LIE GROUPS AND NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS: INVARIANT DISCRETIZATION VERSUS DIFFERENTIAL APPROXIMATION

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Abstract. We briefly review two different methods of applying Lie group theory in the numerical solution of ordinary differential equations. On specific examples we show how the symmetry preserving discretization provides difference schemes for which the "first differential approximation" is invariant under the same Lie group as the original ordinary differential equation.

Keywords: ordinary difference equations, numerical solution, Lie group theory, invariant discretization.

1. Introduction

Lie group theory provides powerful tools for solving ordinary and partial differential equations, specially nonlinear ones [1–3]. The standard approach is to find the Lie point symmetry group G of the equation and then look for invariant solutions, i.e. solutions that are invariant under some subgroup $G_0 \subset G$. For Ordinary Differential Equations (ODEs) this leads to a reduction of the order of the equation (without any loss of information). If the dimension of the symmetry group is large enough (at least equal to the order of the equation) and the group is solvable, then the order of the equation can be reduced to zero. This can be viewed as obtaining the general solution of the equation, explicitly or implicitly.

However, for an ODE obtaining an implicit solution is essentially equivalent to replacing the differential equation by an algebraic or a functional one. From the point of view of visualizing the solution, or presenting a graph of the solution, it maybe easier to solve the ODE numerically than to do the same for the functional equation.

For Partial Differential Equations (PDEs) symmetry reduction reduces the number of independent variables in the equation and leads to particular solutions, rather than the general solution.

Both for ODEs and PDEs with nontrivial symmetry groups it may still be necessary to resort to numerical solutions. The question then arises of making good use of the group G. Any numerical method involves replacing the differential equation by a difference one. In standard discretizations no heed is paid to the symmetry group G and some, or all of the symmetries are lost. Since the Lie point symmetry group encodes many of the properties of the solution space of a

differential equation, it seems desirable to preserve it, or at least some of its features in the discretization process.

Two different methods for incorporating symmetry concepts into the discretization of differential equations exist in the literature. One was proposed and explored by Shokin and Yanenko [4–8] for PDEs and has been implemented in several recent studies [9, 10]. It is called the *differential approximation method* and the basic idea is the following. A uniform orthogonal lattice in x_i is introduced and the considered differential equation

$$E(\vec{x}, u, u_{x_i}, u_{x_i,x_j}, \cdots) = 0, \tag{1}$$

is approximated by some difference equation $E^\Delta = 0$. The derivatives are replaced by discrete derivatives. All known and unknown functions in (1) are then expanded in Taylor series about some reference point ($\vec{x}$), in terms of the lattice spacings. In the simplest case of 2 variables, x and t, we have $\sigma = x_{n+1} - x_n$, $\tau = t_{n+1} - t_n$

$$E^\Delta(x, t, u, \Delta_x u, \Delta_t u, \Delta_{xx} u, \Delta_{tt} u, \Delta_{xt} u, \cdots)$$
$$= E + \sigma E_1 + \tau E_2 + \sigma^2 E_3 + 2\sigma\tau E_4 + \tau^2 E_5 + \cdots. \tag{2}$$

The expansion on the right hand side of (2) is a "differential approximation" of the difference equation

$$E^\Delta = 0. \tag{3}$$

The "zero order differential approximation" of (3) is the original differential equation (1) and hence is invariant under the symmetry group G. Keeping terms of order σ or τ in (2) we obtain the first differential approximation, etc. The idea is to take a higher

order differential approximation, at least the first order one, and require that is also invariant under G, or at least under a subgroup $G_0 \subset G$. This is done by constructing different possible difference schemes approximating eq. (1) and choosing among them the one for which the first (or higher) differential approximation has the "best" symmetry properties.

The second approach, the *invariant discretization* method is part of a program devoted to the study of continuous symmetries of discrete equations, i.e. the application of Lie groups to difference equations [11–24]. As far as applications to the numerical solutions of differential equations are concerned, the idea, originally due to Dorodnitsyn [12, 13], is to start from the differential equation, its symmetry group G and the Lie algebra L of G, realized by vector fields. The differential equation can then be expressed in terms of differential invariants of G. The differential equation is then approximated by a finite difference scheme that is constructed so as to be invariant under the same group G as the differential equation. The difference scheme will consist of several equations establishing relations between points in the space of independent and dependent variables. These equations determine both the evolution of the dependent variables and the form of the lattice. The equations are written in terms of group invariants of the group G acting via its prolongation to all points on the lattice (rather than to derivatives of the dependent functions). As pointed out by P. Olver, this amounts to prolonging the group action to "multi-space" for difference equations [24] rather than to "jet space" as for differential equations [25].

The purpose of this paper is to compare the two different methods of incorporating Lie symmetries into the numerical analysis of differential equations. For simplicity we restrict ourselves to the case of ODEs and analyze difference schemes that were used in recent articles [26–28] to solve numerically some third order nonlinear ODEs with three or four dimensional symmetry algebras. In this article we take invariant difference schemes (on symmetry adapted lattices) and construct its first differential approximation. We then verify that in all examples this first differential approximation is invariant under the entire symmetry group G.

2. Differential approximations of ordinary difference equations and invariant discretization of ODEs.

Let us consider the case of a third order ODE

$$E \equiv E(x, y, y', y'', y''') = 0, \qquad (4)$$

(the generalization to order $n \geq 3$ is straightforward). We can approximate (4) on a 4 point stencil with points (x_k, y_k), $(k = n-1, n, n+1, n+2)$. Alternative coordinates on the stencil are the coordinates of one

reference point, say (x_n, y_n), the distances between the points, and the discrete derivatives up to order 3,

$$\left\{ \begin{aligned} & x_n, \, y_n, \, h_{n+k} = x_{n+k} - x_{n+k-1}, \\ & p_{n+1}^{(1)} = \frac{y_{n+1} - y_n}{x_{n+1} - x_n}, \, p_{n+2}^{(2)} = 2\frac{p_{n+2}^{(1)} - p_{n+1}^{(1)}}{x_{n+2} - x_n}, \\ & \qquad p_{n+3}^{(3)} = 3\frac{p_{n+3}^{(2)} - p_{n+2}^{(2)}}{x_{n+3} - x_n} \right\}. \quad (5) \end{aligned} \right.$$

The one-dimensional lattice can be chosen to be uniform and then

$$h_{n+1} = h_n \equiv h, \qquad (6)$$

or some other distribution of points can be chosen.

The Lie point symmetry group G transforms the variables (x, y) into $(\tilde{x}, \tilde{y}) = (\Lambda(x, y), \Omega(x, y))$ and its Lie point symmetry algebra L is represented by vector fields of the form

$$\hat{X}_\mu = \xi_\mu(x, y)\partial_x + \phi_\mu(x, y)\partial_y, \quad 1 \leq \mu \leq \dim L. \quad (7)$$

The vector fields must be prolonged in the standard manner [1] to derivatives

$$\begin{aligned} \mathrm{pr}\hat{X}_\mu = {} & \hat{X}_\mu + \phi^x(x, y, y_x)\partial_{y_x} \\ & + \phi^{xx}(x, y, y_x, y_{xx})\partial_{y_{xx}} + \cdots, \quad (8) \end{aligned}$$

when acting on a differential equation, or to all points when acting on a difference scheme [15, 18, 20]

$$\mathrm{pr}^\Delta \hat{X}_\mu = \sum_i \big[\xi_\mu(x_i, y_i)\partial_{x_i} + \phi_\mu(x_i, y_i)\partial_{y_i} \big]. \quad (9)$$

In the differential approximation method we start with a difference equation, usually on a uniform lattice (6)

$$E^\Delta\big(x_n, h_{n+1}, h_{n+2}, y_n, p_{n+1}^{(1)}, p_{n+2}^{(2)}, p_{n+3}^{(3)}\big) = 0, \quad (10)$$

and expand it into a Taylor series in the spacing h:

$$E^\Delta = E_0 + hE_1 + h^2 E_2 + \cdots = 0. \qquad (11)$$

For any difference equation approximating the differential equation $E_0 = 0$ the lowest order term will be invariant under the group G. Different schemes (10) can then be compared with respect to the invariance of the first differential approximation

$$E_0 + hE_1 = 0, \qquad (12)$$

or of some higher order differential approximations. Better results can be expected for schemes for which (12) is invariant under all of G or under some subgroup $G_0 \subset G$ that is relevant for the problem.

When studying the invariance of (12) it must be remembered that the prolongations of $\hat{X}_\mu$ also act on the lattice parameters h.

In the invariant discretization method one constructs an invariant difference scheme

$$E_a^\Delta \big(x_n, y_n, h_n, h_{n+1}, h_{n+2}, y_n,$$
$$p_{n+1}^{(1)}, p_{n+2}^{(2)}, p_{n+3}^{(3)} \big) = 0, \quad a = 1, 2 \quad (13)$$
$$\mathrm{pr}\hat{X}_\mu E_a^\Delta \big|_{E_1^\Delta = 0, E_2^\Delta = 0} = 0. \quad (14)$$

The two equations (13) determine both the lattice and the difference equation. Both are constructed out of the invariants of the group G prolonged to the lattice as indicated in eq. (14). Thus, the difference scheme is by construction invariant under the entire group G acting on the equation and lattice. In the continuous limit we have

$$E_1^\Delta = E + h_n E_1^{(1)} + h_{n+1} E_1^{(2)} + h_{n+2} E_1^{(3)} + \text{h.o.t.},$$
$$(15)$$
$$E_2^\Delta = 0 + h_n E_2^{(1)} + h_{n+1} E_2^{(2)} + h_{n+2} E_2^{(3)} + \text{h.o.t.}.$$
$$(16)$$

The terms spelled out in (15) correspond to the first differential approximation.

Since the left hand side of (15), (16) is invariant under G, the series on the right hand side must also be invariant. This does not guarantee that the first (or n-th) differential approximation will be invariant. In the next three sections we will show on examples that the first differential approximation is indeed invariant. Thus, choosing an invariant difference scheme guarantees, at least in the considered cases, that the aims of the differential approximation method are fully achieved.

The examples are all third order ODEs with 3 or 4 dimensional symmetry groups. In each case we write an invariant difference scheme of the form (13) and its first differential approximation (15). The terms $E_1^{(k)}$, $(k = 1, 2, 3)$ in (15) are differential expressions containing y''' and y''''. These expressions will be simplified by removing y''' and y'''', using the ODE (4) and its first differential consequence.

3. EQUATIONS INVARIANT UNDER THE SIMILITUDE GROUP $Sim(2)$.

Let us consider the group of translations, rotations and uniform dilations of an Euclidean plane. Its Lie algebra $sim(2)$ is realized by the vector fields

$$\hat{X}_1 = \frac{\partial}{\partial x}, \qquad \hat{X}_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$\hat{X}_2 = \frac{\partial}{\partial y}, \qquad \hat{X}_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad (17)$$

This group has no second order differential invariant and precisely one third order one, namely[26]

$$I = \frac{(1 + y'^2)y''' - 3y'y''^2}{y''^2}. \quad (18)$$

The expressions

$$I_1 = \frac{y''}{(1 + y'^2)^{3/2}}, \quad I_2 = \frac{(1 + y'^2)y''' - 3y'y''^2}{(1 + y'^2)^3}$$

are invariant under the Euclidean group, with Lie algebra $\{\hat{X}_1, \hat{X}_2, \hat{X}_3\}$, but only the ratio $I_2/I_1^2 = I$ is invariant under dilations.

Thus the lowest order ODE invariant under $Sim(2)$ is

$$(1 + y'^2)y''' - 3y'y''^2 = Ky''^2 \quad (19)$$

where K is an arbitrary constant.

To discretize (19) (or any third order ODE) we need (at least) a four-point stencil. The Euclidean group has 5 independent invariants depending on 4 points $(x_{n+k}, y_{n+k}), k = -1, 0, 1, 2$ namely [26]

$$\xi_1 = h_{n+2}\left[1 + \left(\frac{y_{n+2} - y_{n+1}}{h_{n+2}}\right)^2\right]^{1/2},$$
$$\xi_2 = h_{n+1}\left[1 + \left(\frac{y_{n+1} - y_n}{h_{n+1}}\right)^2\right]^{1/2},$$
$$\xi_3 = h_n\left[1 + \left(\frac{y_n - y_{n-1}}{h_n}\right)^2\right]^{1/2},$$
$$\xi_4 = (y_{n+2} - y_{n+1})h_{n+1} - (y_{n+1} - y_n)h_{n+2},$$
$$\xi_5 = (y_{n+1} - y_n)h_n - (y_n - y_{n-1})h_{n+1}. \quad (20)$$

Out of them we can construct 4 independent $Sim(2)$ invariants, for instance

$$J_1 = \frac{2\alpha\xi_4}{\xi_1\xi_2(\xi_1 + \xi_2)} + \frac{2\beta\xi_5}{(\xi_2\xi_3)(\xi_2 + \xi_3)}, \quad \alpha + \beta = 1,$$
$$J_2 = \frac{6}{\xi_1 + \xi_2 + \xi_3}\left[\frac{\xi_4}{\xi_1\xi_2(\xi_1 + \xi_2)} - \frac{\xi_5}{\xi_2\xi_3(\xi_2 + \xi_3)}\right],$$
$$(21)$$

and the two ratios ξ_1/ξ_2, ξ_2/ξ_3. To obtain the continuous limit we put

$$x_{n-1} = x_n - h_n, \quad x_{n+1} = x_n + h_{n+1},$$
$$x_{n+2} = x_n + h_{n+1} + h_{n+2},$$
$$y_{n+k} = y(x_{n+k}), \quad h_{n+k} = \alpha_k\epsilon, \quad \alpha_k \sim 1, \quad (22)$$

and expand y_{n+k} into a Taylor series about $x_n \equiv x$.

The invariants J_1, J_2 where so chosen that their continuous limits are I_1 and I_2 respectively.

The invariant scheme used in [26] to solve the ODE (19) numerically was

$$E_2^\Delta = \xi_1\xi_3 - \xi_2^2 = 0, \quad (23)$$
$$E_1^\Delta = J_2 - KJ_1^2 = 0. \quad (24)$$

The first differential approximation of (23) is

$$E_2^\Delta \approx 2(-h_{n+1}^2 + h_n h_{n+2})(y'^2 + 1)$$
$$+ (2h_n h_{n+1}h_{n+2} - 2h_{n+1}^3 - h_n^2 h_{n+2}$$
$$+ h_n h_{n+2}^2)y'y'' = 0 \quad (25)$$

Applying $\mathrm{pr}^D \hat{X}_i$ to eq. (25) we find that the equation is invariant under the entire group $Sim(2)$, as are the terms of order ϵ^2 and ϵ^3 separately.

The first order differential approximation of the difference equation (24) is quite complicated. However, if we substitute the ODE (19) and its differential consequences into the first nonvanishing term of the approximation, we obtain a manageable expression

$$
\begin{aligned}
E_1^\Delta \approx &\, (1 + y'^2)y''' - 3y'y''^2 - Ky''^2 \\
&- \frac{1}{24} \frac{y''^3}{[1 + y'^4](h_n + h_{n+1} + h_{n+2})} \\
&\times \Big\{ K^2 \big[16\alpha(h_n + h_{n+1} + h_{n+2})^2 \\
&- 4h_n^2 - 12h_{n+2}^2 - 8h_{n+1}^2 - 16h_n h_{n+2} \\
&- 20h_{n+1}h_{n+2} + 12h_n h_{n+1} \big] \\
&\qquad + 9h_{n+1}(h_{n+2} - h_n) \Big\} = 0. \quad (26)
\end{aligned}
$$

Expression (26) also satisfies

$$
\mathrm{pr}^\Delta \hat{X}_i E_1^\Delta \Big|_{E_1^\Delta = E_2^\Delta = 0} = 0, \quad i = 1, \cdots, 4 \quad (27)
$$

so the first differential approximation of the entire scheme (23), (24) is invariant under $Sim(2)$.

4. Equations invariant under a one-dimensional realization of $SL(2, \mathbb{R})$.

Four non-equivalent realizations of $sl(2, \mathbb{R})$ by vector fields in two variables exist [29]. Invariant difference schemes for second and third order ODEs have been constructed and tested for all of them [26–28]. In this section we will consider the first one, called $sl_1(2, \mathbb{R})$, or alternatively $sl_y(2, \mathbb{R})$, which actually involves one variable only:

$$
\hat{X}_1 = \frac{\partial}{\partial y}, \quad \hat{X}_2 = y\frac{\partial}{\partial y}, \quad \hat{X}_3 = y^2 \frac{\partial}{\partial y}. \quad (28)
$$

The corresponding Lie group acts by Mobius transformations (fractional linear transformations) on y. The third order differential invariants of this action are the Schwarzian derivatives of y and the independent variable x. The most general third order invariant ODE is

$$
\frac{1}{y'^2}\Big(y'y''' - \frac{3}{2}y''^2 \Big) = F(x). \quad (29)
$$

where $F(x)$ is arbitrary. For $F(x) = K = \mathrm{const}$ the group is $GL(2, \mathbb{R})$ and (28) is extended by the vector field $\hat{X}_4 = \partial_x$. For $F(x) = 0$ the symmetry group is $SL_y(2, \mathbb{R}) \otimes SL_x(2, \mathbb{R})$.

The difference invariants on a four point stencil are

$$
R = \frac{(y_{n+2} - y_n)(y_{n+1} - y_{n-1})}{(y_{n+2} - y_{n+1})(y_n - y_{n-1})}, \quad x_n, h_{n+2}, h_{n+1}, h_n. \quad (30)
$$

The discrete invariant approximating the left hand side of (29) is

$$
\begin{aligned}
J_1 = &\, \frac{6h_{n+2}h_n}{h_{n+1}(h_{n+1}+h_{n+2})(h_n+h_{n+1})(h_{n+2}+h_{n+1}+h_n)} \\
&\times \left[\frac{(h_{n+2}+h_{n+1})(h_{n+1}+h_n)}{h_n h_{n+2}} - R \right]. \quad (31)
\end{aligned}
$$

Any lattice depending only on x_k will be invariant, in particular the lattice equation can be chosen to be

$$
x_{n+1} - 2x_n + x_{n-1} = 0. \quad (32)
$$

The general solution of (32) is

$$
x_n = nh + c_0, \quad (33)
$$

i.e. a uniform lattice with origin x_0 and spacing $x_{n+1} - x_n = h$.

Let us consider the invariant ODE

$$
\frac{1}{y'^2}\Big(y'y''' - \frac{3}{2}y''^2 \Big) = \sin(x). \quad (34)
$$

and approximate it on an a priori arbitrary lattice. Such a scheme is given by

$$
E^\Delta = J_1 - \sin(\xi) = 0, \quad (35)
$$
$$
\xi = x_n + ah_n + bh_{n+1} + ch_{n+2},
$$
$$
\phi(x_n, h_n, h_{n+1}, h_{n+2}) = 0, \quad (36)
$$

where a, b, c are constants and ϕ satisfies the condition

$$
\phi(x_n, 0, 0, 0) \equiv 0, \quad (37)
$$

(for instance ϕ can be linear as in (32). The first differential approximation of (35), after the usual simplifications, is

$$
\begin{aligned}
E_0 \approx &\, \frac{1}{y'^2}\Big(y'y''' - \frac{3}{2}y''^2 \Big) - \sin(x) \\
&+ \cos(x)\big\{ h_n(1 + 4a) - 2h_{n+1}(1 - 2b) \\
&\qquad - h_{n+2}(1 - 4c) \big\}. \quad (38)
\end{aligned}
$$

Eq. (38) is invariant under $SL_y(2, \mathbb{R})$. Moreover, if we choose

$$
a = -\frac{1}{4}, \quad b = \frac{1}{2}, \quad c = \frac{1}{4}, \quad (39)
$$

the second term in (38) vanishes completely and $E_0 = 0$ is a second order approximation of the ODE (34). We mention that the uniform lattice (33), used in [26] satisfies (39).

5. Equations invariant under a two-dimensional realization of $GL(2, \mathbb{R})$.

A genuinely two-dimensional realization of the algebra $gl(2, \mathbb{R})$ is given by the vector fields

$$
\begin{aligned}
\hat{X}_1 &= \frac{\partial}{\partial y}, & \hat{X}_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\
\hat{X}_3 &= 2xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, & \hat{X}_4 &= x\frac{\partial}{\partial x}. \quad (40)
\end{aligned}
$$

The $SL(2,\mathbb{R})$ group corresponding to $\{\hat{X}_1, \hat{X}_2, \hat{X}_3\}$ has two differential invariants of order $m \leq 3$:

$$I_1 = \frac{2xy'' + y'}{y'^3}, \quad I_2 = \frac{x^2(y'y''' - 3y''^2)}{y'^5}. \quad (41)$$

The expression $I_2/I_1^{\frac{3}{2}}$ is also invariant under the dilations generated by $\hat{X}_4$ and hence under $GL(2,\mathbb{R})$. The corresponding ODE invariant under $GL(2,\mathbb{R})$ is

$$I_2^2 = A^2 I_1^3 \quad (42)$$

which is equivalent to

$$E = x^4(y'y''' - 3y''^2)^2 - A^2 y'(2xy'' + y')^3 = 0. \quad (43)$$

Five independent $SL(2,\mathbb{R})$ difference invariants are

$$\xi_1 = \frac{1}{\sqrt{x_{n+1}x_{n+2}}}(y_{n+2} - y_{n+1}),$$

$$\xi_2 = \frac{1}{\sqrt{x_n x_{n+1}}}(y_{n+1} - y_n),$$

$$\xi_3 = \frac{1}{\sqrt{x_{n-1}x_n}}(y_n - y_{n-1}),$$

$$\xi_4 = \frac{1}{\sqrt{x_n x_{n+2}}}(y_{n+2} - y_n),$$

$$\xi_5 = \frac{1}{\sqrt{x_{n+1}x_{n-1}}}(y_{n+1} - y_{n-1}). \quad (44)$$

Any 4 ratios ξ_i/ξ_k will be $GL(2,\mathbb{R})$ invariants. The $SL(2,\mathbb{R})$ invariants that have the correct continuous limits are

$$J_2 = \frac{12}{\xi_2(\xi_1 + \xi_2 + \xi_3)}\left[\frac{\xi_4 - \xi_1 - \xi_2}{\xi_1(\xi_1 + \xi_2)} - \frac{\xi_5 - \xi_2 - \xi_3}{\xi_3(\xi_2 + \xi_3)}\right]$$

$$J_1 = 8\left[\alpha \frac{\xi_4 - \xi_1 - \xi_2}{\xi_1\xi_2(\xi_1 + \xi_2)} + (1-\alpha)\frac{\xi_5 - \xi_2 - \xi_3}{\xi_2\xi_3(\xi_2 + \xi_3)}\right]. \quad (45)$$

The difference scheme for the ODE (43) with $A = -1$ used in Ref. [26] was actually the square root of (42), equivalent to

$$E_1^\Delta = J_2 + J_1^{\frac{3}{2}} = 0, \quad (46)$$

$$E_2^\Delta = \frac{\xi_1}{\xi_2} = \gamma = \text{const.} \quad (47)$$

The first differential approximation of the lattice equation (47) is

$$E_1^{(\Delta)} \approx (\gamma h_{n+1} - h_{n+2})y'$$

$$+ \frac{1}{2x^2}\left\{[-\gamma h_{n+1}^2 + h_{n+2}(h_{n+2} + 2h_{n+1})]y'\right.$$

$$\left. + xy''[\gamma h_{n+1}^2 - h_{n+1}h_{n+2} - h_{n+2}^2]\right\} = 0. \quad (48)$$

Both terms in (48) are invariant under $GL(2,\mathbb{R})$.

The first differential approximation to the difference equation (46) is

$$E_1^{(\Delta)} \approx \frac{E}{y'^{10}} - \frac{\sqrt{y'}(y' + 2xy'')^{\frac{7}{2}}}{16xy'^{10}(h_n + h_{n+1} + h_{n+2})}$$

$$\times \left[h_n^2(32\alpha - 11) + 16h_{n+1}^2(2\alpha - 1)\right.$$

$$+ h_{n+2}^2(32\alpha - 21) + h_n h_{n+1}(64\alpha - 21)$$

$$\left. + 32h_n h_{n+2}(2\alpha - 1) + h_{n+1}h_{n+2}(64\alpha - 43)\right]. \quad (49)$$

The second term of expression (49) has been simplified using (43) and its differential consequences. Again, (49) is invariant under the entire $GL(2,\mathbb{R})$ group.

6. CONCLUSIONS

The three examples considered above in Sections 3, 4 and 5 confirm that the method of invariant discretization provides a systematic way of constructing difference schemes for which the first differential approximation is invariant under the entire symmetry group of the original ODE.

The two equations (13) determining the invariant difference scheme for an ODE are not unique since there are more difference invariants than differential ones. The differential approximation of an invariant scheme can be used to benefit from this freedom and to choose an invariant scheme with a higher degree of accuracy. An example of this is given in Section 3, eq. (38) where the choice of the parameters (39) assures that the terms of order ϵ in (38) vanish identically.

Previous numerical comparisons between invariant discretization and standard noninvariant numerical methods for ODEs [26–28] have shown two features. The first is that the discretization errors for invariant schemes are significantly smaller [26] (by 3 orders of magnitude for eq. (18), 1 order of magnitude for eq. (43)). The second feature is that the qualitative behaviour of solutions close to singularities is described much more accurately by the invariant schemes [26–28].

An analysis of the relation between invariant discretization and the differential approximation method for PDEs is in progress.

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