Two Remarks to Bifullness of Centers of Archimedean Atomic Lattice Effect Algebras

M. Kalina

Abstract

Lattice effect algebras generalize orthomodular lattices as well as MV-algebras. This means that within lattice effect algebras it is possible to model such effects as unsharpness (fuzziness) and/or non-compatibility. The main problem is the existence of a state. There are lattice effect algebras with no state. For this reason we need some conditions that simplify checking the existence of a state. If we know that the center $C(E)$ of an atomic Archimedean lattice effect algebra E (which is again atomic) is a bifull sublattice of E, then we are able to represent E as a subdirect product of lattice effect algebras E_i where the top element of each one of E_i is an atom of $C(E)$. In this case it is enough if we find a state at least in one of E_i and we are able to extend this state to the whole lattice effect algebra E. In [8] an atomic lattice effect algebra E (in fact, an atomic orthomodular lattice) with atomic center $C(E)$ was constructed, where $C(E)$ is not a bifull sublattice of E. In this paper we show that for atomic lattice effect algebras E (atomic orthomodular lattices) neither completeness (and atomicity) of $C(E)$ nor σ-completeness of E are sufficient conditions for $C(E)$ to be a bifull sublattice of E.

Keywords: lattice effect algebra, orthomodular lattice, center, atom, bifullness.

1 Preliminaries

Effect algebras, introduced by D. J. Foulis and M. K. Bennett [3], have their importance in the investigation of uncertainty. Lattice ordered effect algebras generalize orthomodular lattices and MV-algebras. Thus they may include non-compatible pairs of elements as well as unsharp elements.

Definition 1 *(Foulis and Bennett [3]) An effect algebra is a system* $(E; \oplus, \mathbf{0}, \mathbf{1})$ *consisting of a set* E *with two different elements* $\mathbf{0}$ *and* $\mathbf{1}$*, called zero and unit, respectively and* $\oplus$ *is a partially defined binary operation satisfying the following conditions for all* $p, q, r \in E$:

(E1) If $p \oplus q$ *is defined, then* $q \oplus p$ *is defined and* $p \oplus q = q \oplus p$.

(E2) If $q \oplus r$ *is defined and* $p \oplus (q \oplus r)$ *is defined, then* $p \oplus q$ *and* $(p \oplus q) \oplus r$ *are defined and* $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(E3) For every $p \in E$ *there exists a unique* $q \in E$ *such that* $p \oplus q$ *is defined and* $p \oplus q = \mathbf{1}$.

(E4) If $p \oplus \mathbf{1}$ *is defined then* $p = \mathbf{0}$.

The element q in (E3) will be called the *supplement* of p, and will be denoted as p'.

In the whole paper, for an effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$, writing $a \oplus b$ for arbitrary $a, b \in E$ will mean that $a \oplus b$ exists. On an effect algebra E we may define another partial binary operation $\ominus$ by

$$a \ominus b = c \quad \Leftrightarrow \quad b \oplus c = a.$$

The operation $\ominus$ induces a partial order on E. Namely, for $a, b \in E$ $b \leq a$ if there exists a $c \in E$ such that $a \ominus b = c$. If E with respect to $\leq$ is lattice ordered, we say that E is a *lattice effect algebra*. For the sake of brevity we will write just LEA. Further, in this article we often briefly write 'an effect algebra E' skipping the operations.

S. P. Gudder ([5, 6]) introduced the notion of sharp elements and sharply dominating lattice effect algebras. Recall that an element x of the LEA E is called *sharp* if $x \wedge x' = \mathbf{0}$. Jenča and Riečanová in [7] proved that in every lattice effect algebra E the set $S(E) = \{x \in E; x \wedge x' = \mathbf{0}\}$ of sharp elements is an orthomodular lattice which is a *sub-effect algebra* of E, meaning that if among $x, y, z \in E$ with $x \oplus y = z$ at least two elements are in $S(E)$ then $x, y, z \in S(E)$. Moreover $S(E)$ is a *full sublattice* of E, hence a supremum of any set of sharp elements, which exists in E, is again a sharp element. Further, each maximal subset M of pairwise compatible elements of E, called a *block* of E, is a sub-effect algebra and a full sublattice of E and $E = \bigcup\{M \subseteq E; M \text{ is a block of } E\}$ (see [16, 17]). *Central elements* and centers of effect algebras were defined in [4]. In [14, 15] it was proved that in every lattice effect algebra E the *center*

$$\begin{aligned} C(E) &= \{x \in E; (\forall y \in E) y = (y \wedge x) \vee (y \wedge x')\} \\ &= S(E) \cap B(E), \end{aligned} \tag{1}$$

where $B(E) = \bigcap\{M \subseteq E; M \text{ is a block of } E\}$. Since $S(E)$ is an orthomodular lattice and $B(E)$ is an

MV-effect algebra, we obtain that $C(E)$ is a Boolean algebra. Note that E is an orthomodular lattice if and only if $E = S(E)$ and E is an MV-effect algebra if and only if $E = B(E)$. Thus E is a Boolean algebra if and only if $E = S(E) = B(E) = C(E)$.

Recall that an element p of an effect algebra E is called an *atom* if and only if p is a minimal non-zero element of E and E is *atomic* if for each $x \in E$, $x \neq \mathbf{0}$, there exists an atom $p \leq x$.

Definition 2 *Let $(E, \oplus, 0)$ be an effect algebra. To each $a \in E$ we define its isotropic index, notation $ord(a)$, as the maximal positive integer n such that*

$$na := \underbrace{a \oplus \ldots \oplus a}_{n\text{-times}}$$

exists. We set $ord(a) = \infty$ if na exists for each positive integer n. We say that E is Archimedean, if for each $a \in E$, $a \neq \mathbf{0}$, $ord(a)$ is finite.

An element $u \in E$ is called *finite*, if there exists a finite system of atoms $a_1, \ldots, a_n$ (which are not necessarily distinct) such that $u = a_1 \oplus \ldots \oplus a_n$. An element $v \in E$ is called *cofinite*, if there exists a finite element $u \in E$ such that $v = u'$.

We say that for a finite system $F = (x_j)_{j=1}^k$ of not necessarily different elements of an effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is $\oplus$-*orthogonal* if for all $n \leq k$ $x_1 \oplus x_2 \oplus \cdots \oplus x_n = (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n$ exists in E (briefly we will write $\bigoplus_{j=1}^{n} x_j$). We define also $\oplus\emptyset = \mathbf{0}$.

Definition 3 *For a lattice $(L, \wedge, \vee)$ and a subset $D \subseteq L$ we say that D is a bifull sublattice of L, if and only if for any $X \subseteq D$, $\bigvee_L X$ exists if and only if $\bigvee_D X$ exists and $\bigwedge_L X$ exists if and only if $\bigwedge_D X$ exists, in which case $\bigvee_L X = \bigvee_D X$ and $\bigwedge_L X = \bigwedge_D X$.*

It is known that if E is a distributive effect algebra (i.e., the effect algebra E is a distributive lattice — e.g., if E is an MV-effect algebra) then $C(E) = S(E)$. If moreover E is Archimedean and atomic then the set of atoms of $C(E) = S(E)$ is the set $\{n_a a; a \in E \text{ is an atom of } E\}$, where $n_a = ord(a)$ (see [20]). Since $S(E)$ is a bifull sublattice of E if E is an Archimedean atomic LEA (see [13]), we obtain that

$$\begin{aligned}\mathbf{1} &= \bigvee_{C(E)} \{p \in C(E); p \text{ is an atom of } C(E)\} \\ &= \bigvee_E \{p \in C(E); p \text{ is an atom of } C(E)\}\end{aligned}$$

for every Archimedean atomic distributive lattice effect algebra E. In [8] it was shown that there exists an LEA E for which this property fails to be true. Important properties of Archimedean atomic lattice effect algebras with an atomic center were proven by Riečanová in [21].

Theorem 1 *(Riečanová [21]) Let E be an Archimedean atomic lattice effect algebra with an atomic center $C(E)$. Let A_E be the set of all atoms of E and $A_{C(E)}$ the set of all atoms of $C(E)$. The following conditions are equivalent:*

1. $\bigvee_E A_{C(E)} = \mathbf{1}$.
2. *For every atom $a \in A_E$ there exists an atom $p_a \in A_{C(E)}$ such that $a \leq p_a$.*
3. *For every $z \in C(E)$ it holds*

$$\begin{aligned} z &= \bigvee_{C(E)} \{p \in A_{C(E)}; p \leq z\} \\ &= \bigvee_E \{p \in A_{C(E)}; p \leq z\}. \end{aligned}$$

4. *$C(E)$ is a bifull sub-lattice of E.*

In this case E is isomorphic to a subdirect product of Archimedean atomic irreducible lattice effect algebras.

Theorem 2 *(Paseka, Riečanová [13]) Let E be an atomic Archimedean lattice effect algebra. Then the set $S(E)$ of all sharp elements of E is a bifull sublattice of E.*

We will deal only with atomic Archimedean lattice effect algebras E. We have $C(E) \subset S(E) \subset E$. Because of this inclusion and Theorem 2, considering the bifullness of the center $C(E)$ in E is equivalent to considering the bifullness of $C(E)$ in $S(E)$. And $S(E)$ is an orthomodular lattice. For this reason, in the rest of the paper we will restrict our attention to atomic orthomodular lattices L and their centers $C(L)$. For the sake of completeness, we give the definition of an orthomodular lattice.

Definition 4 *Let L be a bounded lattice with a unary operation $'$ (called complementation) satisfying the following conditions*

1. *for all $a \in L$ $(a')' = a$,*
2. *for all $a, b \in L$ if $a \leq b$ then $b' \leq a'$,*
3. *for all $a, b \in L$ if $a \leq b$ then $a \vee (a' \wedge b) = b$.*

Then L is said to be an orthomodular lattice (OML for brevity).

Remark 1 Though in OML's we have just lattice-theoretical operations $\vee$ and $\wedge$, we will use also effect algebraic operations $\oplus$ and $\ominus$ with the meaning $a \oplus b = a \vee b$ iff $a \leq b'$ and $a \ominus b = c$ iff $b \oplus c = a$.

2 Orthomodular lattice L whose center is not a bifull sublattice

Let us have the following sequences of atoms (sets):

$$\begin{aligned} a_0 &= \{(x,y) \in \mathbb{R}^2; 0 \le x \le 1,\ y \in \mathbb{R}\}, \\ a_l &= \{(x,y) \in \mathbb{R}^2; l < x \le l+1,\ y \in \mathbb{R}\}, \\ &\quad \text{for } l = 1,2,\ldots, \\ b_0 &= \{(x,y) \in \mathbb{R}^2; -1 \le x < 0,\ y \in \mathbb{R}\}, \\ b_l &= \{(x,y) \in \mathbb{R}^2; -l-1 \le x < -l,\ y \in \mathbb{R}\}, \\ &\quad \text{for } l = 1,2,\ldots, \qquad (2) \\ c_j &= \{(x,y) \in \mathbb{R}^2; -j \le x \le j,\ y \le j \cdot x\}, \\ &\quad \text{for } j = 1,2,\ldots, \\ d_j &= \{(x,y) \in \mathbb{R}^2; -j \le x \le j,\ y > j \cdot x\}, \\ &\quad \text{for } j = 1,2,\ldots, \\ p_j &= \{j\}, \quad \text{for } j = 1,2,\ldots. \end{aligned}$$

For such a choice of atoms, $q_1 \ne q_2$ are compatible if and only if $q_1 \cap q_2 = \emptyset$. Fig. 1 shows the compatibility among atoms. For their non-compatibility (denoted by $\not\leftrightarrow$) the following rules hold

$$\begin{aligned} &c_j \not\leftrightarrow a_i, \quad c_j \not\leftrightarrow b_i && \text{for all} \quad j = 1,2,\ldots \\ & && \text{and} \quad i = 0,\ldots,j-1, \\ &d_j \not\leftrightarrow a_i, \quad d_j \not\leftrightarrow b_i && \text{for all} \quad j = 1,2,\ldots \\ & && \text{and} \quad i = 0,\ldots,j-1, \\ &c_j \not\leftrightarrow d_i && \text{for all} \quad i,j = 1,2,\ldots \\ & && \text{such that } i \ne j, \\ &c_j \not\leftrightarrow c_i, \quad d_j \not\leftrightarrow d_i && \text{for all} \quad i,j = 1,2,\ldots \\ & && \text{such that } i \ne j. \end{aligned}$$

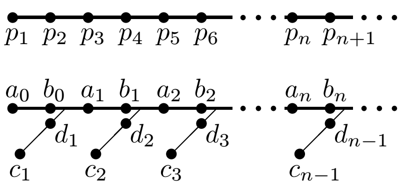

Fig. 1: Greechie diagram of sets of atoms

For non-compatible atoms the following equalities hold

$$\begin{aligned} c_j \oplus d_j &= \bigoplus_{i=0}^{j-1}(a_i \oplus b_i) \\ &= c_k \vee c_j = d_k \vee d_j \\ &= c_k \vee d_j = d_k \vee c_j \\ &= c_j \vee a_l = c_j \vee b_l \\ &= d_j \vee a_l = d_j \vee b_l \end{aligned}$$

for $1 \le k < j$ and $0 \le l < j$.

Denote $\hat{B}_0$, $\hat{B}_j$ (for $j = 1,2,\ldots$) complete atomic Boolean algebras with the corresponding sets of atoms A_0, A_j ($j = 1,2,\ldots$), given by

$$A_0 = \bigcup_{i=0}^{\infty}\{a_i\} \cup \bigcup_{i=0}^{\infty}\{b_i\} \cup \bigcup_{j=1}^{\infty}\{p_j\}, \qquad (3)$$

$$A_j = \bigcup_{i=j}^{\infty}\{a_i\} \cup \bigcup_{i=j}^{\infty}\{b_i\} \cup \bigcup_{j=1}^{\infty}\{p_j\} \cup \{c_j, d_j\}. \qquad (4)$$

Disjointness occurring among some atoms of the system (2) is equivalent to the fact that A_0 and A_j ($j = 1,2,\ldots$) are unique maximal sets of pairwise compatible atoms.

Theorem 3 *(Kalina [9]) Let* $\hat{L} = \bigcup_{i=0}^{\infty} \hat{B}_i$. *Let* L_1 *be the complete OML generated by sets of atoms* $\bigcup_{i=0}^{\infty}\{a_i, b_i\} \cup \bigcup_{j=1}^{\infty}\{c_j, d_j\}$ *and* $\mathbf{N}$ *the complete Boolean algebra generated by the set of atoms* $\bigcup_{j=1}^{\infty}\{p_j\}$. *Then* $(\hat{L}, \vee, \wedge, \mathbf{0}, \mathbf{1})$ *is a complete OML and* $\hat{L} \cong L_1 \times \mathbf{N}$.

An element $u \in \hat{B}_l$ is finite if and only if $u = q_1 \oplus q_2 \oplus \ldots \oplus q_n$ for an $n \in \mathbb{N}$ and $q_1, q_2, \ldots, q_n \in A_l$. Set $Q_l = \{u \in B_l; u \text{ is finite}\}$, $l = 0,1,2,\ldots$. Then Q_l is a generalized Boolean algebra, since $B_l = Q_l \,\dot{\cup}\, Q_l^*$ is a Boolean algebra, where $Q_l^* = \{u^*; u^* = 1_l \ominus u$ and $u \in Q_l\}$ (see [22], or [2, pp. 18-19]). This means that B_l is a Boolean subalgebra of finite and cofinite elements of $\hat{B}_l$ ($l = 0,1,2,\ldots$).

Theorem 4 *(Kalina [8]) Denote* $L = \bigcup_{l=0}^{\infty} B_l$. *Then* $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ *is a compactly generated orthomodular lattice with the family* $(B_l)_{l=0}^{\infty}$ *of atomic blocks of* L. *The center of* L, $C(L)$, *is not a bifull sublattice of* L.

3 Completion of the center of L

We are going to show that it is possible to extend the orthomodular lattice L from Theorem 4 to $\bar{L}$, whose center, $C(\bar{L})$, is a complete Boolean algebra which is not a bifull sublattice of $\bar{L}$.

Denote $\mathcal{F}$ a fixed non-trivial ultrafilter on $\mathbb{N}$ (the index set of atoms p_j). Then $\mathcal{F}$ has the following properties which will be important for our construction:

- Let $F \subset \mathbb{N}$. Then either $F \in \mathcal{F}$ or $\mathbb{N} \setminus F \in \mathcal{F}$.
- Let $F \subset \mathbb{N}$ be a finite set. Then $F \notin \mathcal{F}$.
- If $F_1 \in \mathcal{F}$ and $F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$.
- If $F_1 \in \mathcal{F}$ and $F_2 \supset F_1$ then $F_2 \in \mathcal{F}$.

Let Q_{L_1} denote the set of all finite elements of L_1. Further set

$$P_{\mathcal{F}} = \left\{ \bigoplus_{i\in F} p_i; F \notin \mathcal{F} \right\} \tag{5}$$

and

$$\begin{aligned} G &= \{f \oplus g; g \in Q_{L_1}, f \in P_{\mathcal{F}}\}, \\ G^{\perp} &= \{h' \in \hat{L}; h \in G\}. \end{aligned}$$

Theorem 5 *Let $\tilde{L} = G \dot{\cup} G^{\perp}$. Then the system $\left(\tilde{L}, \vee, \wedge, \mathbf{0}, \mathbf{1}\right)$ is an orthomodular lattice. The center*

$$C(\tilde{L}) = \{f \in \tilde{L}; f \in P_{\mathcal{F}} \text{ or } f' \in P_{\mathcal{F}}\},$$

and $C(\tilde{L})$ is a complete Boolean algebra which is not bifull in $\tilde{L}$.

Proof. First we show that $\tilde{L}$ is a bounded lattice. Consider elements $h_1, h_2 \in G$. Then there exist elements $g_1, g_2 \in Q_{L_1}$ and elements $f_1, f_2 \in P_{\mathcal{F}}$ such that

$$h_1 = f_1 \oplus g_1, \quad h_2 = f_2 \oplus g_2. \tag{6}$$

By the properties of the non-trivial ultrafilter $\mathcal{F}$ we get that $f_1 \vee f_2 \in P_{\mathcal{F}}$ and $f_1 \wedge f_2 \in P_{\mathcal{F}}$. Since g_1, g_2 are finite elements of L_1, we get that $g_1 \vee g_2 \in Q_{L_1}$ and also $g_1 \wedge g_2 \in Q_{L_1}$. Since L_1 is generated by the sets of atoms $\bigcup_{i=0}^{\infty}\{a_i, b_i\}$ and $\bigcup_{j=1}^{\infty}\{c_j, d_j\}$, each $g \in Q_{L_1}$ is $\oplus$-orthogonal to each $f \in P_{\mathcal{F}}$. This implies that G is closed under $\vee$ and $\wedge$. Because $G^{\perp}$ consists of complements of elements of G, we have that also $G^{\perp}$ is closed under $\vee$ and $\wedge$. Now assume that $h_1 \in G$ and $h_2 \in G^{\perp}$. Then $h_2' \in G$ and we can write

$$h_1 = f_1 \oplus g_1, \quad h_2' = f_2 \oplus g_2$$

with the same meaning of f_1, f_2, g_1, g_2 as in formula (6). This means that $h_2 = f_2' \ominus g_2$. Then, because of the monotonicity of the ultrafilter $\mathcal{F}$, we have $(f_1 \vee f_2')' \in P_{\mathcal{F}}$ and hence $f_1 \vee f_2' \in G^{\perp}$. Moreover, $g_2 \in Q_{L_1}$ is orthogonal to f_1 which implies

$$(f_1 \vee f_2') \ominus g_2 = f_1 \vee (f_2' \ominus g_2) \in G^{\perp}.$$

Since G is a monotone system (meaning that with an arbitrary element $\delta_1 \in G$ it contains also all elements $\delta_2 \in \hat{L}$ such that $\delta_2 \leq \delta_1$), we get from the duality between G and $G^{\perp}$ that

$$(f_1 \vee g_1) \vee (f_2' \ominus g_2) = h_1 \vee h_2 \in G^{\perp}$$

Dually we get that $h_1 \wedge h_2 \in G$. This implies that $\tilde{L} = G \dot{\cup} G^{\perp}$ is a lattice. Obviously it is a bounded and orthocomplemented lattice. Showing that it is an OML is a matter of routine. We will omit the detailed proof.

Let us consider an element $f \in \tilde{L}$ such that $f \in P_{\mathcal{F}}$ or $f' \in P_{\mathcal{F}}$. Then f is a central element. If f is such that neither $f \in P_{\mathcal{F}}$ nor $f' \in P_{\mathcal{F}}$, then there exist atoms $\alpha_1, \alpha_2 \in \bigcup_{i=0}^{\infty}\{a_i, b_i\} \cup \bigcup_{j=1}^{\infty}\{c_j, d_j\}$ fulfilling $\alpha_1 \not\leftrightarrow \alpha_2$ and $\alpha_1 \leq f$, $\alpha_2 \not\leq f$. Then f is not a central element. This proves that

$$C(\tilde{L}) = \{f \in \tilde{L}; f \in P_{\mathcal{F}} \text{ or } f' \in P_{\mathcal{F}}\}.$$

Due to the fact that $\mathcal{F}$ is a non-trivial ultrafilter, $C(\tilde{L})$ is a complete Boolean algebra.

The only central element that is greater than all atoms p_j for $j = 1, 2, \ldots$, is $\mathbf{1}$, hence we have that

$$\bigvee_{C(\tilde{L})} \{p_j; j = 1, 2, \ldots\} = \mathbf{1}.$$

On the other hand, let us take an arbitrary atom $\alpha \in \bigcup_{i=0}^{\infty}\{a_i, b_i\} \cup \bigcup_{j=1}^{\infty}\{c_j, d_j\}$ and assume that $\bigvee_{\tilde{L}}\{p_j; j = 1, 2, \ldots\}$ does exist. Since α is orthogonal to all atoms from the set $\bigcup_{j=1}^{\infty}\{p_j\}$, we have that α is orthogonal to $\bigvee_{\tilde{L}}\{p_j; j = 1, 2, \ldots\}$ and hence

$$\bigvee_{\tilde{L}}\{p_j; j = 1, 2, \ldots\} \neq \mathbf{1}.$$

It can be shown (see [8]) that $\bigvee_{\tilde{L}}\{p_j; j = 1, 2, \ldots\}$ does not exist. This implies that $C(\tilde{L})$ is not a bifull sublattice of $\tilde{L}$. □

4 σ-complete orthomodular lattice $\tilde{L}_\sigma$ whose center is not a bifull sublattice

Let $\mathcal{I}$ denote the set of all ordinal numbers less than Ω (the first uncountable ordinal number). Further, denote $\mathcal{E}$ the set of all limit ordinal numbers up to Ω and $\mathcal{J} = \mathcal{I} \setminus \mathcal{E}$.

Assume sets of elements $\{p_i; i \in \mathcal{I}\}$, $\{a_i; i \in \mathcal{I}\}$, $\{b_i; i \in \mathcal{I}\}$, $\{c_i; i \in \mathcal{I}\}$, $\{d_i; i \in \mathcal{I}\}$, where the corresponding elements for $i \in \mathcal{J}$ will act as atoms. We will have a partial relation $\not\leftrightarrow$ modelling non-compatibility. This partial relation will have the following form among atoms

$$\begin{aligned} &c_j \not\leftrightarrow a_i, \quad c_j \not\leftrightarrow b_i \quad \text{for all} \quad j \in \mathcal{J} \quad \text{and} \quad i \leq j, \\ &d_j \not\leftrightarrow a_i, \quad d_j \not\leftrightarrow b_i \quad \text{for all} \quad j \in \mathcal{J} \quad \text{and} \quad i \leq j, \end{aligned}$$

$$\begin{aligned} c_j &\not\leftrightarrow d_i && \text{for all } i,j \in \mathcal{J} \\ & && \text{such that } i \neq j, \\ c_j &\not\leftrightarrow c_i, \quad d_j \not\leftrightarrow d_i && \text{for all } i,j \in \mathcal{J} \\ & && \text{such that } i \neq j. \end{aligned}$$

Sets of elements $\{p_i; i \in \mathcal{I}\}$, $\{a_i; i \in \mathcal{I}\}$, $\{b_i; i \in \mathcal{I}\}$, $\{c_i; i \in \mathcal{I}\}$, $\{d_i; i \in \mathcal{I}\}$ will present atoms for $i \in \mathcal{J}$ and for $\kappa \in \mathcal{E}$ we will have

$$p_\kappa = \bigvee_{i<\kappa} p_i, \tag{7}$$

$$a_\kappa = \bigvee_{i<\kappa} a_i, \tag{8}$$

$$b_\kappa = \bigvee_{i<\kappa} b_i, \tag{9}$$

$$c_\kappa = \bigvee_{i<\kappa} c_i = \bigvee_{i<\kappa} d_i = d_\kappa = a_\kappa \oplus b_\kappa. \tag{10}$$

As a possible model for the just presented sets of elements fulfilling the non-compatibility relation we may have the following:

Let us choose a good order of positive real numbers of type Ω, i.e., positive real numbers will be enumerated by ordinal numbers from $\mathcal{J}$. For $i \in \mathcal{J}$ and $r > 0$, $r \in \mathbb{R}$, we denote r_i the i-th number in the chosen good order. Then we identify the set $\{p_i; i \in \mathcal{J}\}$ with the set of all positive real numbers, i.e., $p_i = r_i$. Further we put for $i, j \in \mathcal{J}$

$$\begin{aligned} a_i &= \{(r_i, y) \in \mathbb{R}^2; y \in \mathbb{R}\}, \\ b_i &= \{(-r_i, y) \in \mathbb{R}^2; y \in \mathbb{R}\}, \\ c_i &= \{(r_j, y) \in \mathbb{R}^2; j \leq i, y \leq r_i \cdot r_j\} \\ &\quad \cup \{(-r_j, y) \in \mathbb{R}^2; j \leq i, y \leq -r_i \cdot r_j\}, \\ d_i &= \{(r_j, y) \in \mathbb{R}^2; j \leq i, y > r_i \cdot r_j\} \\ &\quad \cup \{(-r_j, y) \in \mathbb{R}^2; j \leq i, y > -r_i \cdot r_j\}. \end{aligned}$$

For $\kappa \in \mathcal{E}$ we define the corresponding elements $p_\kappa, a_\kappa, b_\kappa, c_\kappa, d_\kappa$ by equalities 7, 8, 9, 10, respectively. Compatibility among different atoms is given by disjointness of the corresponding sets. This implies that the uniquely given maximal sets of pairwise compatible atoms are

$$\tilde{A}_0 = \bigcup_{i \in \mathcal{J}} \{a_i, b_i, p_i\},$$

$$\tilde{A}_j = \bigcup_{\substack{i \in \mathcal{J} \\ i > j}} \{a_i, b_i\} \cup \bigcup_{i \in \mathcal{J}} \{p_i\} \cup \{c_j, d_j\}$$

for $j \in \mathcal{J}$. Sets of atoms $\tilde{A}_0$ and $\tilde{A}_j$ for $j \in \mathcal{J}$, generate complete Boolean algebras $\tilde{B}_0$ and $\tilde{B}_j$ for $j \in \mathcal{J}$, respectively. For $\kappa \in \mathcal{E}$ we get complete atomic Boolean algebras $\tilde{B}_\kappa$ generated by sets of atoms

$$\tilde{A}_\kappa = \bigcup_{i \in \mathcal{J}} \{p_i\} \cup \{a_\kappa, b_\kappa\} \cup \bigcup_{\substack{i \in \mathcal{J} \\ i > \kappa}} \{a_i, b_i\}.$$

This means that for $\kappa \in \mathcal{E}$ $\tilde{B}_\kappa \subset \tilde{B}_0$. The union of all complete atomic Boolean algebras, $\tilde{L} = \tilde{B}_0 \cup \bigcup_{i \in \mathcal{I}} \tilde{B}_i$, is a complete OML. An element $f \in \tilde{L}$ will be called *countable* if there exists an at most countable set of atoms (an at most countable set of indices K) $\{q_k\}_{k \in K} \subset \tilde{A}_0$ or $\{q_k\}_{k \in K} \subset \tilde{A}_i$ for $i \in \mathcal{J}$, such that

$$f = \bigoplus_{k \in K} q_k.$$

By definition of elements p_i, a_i, b_i, c_i, d_i for $i \in \mathcal{I}$ we get that each of these elements is countable.

Let $\mathcal{K}$ denote the set of all countable elements of $\tilde{L}$ and $\mathcal{K}^\perp = \{f \in \tilde{L}; f' \in \mathcal{K}\}$. Further, let $\mathcal{P}$ denote the set of all countable elements generated by $\{p_i, i \in \mathcal{J}\}$, and $\mathcal{P}^\perp = \{f \in \tilde{L}; f' \in \mathcal{P}\}$.

Theorem 6 *Let $\tilde{L}_\sigma = \mathcal{K} \dot{\cup} \mathcal{K}^\perp$. Then $\left(\tilde{L}_\sigma, \vee, \wedge, \mathbf{0}, \mathbf{1}\right)$ is a σ-complete OML. The center $C(\tilde{L}_\sigma) = \mathcal{P} \dot{\cup} \mathcal{P}^\perp$ and it is not a bifull sublattice of $\tilde{L}_\sigma$.*

Proof. Each of the atoms p_i, a_i, b_i, c_i, d_i for $i \in \mathcal{J}$ (and hence also each of the elements p_i, a_i, b_i, c_i, d_i for $i \in \mathcal{I}$) is countable. This implies that $\tilde{L}_\sigma$ is an OML. Since it is by definition closed under countable meets and joins, it is σ-complete.

Elements p_i for $i \in \mathcal{I}$ are central because each of the elements p_i is compatible with all atoms of $\tilde{L}_\sigma$. This implies that $\mathcal{P} \dot{\cup} \mathcal{P}^\perp \subset C(\tilde{L}_\sigma)$. On the other hand, let f be a countable element, $f \notin \mathcal{P}$. Then there exists c_i such that $c_i \not\leq f$ for $i \in \mathcal{J}$ and an atom out of $e \in \{a_j, b_j, c_j, d_j\}$ for $j < i$, $e \leq f$. Then $c_i \not\leftrightarrow e$ and hence $c_j \not\leftrightarrow f$. Similarly, if $f \in \mathcal{K}^\perp$, there exists $c_i \leq f$ and an atom out of $e \in \{a_j, b_j, c_j, d_j\}$ for $j < i$ such that $e \not\leq f$. In this case $e \not\leftrightarrow c_i$ and hence also $e \not\leftrightarrow f$. We conclude that $C(\tilde{L}_\sigma) = \mathcal{P} \dot{\cup} \mathcal{P}^\perp$.

We show that $C(\tilde{L}_\sigma)$ is not a bifull sublattice of $\tilde{L}_\sigma$. Obviously

$$\bigvee_{C(\tilde{L}_\sigma)} \{p_i, i \in \mathcal{I}\} = \mathbf{1}.$$

Assume that $\bigvee_{\tilde{L}_\sigma} \{p_i, i \in \mathcal{I}\}$ does exist. Then all elements $e \in \bigcup_{i \in \mathcal{I}} \{a_i, b_i, c_i, d_i\}$ are orthogonal with all elements from the set $\bigcup_{\mathcal{I}} \{p_j\}$ and consequently also with $\bigvee_{\tilde{L}_\sigma} \{p_i, i \in \mathcal{I}\}$. This implies $\bigvee_{\tilde{L}_\sigma} \{p_i, i \in \mathcal{I}\} \neq \mathbf{1}$. This means that $C(\tilde{L}_\sigma)$ is not a bifull sublattice of $\tilde{L}_\sigma$. □

Acknowledgement

Support from the Science and Technology Assistance Agency under contract No. APVV-0073-10, and from the VEGA grant agency, grant number 1/0297/11, is gratefully acknowledged.

References

[1] Chang, C. C.: Algebraic analysis of many-valued logics. *Trans. Amer. Math. Soc.* **88** (1958), 467–490.

[2] Dvurečenskij, A., Pulmannová, S.: *New Trends in Quantum Structures.* Dordrecht, Boston, London, and Isterscience, Bratislava : Kluwer Acad. Publisher, 2000.

[3] Foulis, D. J., Bennett, M. K.: Effect algebras and unsharp quantum logics. *Found. Phys.* **24** (1994), 1 325–1 346.

[4] Greechie, R. J., Foulis, D. J., Pulmannová, S.: The center of an effect algebra. *Order* **12** (1995), 91–106.

[5] Gudder, S. P.: Sharply dominating effect algebras. *Tatra Mountains Math. Publ.* **15** (1998), 23–30.

[6] Gudder, S. P.: S-dominating effect algebras. *Internat. J. Theor. Phys.* **37** (1998), 915–923.

[7] Jenča, G., Riečanová, Z.: On sharp elements in lattice ordered effect algebras. *BUSEFAL* **80** (1999), 24–29.

[8] Kalina, M.: On central atoms of Archimedean atomic lattice effect algebras. *Kybernetika* **46** (2010), 4, 609–620.

[9] Kalina, M.: Mac Neille completion of centers and centers of Mac Neille completions of lattice effect algebras. *Kybernetika* **46** (2010), 6, 635–647.

[10] Kôpka, F.: Compatibility in D-posets. *Internat. J. Theor. Phys.* **34** (1995), 1 525–1 531.

[11] Mosná, K.: About atoms in generalized efect algebras and their effect algebraic extensions. *J. Electr. Engrg.* **57** (2006), 7/s, 110–113.

[12] Mosná, K., Paseka, J., Riečanová, Z.: Order convergence and order and interval topologies on posets and lattice effect algebras. In *Proc. internat. seminar UNCERTAINTY 2008*, Publishing House of STU 2008, 45–62.

[13] Paseka, J., Riečanová, Z.: The inheritance of BDE-property in sharply dominating lattice effect algebras and (*o*)-continuous states. *Soft Computing*, **15** (2011), 543–555.

[14] Riečanová, Z.: Compatibility and central elements in effect algebras. *Tatra Mountains Math. Publ.* **16** (1999), 151–158.

[15] Riečanová, Z.: Subalgebras, intervals and central elements of generalized effect algebras. *Internat. J. Theor. Phys.*, **38** (1999), 3 209–3 220.

[16] Riečanová, Z.: Generalization of blocks for D-lattices and lattice ordered effect algebras. *Internat. J. Theor. Phys.* **39** (2000), 231–237.

[17] Riečanová, Z.: Orthogonal sets in effect algebras. *Demontratio Mathematica* **34** (2001), 525–532.

[18] Riečanová, Z.: Smearing of states defined on sharp elements onto effect algebras. *Internat. J. Theor. Phys.* **41** (2002), 1 511–1524.

[19] Riečanová, Z.: Subdirect decompositions of lattice effect algebras. *Internat. J. Theor. Phys.* **42** (2003), 1 425–1 433.

[20] Riečanová, Z.: Distributive atomic effect akgebras. *Demontratio Mathematica* **36** (2003), 247–259.

[21] Riečanová, Z.: Lattice effect algebras densely embeddable into complete ones. *Kybernetika*, **47** (2011), 1, 100–109.

[22] Riečanová, Z., Marinová, I.: Generalized homogenous, prelattice and MV-effect algebras. *Kybernetika* **41** (2005), 129–142.

Martin Kalina
E-mail: kalina@math.sk
Dept. of Mathematics
Faculty of Civil Engineering
Slovak Univ. of Technology
Radlinského 11, Sk-813 68 Bratislava, Slovakia