldsymbol{w}, n)$, the *complexity function* of $\boldsymbol{w}$, to be the map that counts, for all integer $n \geq 0$, the number of subwords of length n in $\boldsymbol{w}$. An infinite word $\boldsymbol{w}$ is a *Sturmian word* if $P(\boldsymbol{w}, n) = n + 1$ for all integers $n \geq 0$. Since for any Sturmian word $P(\boldsymbol{w}, 1) = 2$, Sturmian words are over two symbols. The word $\boldsymbol{p}$, in Example 1, is not a Sturmian word because $P(\boldsymbol{p}, 2) = 4$.

Given two real numbers $\theta, \rho \in \mathbb{R}$ with θ irrational and $0 < \theta < 1$, $0 \leq \rho < 1$, we define the infinite word $\boldsymbol{w} = w_1 w_2 w_3 \cdots$ by $w_n = \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor$.

The numbers θ and ρ are called the *slope* and the *intercept*, respectively. Words of this form are called *lower mechanical words* and are known to be equivalent to Sturmian words [14]. As a special case, when $\rho = 0$, we obtain *characteristic words*.

Definition 2. Let θ be an irrational number with $0 < \theta < 1$. For $n \geq 1$, define $w_\theta(n) := \lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor$, and $\boldsymbol{w}(\theta) := w_\theta(1)w_\theta(2)w_\theta(3)\cdots$. Then $\boldsymbol{w}(\theta)$ is called the characteristic word with slope θ.

Note that every irrational $\theta \in (0, 1)$ has a unique continued fraction expansion

$$\theta = [0, a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where each a_i is a positive integer. Let $\theta = [0, 1 + d_1, d_2, \ldots]$ be an irrational number with $d_1 \geq 0$ and $d_n > 0$ for $n > 1$. We use the continued fraction expansion of θ as a directive sequence in the following way. We associate a sequence $(s_n)_{n \geq -1}$ of words defined by $s_{-1} = 1$, $s_0 = 0$, $s_n = s_{n-1}^{d_n} s_{n-2}$, for $n \geq 1$. Such a sequence of words is called a *standard sequence*. This sequence is related to characteristic words in the following way. Observe that, for any $n \geq 0$, s_n is a prefix of s_{n+1}, which gives meaning to $\lim_{n\to\infty} s_n$ as

a	b	$\boldsymbol{f}_{(a,b)}$
1	2	$11011101110110111011101110111011\cdots$
2	1	$01001001010010010010100100100100\cdots$
2	3	$01010100101010010101001010100\cdots$
3	2	$00100100010010001001000100100100\cdots$
3	5	$00100100100100100010010010010010010\cdots$

TABLE 3. Some biperiodic Fibonacci words.

an infinite word. In fact, one can prove [14] that each s_n is a prefix of $\boldsymbol{w}(\theta)$ for all $n \geq 0$ and

$$\boldsymbol{w}(\theta) = \lim_{n \to \infty} s_n. \tag{2}$$

3. BIPERIODIC FIBONACCI WORDS

Let $f_{(a,b,n)}$ and $\boldsymbol{f}_{(a,b)}$ be the nth biperiodic Fibonacci word and the infinite biperiodic Fibonacci word, respectively. Note that, for $a = b = 1$ we have the word $\overline{\boldsymbol{f}} = \texttt{1011010110110}\ldots$. If $a = b = k$, we obtain k-Fibonacci words, i.e., $\boldsymbol{f}_{(k,k)} = \boldsymbol{f}_k$.

Definition 3. The (a,b)-Fibonacci morphism $\sigma_{(a,b)} : \{0,1\} \to \{0,1\}^*$ is defined by $\sigma_{(a,b)}(0) = (0^{a-1}1)^b 0$ and $\sigma_{(a,b)}(1) = (0^{a-1}1)^b 0^a 1$.

Theorem 4. For all $n \geq 0$, $\sigma_{(a,b)}^n(0) = f_{(a,b,2n+1)}$ and $\sigma_{(a,b)}^n(0^{a-1}1) = f_{(a,b,2n+2)}$. Hence, the biperiodic Fibonacci word $\boldsymbol{f}_{(a,b)}$ satisfies that

$$\lim_{n \to \infty} \sigma_{(a,b)}^n(0) = \boldsymbol{f}_{(a,b)}.$$

Proof. We prove the two assertions about $\sigma_{(a,b)}^n$ by induction on n. They are clearly true for $n = 0, 1$. Now assume the result holds for n, and let us prove it for $n + 1$:

$$\begin{aligned}
\sigma_{(a,b)}^{n+1}(0) &= \sigma_{(a,b)}^n((0^{a-1}1)^b 0) \\
&= (\sigma_{(a,b)}^n(0^{a-1}1))^b \sigma_{(a,b)}^n(0) \\
&= f_{(a,b,2n+2)}^b f_{(a,b,2n+1)} = f_{(a,b,2n+3)},
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{(a,b)}^{n+1}(0^{a-1}1) &= \sigma_{(a,b)}^n(((0^{a-1}1)^b 0)^{a-1}(0^{a-1}1)^b 0^a 1) \\
&= \sigma_{(a,b)}^n(((0^{a-1}1)^b 0)^a 0^{a-1}1) \\
&= ((\sigma_{(a,b)}^n(0^{a-1}1))^b \sigma_{(a,b)}^n(0))^a \sigma_{(a,b)}^n(0^{a-1}1) \\
&= (f_{(a,b,2n+2)}^b f_{(a,b,2n+1)})^a f_{(a,b,2n+2)} \\
&= f_{(a,b,2n+3)}^a f_{(a,b,2n+2)} = f_{(a,b,2n+4)}. \quad \blacksquare
\end{aligned}$$

Example 5. In Table 3 we show some biperiodic Fibonacci words for specific values of a and b.

Proposition 6. *The number of occurrences of* 1 *and* 0 *in the finite biperiodic Fibonacci words are given*

by p_n and h_n, where p_n and h_n satisfy the following recurrences:

$$p_0 = 0, \quad p_1 = 0, \quad p_2 = 1,$$
$$p_n = \begin{cases} ap_{n-1} + p_{n-2}, & \text{if } n \geq 3 \text{ is even,} \\ bp_{n-1} + p_{n-2}, & \text{if } n \geq 3 \text{ is odd.} \end{cases} \tag{3}$$

Moreover,

$$p_n = \left(\frac{b^{1-\xi(n-1)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}}\right)\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}, \quad \text{for } n \geq 1, \tag{4}$$

and

$$h_0 = 0, \quad h_1 = 1, \quad h_2 = a - 1,$$
$$h_n = \begin{cases} ah_{n-1} + h_{n-2}, & \text{if } n \geq 3 \text{ is even,} \\ bh_{n-1} + h_{n-2}, & \text{if } n \geq 3 \text{ is odd.} \end{cases} \tag{5}$$

Moreover, for $n \geq 1$,

$$h_n = (a - 1)F_{n-1}^{(b,a)} + \left(\frac{a}{b}\right)^{\xi(n-1)} F_{n-2}^{(b,a)}. \tag{6}$$

Proof. Recurrences (3) and (5) are clear from definition of $f_{(a,b,n)}$. Equation (4) is clear from the Binet-like formula; see Equation (1). We obtain Equation (6) from [12, Theorem 8]. $\blacksquare$

Proposition 7. *One has*

(1.) $\lim_{n \to \infty} \frac{|f_{(a,b,n)}|}{|f_{(a,b,n)}|_1} = \frac{\alpha}{b} = \frac{ab + \sqrt{(ab)^2 + 4ab}}{2b}$.

(2.) $\lim_{n \to \infty} \frac{|f_{(a,b,n)}|}{|f_{(a,b,n)}|_0} = \frac{a(\alpha+1)}{\alpha(a-1)+a}$.

(3.) $\lim_{n \to \infty} \frac{|f_{(a,b,n)}|_0}{|f_{(a,b,n)}|_1} = a\left(1 + \frac{1}{\alpha}\right) - 1$.

Proof. (1.) From Equations (1) and (4) we obtain

$$\begin{aligned}
\lim_{n \to \infty} \frac{|f_{(a,b,n)}|}{|f_{(a,b,n)}|_1} &= \lim_{n \to \infty} \frac{q_n}{p_n} \\
&= \lim_{n \to \infty} \frac{\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right)\frac{\alpha^n - \beta^n}{\alpha - \beta}}{\left(\frac{b^{1-\xi(n-1)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}}\right)\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}} \\
&= \lim_{n \to \infty} \frac{(ab)^{\lfloor \frac{n-1}{2} \rfloor}(a^{1-\xi(n)})(\alpha^n - \beta^n)}{(ab)^{\lfloor \frac{n}{2} \rfloor}(b^{1-\xi(n-1)})(\alpha^{n-1} - \beta^{n-1})}.
\end{aligned}$$

If n is even, then

$$\begin{aligned}
\lim_{n \to \infty} \frac{|f_{(a,b,n)}|}{|f_{(a,b,n)}|_1} &= \frac{1}{b} \lim_{n \to \infty} \frac{\alpha^n - \beta^n}{\alpha^{n-1} - \beta^{n-1}} \\
&= \frac{1}{b} \lim_{n \to \infty} \alpha \frac{1 - \left(\frac{\beta}{\alpha}\right)^n}{1 - \left(\frac{\beta}{\alpha}\right)^{n-1}} = \frac{\alpha}{b}.
\end{aligned}$$

If n is odd, the limit is the same.

(2.) From Equations (1) and (6) we obtain

$$\lim_{n\to\infty} \frac{|f_{(a,b,n)}|}{|f_{(a,b,n)}|_0} = \lim_{n\to\infty} \frac{q_n}{p_n}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right)\frac{\alpha^n-\beta^n}{\alpha-\beta}}{(a-1)B(n) + \left(\frac{a}{b}\right)^{\xi(n-1)}B(n-1)},$$

where

$$B(n) = \left(\frac{b^{1-\xi(n)}}{(ab)^{\lfloor \frac{n-2}{2} \rfloor}}\right)\frac{\alpha^{n-2}-\beta^{n-2}}{\alpha-\beta}.$$

If n is even, then

$$\lim_{n\to\infty} \frac{|f_{(a,b,n)}|}{|f_{(a,b,n)}|_0}$$

$$= a \lim_{n\to\infty} \frac{\alpha^n - \beta^n}{(a-1)(ab)(\alpha^{n-1}-\beta^{n-1})+a^2 b(\alpha^{n-2}-\beta^{n-2})}$$

$$= a\frac{1}{\frac{1}{\alpha}(a-1)(ab)+a^2 b\frac{1}{\alpha^2}} = \frac{a(\alpha+1)}{\alpha(a-1)+a}.$$

If n is odd, the limit is the same.

(3.) The proof runs like in the previous two items.   ∎

The previous proposition is a particular result related to the incidence matrix of a substitution, see, e.g., [7].

Proposition 8. *The biperiodic Fibonacci word and the nth biperiodic Fibonacci word satisfy the following properties.*

(1.) *Word 11 is not a subword of the biperiodic Fibonacci word, for $a \geq 2$, and $b \geq 1$.*

(2.) *Let xy be the last two symbols of $f_{(a,b,n)}$. For $n, a \geq 2$, and $b \geq 1$, we have $xy = 01$ if n is even, and $xy = 10$ if n is odd.*

(3.) *The concatenation of two successive biperiodic Fibonacci words is "almost commutative", i.e., $f_{(a,b,n-1)}f_{(a,b,n-2)}$ and $f_{(a,b,n-2)}f_{(a,b,n-1)}$ have a common prefix of length $q_{n-1} + q_{n-2} - 2$, for all $n \geq 3$.*

Proof. (1.) It suffices to prove that 11 is not a subword of $f_{(a,b,n)}$, for all $n \geq 0$. We proceed by induction on n. For $n = 0, 1, 2$ it is clear. Now, assume that it is true for an arbitrary integer $n \geq 2$. If $n+1$ is even, we know that $f_{(a,b,n+1)} = f_{(a,b,n)}^a f_{(a,b,n-1)}$, so by the induction hypothesis we have that 11 is not a subword of $f_{(a,b,n)}$ and $f_{(a,b,n-1)}$. Therefore, the only possibility is that 1 is a suffix and a prefix of $f_{(a,b,n)}$ or 1 is a suffix of $f_{(a,b,n)}$ and a prefix of $f_{(a,b,n-1)}$, but these are impossible by the definition of the word $f_{(a,b,n)}$. If $n + 1$ is odd the proof is analogous.

(2.) We proceed by induction on n. For $n = 2$ it is clear. Now, assume that it is true for an arbitrary integer $n \geq 2$. If $n + 1$ is even, we know that $f_{(a,b,n+1)} = f_{(a,b,n)}^a f_{(a,b,n-1)}$, and by the induction

hypothesis the last two symbols of $f_{(a,b,n-1)}$ are 01, therefore the last two symbols of $f_{(a,b,n+1)}$ are 01. Analogously, if $n + 1$ is odd.

(3.) We proceed by induction on n. For $n = 3, 4$ it is clear. Now, assume that it is true for an arbitrary integer $n \geq 4$. If n is even, then by definition of $f_{(a,b,n)}$, we have

$$f_{(a,b,n-1)}f_{(a,b,n-2)} = f_{(a,b,n-2)}^b f_{(a,b,n-3)}$$
$$\cdot f_{(a,b,n-3)}^a f_{(a,b,n-4)} = (f_{(a,b,n-3)}^a f_{(a,b,n-4)})^b$$
$$\cdot f_{(a,b,n-3)}^a f_{(a,b,n-3)} f_{(a,b,n-4)},$$

and

$$f_{(a,b,n-2)}f_{(a,b,n-1)}$$
$$= f_{(a,b,n-3)}^a f_{(a,b,n-4)} \cdot f_{(a,b,n-2)}^b f_{(a,b,n-3)}$$
$$= f_{(a,b,n-3)}^a f_{(a,b,n-4)} \cdot (f_{(a,b,n-3)}^a f_{(a,b,n-4)})^b$$
$$\cdot f_{(a,b,n-3)} = (f_{(a,b,n-3)}^a f_{(a,b,n-4)})^b$$
$$f_{(a,b,n-3)}^a f_{(a,b,n-4)} f_{(a,b,n-3)}.$$

Hence the words have a common prefix of length $b(aq_{n-3} + q_{n-4}) + aq_{n-3} = bq_{n-2} + aq_{n-3}$. By the induction hypothesis $f_{(a,b,n-3)}f_{(a,b,n-4)}$ and $f_{(a,b,n-4)}f_{(a,b,n-3)}$ have a common prefix of length $q_{n-3}+q_{n-4}-2$. Therefore the words have a common prefix of length

$$bq_{n-2} + aq_{n-3} + q_{n-3} + q_{n-4} - 2 = q_{n-1} + q_{n-2} - 2.$$

If n is odd the proof is analogous.   ∎

The above proposition is a particular result related to Sturmian words, see, e.g., [14].

Definition 9. Let $\Phi : \{0,1\}^* \to \{0,1\}^*$ be a map such that Φ deletes the last two symbols, i.e., $\Phi(a_1 a_2 \cdots a_n) = a_1 a_2 \cdots a_{n-2}$, if $n > 2$, and $\Phi(a_1 a_2 \cdots a_n) = \epsilon$ if $n \leq 2$.

Corollary 10. *The nth biperiodic Fibonacci word satisfies for all $n \geq 2$, and $x, y \in \{0, 1\}$ that*

(1.) $\Phi(f_{(a,b,n-1)}f_{(a,b,n-2)}) = \Phi(f_{(a,b,n-2)}f_{(a,b,n-1)}).$

(2.) $\Phi(f_{(a,b,n-1)}f_{(a,b,n-2)}) = f_{(a,b,n-2)}\Phi(f_{(a,b,n-1)})$
$= f_{(a,b,n-1)}\Phi(f_{(a,b,n-2)}).$

(3.) *If $f_{(a,b,n)} = \Phi(f_{(a,b,n)})xy$, then*
$\Phi(f_{(a,b,n-2)})xy\Phi(f_{(a,b,n-1)})$
$= f_{(a,b,n-1)}\Phi(f_{(a,b,n-2)}).$

(4.) *If $f_{(a,b,n)} = \Phi(f_{(a,b,n)})xy$, then*

$$\Phi(f_{(a,b,n)}) = \begin{cases} \Phi(f_{(a,b,n-2)})(10\Phi(f_{(a,b,n-1)}))^a, \\ \qquad\qquad\qquad \textit{if } n \textit{ is even.} \\ \Phi(f_{(a,b,n-2)})(01\Phi(f_{(a,b,n-1)}))^b, \\ \qquad\qquad\qquad \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. (1.) and (2.) follow immediately from Proposition 8.(3.) and $|f_{(a,b,n)}| \geq 2$ for all $n \geq 2$. For (3.), in

Symbol	Action
1	Draw a line forward.
0	Draw a line forward and if the symbol 0 is in a position even, then turn θ degree and if 0 is in a position odd, then turn $-\theta$ degrees.

TABLE 4. Odd-even drawing rule with turn angle θ.

fact, if $f_{(a,b,n)} = \Phi(f_{(a,b,n)})xy$, then from Proposition 8.(2.) we have $f_{(a,b,n-2)} = \Phi(f_{(a,b,n-2)})xy$. Hence

$$\Phi(f_{(a,b,n-2)})xy\Phi(f_{(a,b,n-1)})$$
$$= f_{(a,b,n-2)}\Phi(f_{(a,b,n-1)}) = f_{(a,b,n-1)}\Phi(f_{(a,b,n-2)}).$$

Item (4.) is clear from (3.) and the definition of $f_{(a,b,n)}$. ∎

Theorem 11.

$\Phi(f_{(a,b,n)})$ is a palindrome for all $n, a \geq 2$ and $b \geq 1$.

Proof.

We proceed by induction on n. If $n = 2, 3$, then $\Phi(f_{(a,b,2)}) = 0^{a-1}$ and $\Phi(f_{(a,b,3)}) = (0^{a-1}1)^b 0$ are palindromes. Now, assume that it is true for an arbitrary integer $n \geq 3$. If n is even, then from Corollary 10

$$(\Phi(f_{(a,b,n)}))^R = (\Phi(f_{(a,b,n-1)}^a f_{(a,b,n-2)}))^R$$
$$= (f_{(a,b,n-1)}^a \Phi(f_{(a,b,n-2)}))^R$$
$$= \Phi(f_{(a,b,n-2)})^R (f_{(a,b,n-1)}^a)^R$$
$$= \Phi(f_{(a,b,n-2)})(f_{(a,b,n-1)}^R)^a$$
$$= \Phi(f_{(a,b,n-2)})(\Phi(f_{(a,b,n-1)}10)^R)^a$$
$$= \Phi(f_{(a,b,n-2)})(01\Phi(f_{(a,b,n-1)}))^a = (\Phi(f_{(a,b,n)})).$$

If n is odd, the proof is analogous. ∎

Corollary 12. (1.) If $f_{(a,b,n)} = \Phi(f_{(a,b,n)})xy$, then $yx\Phi(f_{(a,b,n)})xy$ is a palindrome.

(2.) If u is a subword of the biperiodic Fibonacci word, then so is its reversal, u^R.

The above propositions are particular results related to palindromes of Sturmian words, see, e.g., [9].

Theorem 13. Let $\zeta = [0, \overline{a, b}]$ be an irrational number, with a and b positive integers, then $\boldsymbol{w}(\zeta) = \boldsymbol{f}_{(a,b)}$.

Proof. Let $\zeta = [0, \overline{a, b}]$ be an irrational number, then its associated standard sequence is

$$s_{-1} = 1, \quad s_0 = 0, \quad s_1 = s_0^{a-1}s_{-1} = 0^{a-1}1,$$

$$\text{and} \quad s_n = \begin{cases} s_{n-1}^b s_{n-2}, & \text{if } n \geq 2 \text{ is even,} \\ s_{n-1}^a s_{n-2}, & \text{if } n \geq 2 \text{ is odd.} \end{cases}$$

Hence $\{s_n\}_{n\geq 0} = \{f_{(a,b,n+1)}\}_{n\geq 0}$ and from Equation (2), we have

$$\boldsymbol{w}(\zeta) = \lim_{n\to\infty} s_n = \boldsymbol{f}_{(a,b)}.$$ ∎

Remark. Note that

$$\zeta = [0, \overline{a, b}] = \frac{1}{a + \frac{1}{b + \frac{1}{a + \frac{1}{\dots}}}} = \frac{-ab + \sqrt{(ab)^2 + 4ab}}{2a} = -\frac{\alpha}{2}.$$

From the above theorem, we conclude that biperiodic Fibonacci words are Sturmian words.

A fractional power is a word of the form $z = x^n y$, where $n \in \mathbb{Z}^+$, $x \in \Sigma^+$ and y is a prefix of x. If $|z| = p$ and $|x| = q$, we say that z is a p/q-power, or $z = x^{p/q}$. In the expression $x^{p/q}$, the number p/q is the power's exponent. For example, 01201201 is an 8/3-power, $01201201 = (012)^{8/3}$. The index of an infinite word $\boldsymbol{w} \in \Sigma^\omega$ is defined by

$$\text{Ind}(\boldsymbol{w}) := \sup\{r \in \mathbb{Q}_{\geq 1} : \boldsymbol{w} \text{ contains an } r\text{-power.}\}$$

For example, $\text{Ind}(\boldsymbol{f}) > 3$ because the cube $(010)^3$ occurs in $\boldsymbol{f}$ at position 6. Mignosi and Pirillo [15] proved that $\text{Ind}(\boldsymbol{f}) = 2 + \phi \approx 3.618$, where ϕ is the golden ratio. Ramírez and Rubiano [20] proved that the index of the k-Fibonacci word is given by $\text{Ind}(\boldsymbol{f}_k) = 2 + k + 1/r_{k,1}$, where $r_{k,1} = (k + \sqrt{k^2 + 4})/2$. A general formula for the index of a Sturmian word was given by Damanik and Lenz [11].

Theorem 14 [11]**.** *If $\boldsymbol{w}$ is a Sturmian word of slope $\theta = [0, a_1, a_2, a_3, \dots]$, then*

$$\text{Ind}(\boldsymbol{w}) = \sup_{n\geq 0}\{2 + a_{n+1} + \frac{r_{n-1} - 2}{r_n}\},$$

where r_n is the denominator of $\theta = [0, a_1, a_2, a_3, \dots, a_n]$ and satisfies $r_{-1} = 0, r_0 = 1, r_{n+1} = a_{n+1}r_n + r_{n-1}$.

Corollary 15. The index of the biperiodic Fibonacci word is

$$\text{Ind}(\boldsymbol{f}_{(a,b)}) = \max\Big\{2 + a + \frac{a}{\alpha}, 2 + b + \frac{b}{\alpha}\Big\}, \quad (7)$$

where $\alpha = (ab + \sqrt{(ab)^2 + 4ab})/2$.

Proof. The word $\boldsymbol{f}_{(a,b)}$ is a Sturmian word of slope $\zeta = [0, \overline{a, b}]$, then from the above theorem $r_n = q_{n+1}$, and $\text{Ind}(\boldsymbol{f}_{(a,b)}) = \sup_{n\geq 0}\{h_n\}$, where

$$h_n = \begin{cases} 2 + b + \frac{q_{n-1} - 2}{q_n}, & \text{if } n \text{ is even,} \\ 2 + a + \frac{q_{n-1} - 2}{q_n}, & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\text{Ind}(\boldsymbol{f}_{(a,b)}) = \max\Big\{\sup_{n\geq 0}\{2 + a + \frac{q_{2n} - 2}{q_{2n+1}}\},$$
$$\sup_{n\geq 0}\{2 + b + \frac{q_{2n-1} - 2}{q_{2n}}\}\Big\}.$$

Since $q_{2n+1}/q_{2n} \to \alpha/a$ and $q_{2n}/q_{2n-1} \to \alpha/b$ as $n \to \infty$, then Equation (7) follows. ∎

$\mathcal{F}_9^{(2,5)}, \ \theta = 90°$

$\mathcal{F}_7^{(5,5)}, \ \theta = 90°$

$\mathcal{F}_6^{(6,6)}, \ \theta = 90°$

$\mathcal{F}_{10}^{(2,5)}, \ \theta = 90°$

$\mathcal{F}_{10}^{(2,3)}, \ \theta = 60°$
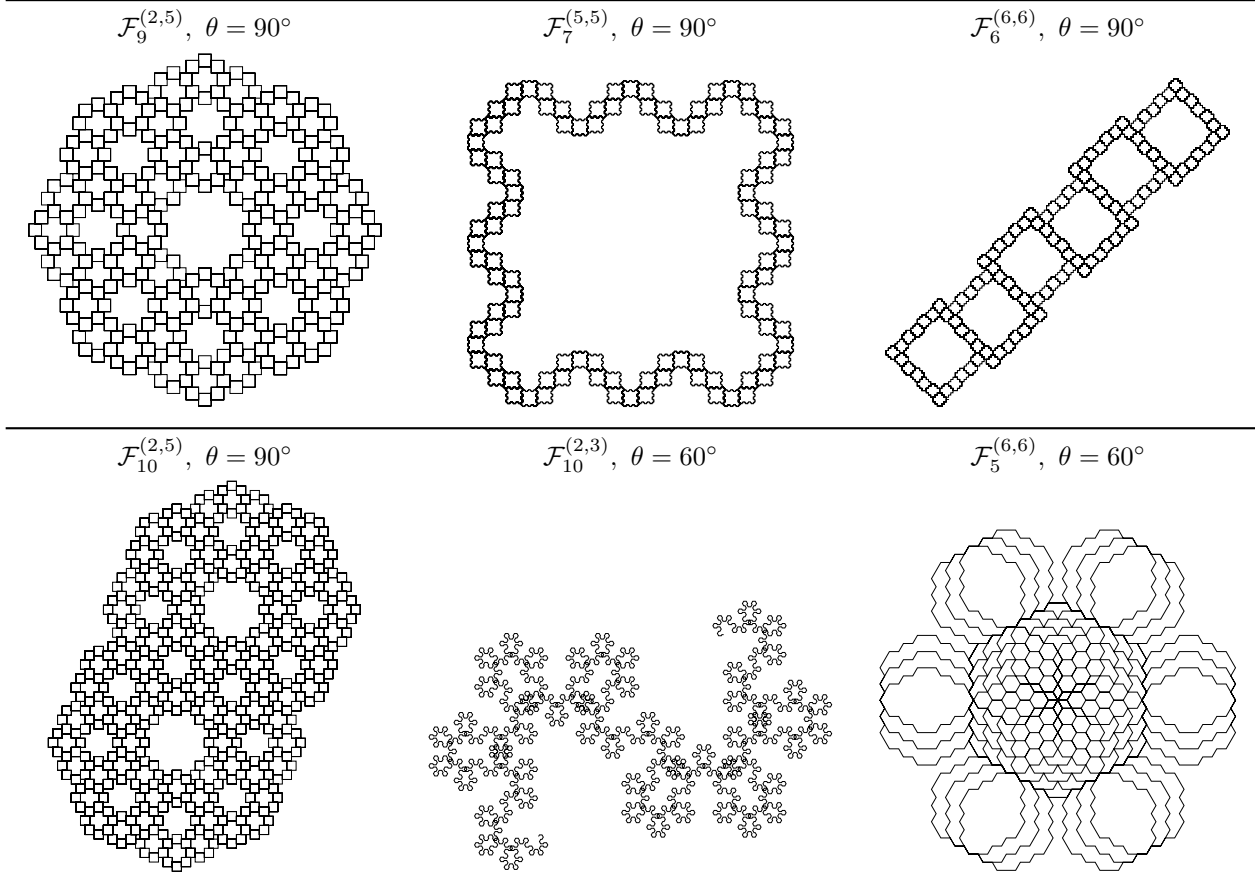
$\mathcal{F}_5^{(6,6)}, \ \theta = 60°$

TABLE 5. Some curves $\mathcal{F}_n^{(a,b)}$ with an angle θ.

4. THE BIPERIODIC FIBONACCI WORD CURVE

The odd-even drawing rule can be extended from a parameter θ, where θ is the turn angle, see Table 4. If $\theta = 90°$, then we obtain the drawing rule in Table 1.

Definition 16. The nth-biperiodic curve of Fibonacci, denoted by $\mathcal{F}_n^{(a,b)}$, is obtained by applying the odd-even drawing rule to the word $f_{(a,b,n)}$. The biperiodic Fibonacci word fractal $\mathcal{F}^{(a,b)}$, is defined as $\mathcal{F}^{(a,b)} := \lim_{n\to\infty} \mathcal{F}_n^{(a,b)}$.

In Table 5, we show some curves $\mathcal{F}_n^{(a,b)}$ with a given angle θ.

Proposition 17. The biperiodic Fibonacci word curve and the curve $\mathcal{F}_n^{(a,b)}$ have the following properties:

(1.) The biperiodic Fibonacci curve $\mathcal{F}^{(a,b)}$ is composed only of segments of length 1 or 2.

(2.) The curve $\mathcal{F}_n^{(a,b)}$ is symmetric.

(3.) The number of turns in the curve $\mathcal{F}_n^{(a,b)}$ is h_n.

Proof. (1.) It is clear from Proposition 8.(1.), because 110 and 111 are not subwords of $\boldsymbol{f}_n^{(a,b)}$.

(2.) It is clear from Theorem 11, because $f_{(a,b,n)} = \Phi(f_{(a,b,n)})xy$, where $\Phi(f_{(a,b,n)})$ is a palindrome.

(3.) It is clear from the definition of the odd-even drawn rule and because $|f_{(a,b,n)}|_0 = h_n$; see Proposition 6. ∎

Proposition 18 (Monnerot-Dumaine [16])**.** The curve $\mathcal{F}_n^{(1,1)} = \mathcal{F}_n$ is similar to the curve $\mathcal{F}_{n+3}^{(1,1)} = \mathcal{F}_{n+3}$.

Proposition 19 (Ramírez and Rubiano [20])**.** If a is even, then the curve $\mathcal{F}_n^{(a,a)} = \mathcal{F}_{a,n}$ is similar to the curve $\mathcal{F}_{n+2}^{(a,a)} = \mathcal{F}_{a,n+2}$. If n is odd, then the curve $\mathcal{F}_n^{(a,a)} = \mathcal{F}_{a,n}$ is similar to the curve $\mathcal{F}_n^{(a,a)} = \mathcal{F}_{a,n+3}$.

Theorem 20. (1.) If a is even, then the curve $\mathcal{F}_n^{(a,b)}$ is similar to the curve $\mathcal{F}_{n+2}^{(a,b)}$, i.e., they have the same shape except for the number of segments.

(2.) If $a \neq b$, and a, b are odd, then the curve $\mathcal{F}_n^{(a,b)}$ is similar to the curve $\mathcal{F}_{n+6}^{(a,b)}$.

(3.) If a is odd, and b is even, then the curve $\mathcal{F}_n^{(a,b)}$ is similar to the curve $\mathcal{F}_{n+4}^{(a,b)}$.

Proof. Suppose a is even. Then it is clear that $\sigma_{(a,b)}(f_{(a,b,n)}) = f_{(a,b,n+2)}$; see Theorem 4. We are going to prove that $\sigma_{(a,b)}$ preserves the odd-even alternation required by the odd-even drawing rule. In fact, $\sigma_{(a,b)}(0) = (0^{a-1}1)^b 0$, and $\sigma_{(a,b)}(1) = (0^{a-1}1)^b 0^a 1$. As a is even, then $|\sigma_{(a,b)}(0)|$ and $|\sigma_{(a,b)}(1)|$ are odd.
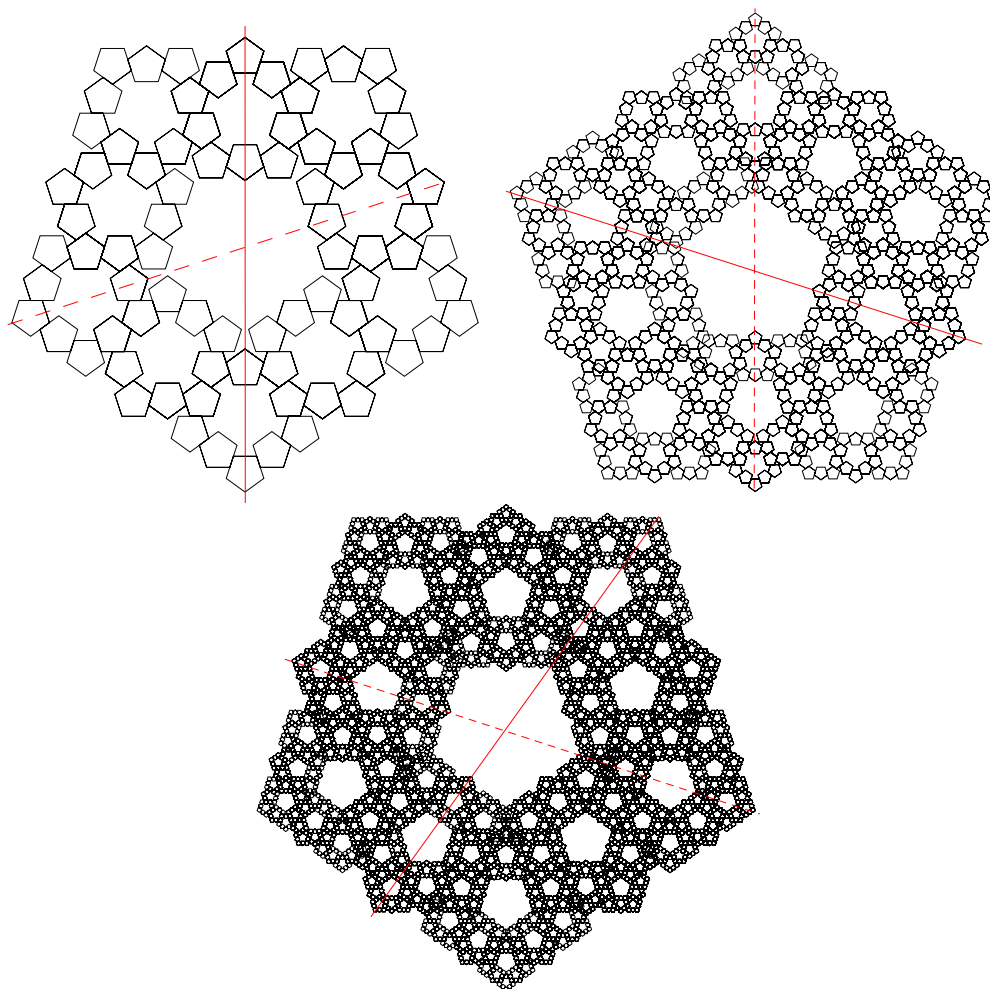
FIGURE 2. Curves $\mathcal{F}_7^{(2,6)}, \mathcal{F}_9^{(2,6)}, \mathcal{F}_{11}^{(2,6)}$ with $\theta = 72°$.

Hence if $|w|$ is even (odd), then $|\sigma_{(a,b)}(w)|$ is even (odd). Since $\sigma_{(a,b)}$ preserves parity of length then any subword in a biperiodic Fibonacci word preserves parity of position.

Finally, let $\mathcal{A}(w)$ be the function that gives the ending or resulting angle of an associate curve to the word w through the odd-even drawing rule of angle θ. We have to prove that the resulting angle of a curve must be preserved or inverted by $\sigma_{(a,b)}$. Note that $\mathcal{A}(00) = 0$, $\mathcal{A}(01) = -\theta$ and $\mathcal{A}(10) = +\theta$. Therefore

- $\mathcal{A}(\sigma_{(a,b)}(00)) = \mathcal{A}((0^{a-1}1)^b 0(0^{a-1}1)^b 0) = \mathcal{A}((0^{a-1}1)^b) + \mathcal{A}(0^a) + \mathcal{A}((10^{a-1})^{b-1}) + \mathcal{A}(10) = \mathcal{A}((0^{a-1}1)^b) + \mathcal{A}(0^a) + \mathcal{A}(10) + \mathcal{A}(0^{a-2}) + \mathcal{A}(10) + \mathcal{A}(0^{a-2}) + \cdots + \mathcal{A}(10) = b(-\theta) + 0 + \theta + 0 + \theta + \cdots + \theta = 0$.

- $\mathcal{A}(\sigma_{(a,b)}(01)) = \mathcal{A}((0^{a-1}1)^b 0(0^{a-1}1)^b 0^a 1) = \mathcal{A}((0^{a-1}1)^b) + \mathcal{A}(0^a) + \mathcal{A}(10) + \mathcal{A}(0^{a-2}) + \cdots + \mathcal{A}(10)\mathcal{A}(0^{a-1}1) = 0 - \theta = -\theta$.

- $\mathcal{A}(\sigma_{(a,b)}(10)) = \mathcal{A}((0^{a-1}1)^b 0^a 1(0^{a-1}1)^b 0) = \mathcal{A}((0^{a-1}1)^b) + \mathcal{A}(0^a) + \mathcal{A}(10) + \mathcal{A}(0^{a-2}) + \mathcal{A}(10) + \mathcal{A}(0^{a-2}) + \cdots + \mathcal{A}(10) = b\theta + 0 + \theta + 0 + \theta + \cdots + \theta = b\theta + (b+1)\theta = +\theta$.

Then $\sigma_{(a,b)}$ preserves the resulting angle, i.e.,

$\mathcal{A}(w) = \mathcal{A}(\sigma_{(a,b)}(w))$ for any word w of even length. Therefore the image of a pattern by $\sigma_{(a,b)}$ is the rotation of this pattern by an angle of $+\theta$. Since $\sigma_{(a,b)}(f_{(a,b,n)}) = f_{(a,b,n+2)}$, then the curve $\mathcal{F}_{(a,b,n)}$ is similar to the curve $\mathcal{F}_{(a,b,n+2)}$.
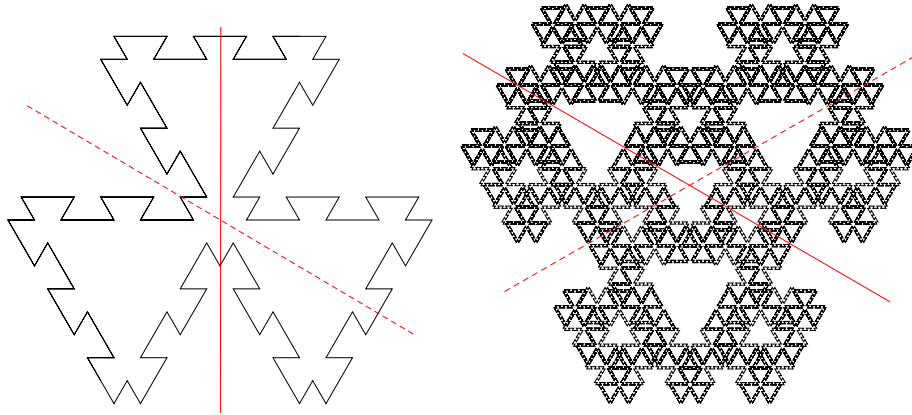
If $a \neq b$, and a, b are odd the proof is similar, but using $\sigma_{(a,b)}^3$. If a is odd, and b is even the proof is similar, but using $\sigma_{(a,b)}^2$. ∎

An open problem is try to find a characterization of similar word curves in terms of words.

Example 21. In Figure 2, $\mathcal{F}_7^{(2,6)}$ is similar to $\mathcal{F}_9^{(2,6)}, \mathcal{F}_{11}^{(2,6)}$ and so on. In Figure 3, $\mathcal{F}_5^{(3,4)}$ is similar to $\mathcal{F}_9^{(3,4)}$ and so on.

FIGURE 3. Curves $\mathcal{F}_5^{(3,4)}, \mathcal{F}_9^{(3,4)}$ with $\theta = 120°$.

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