Path Integral Solution of PT-/non-PT-Symmetric and non-Hermitian Hulthen Potential

N. Kandirmaz, R. Sever

Abstract

The wave functions and the energy spectrum of PT-/non-PT-Symmetric and non-Hermitian Hulthen potential are of an exponential type and are obtained via the path integral. The path integral is constructed using parametric time and point transformation.

Keywords: PT-symmetry, coherent states, path integral, Hulthen Potential.

1 Introduction

A suggestion by Bender and Boetcher on PT-symmetric quantum mechanics has put forward a different point of view from standard quantum mechanics. For a quantum mechanical system have to a real energy spectrum, the Hamiltonian must be Hermitian. Bender and his co-workers showed that even if a Hamiltonian is not Hermitian, it has a real energy spectrum [1]. PT-symmetric and non Hermitian potentials have been studied to prove they have a real energy spectrum, using numerical and analytical techniques. The energy spectrum corresponding to the wave functions is also calculated [2–9].

In this work, we have used Feynman's path integral method to get the energy spectrum and the wave functions of the PT-/Non-PT-Symmetric and non-Hermitian exponential potential. The Feynman Path Integral is a given kernel which has transition amplitudes between the initial and final positions of the energy dependent Green function. A Feynman Path Integral formalism for deriving the kernel of various potentials was developed in [10–16]. Duru derived the wave functions and the energy spectrum of the Wood-Saxon potential for s-waves via the radial path integral. Inomata obtained the energy spectrum and the normalized s-state eigenfunctions for the Hulthen Potential using the Green function [11]. The kernel of the Hulthen potential can be exactly solved given the path integral for the particle motion on the SU(2) manifold S^3 [10–12]. In Sec. II and III, we derive the energy dependent Green's function of the PT-/Non-PT-Symmetric and non-Hermitian q-deformed Hulthen Potential. We obtained the energy eigenvalues and the corresponding wave functions.

2 PT-Symmetric and Non-Hermitian Hulthen Potential

The kernel of a point particle moving in the $V(x)$ potential in one dimension is represented by the following path integral

$$K(x_b, t_b; x_a, t_a) = \int \frac{DxDp}{2\pi} \cdot \exp\{i \int \mathrm{d}t [p\dot{x} - \frac{p^2}{2m} - V(x)]\} \tag{1}$$

where $\hbar = 1$. The kernel expresses the probability amplitude of a particle moving to position x_b at time t_b from position x_a at time t_a. The time interval can be divided into n equal parts

$$t_j - t_{j-1} = t_b - t_a = T \qquad j = 1, 2, 3, \ldots, N \tag{2}$$

and taking initial position is x_a and final position x_b, the kernel [11] can be performed as

$$K(x_b, T; x_a, 0) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \mathrm{d}x_i \prod_{i=1}^{n+1} \frac{\mathrm{d}p_i}{2\pi} \cdot \exp\{i \sum_{i=1}^{n+1} [p_i(x_i - x_{i-1}) - \frac{p_i^2}{2m} - V(x_i)]\}. \tag{3}$$

The PT- symmetric and non-Hermitian potential is

$$V(x) = -\frac{V_o e^{-ix/a}}{1 - qe^{-ix/a}} \tag{4}$$

which is determined by taking $\frac{1}{a} \longrightarrow \frac{i}{a}$ in the q-deformed Hulthen potential [5].

We will start by applying point transformation to get a solvable path integral form for the Hulthen potential

$$\frac{1}{1 - qe^{-ix/a}} = \sin^2\theta \qquad p = \frac{i}{2a} \sin\theta \cos\theta p_\theta \tag{5}$$

Because of this transformation, there is a contribution to the Jacobi performed kernel

$$K(x_b,T;x_a,0)=\frac{q}{2a}\sin\theta_b\cos\theta_b\int D\theta Dp_\theta\times$$
$$\exp[i\int \mathrm{d}t(p_\theta\dot{\theta}+\frac{\sin^2\theta\cos^2\theta}{4\alpha^2}\frac{p_\theta^2}{2\mu}-V_0\cos^2\theta)]. \quad (6)$$

Here, the kinetic energy term becomes positive. We define a new time parameter s [12] to eliminate the $\frac{\sin^2\theta\cos^2\theta}{4\alpha^2}$ part in the kinetic energy term

$$\frac{\mathrm{d}t}{\mathrm{d}s}=-\frac{4a^2}{\sin^2\theta\cos^2\theta} \text{ or } t=-4a^2\int\frac{\mathrm{d}s'}{\sin^2\theta\cos^2\theta}. \quad (7)$$

If we use the Fourier transform of the $\delta-$ function, we can write

$$1 = \int \mathrm{d}S\int\frac{\mathrm{d}E}{2\pi}\frac{4a^2}{\sin^2\theta_b\cos^2\theta_b}\cdot$$
$$\exp\left[-i\left(ET-\int \mathrm{d}s\frac{4a^2E}{\sin^2\theta\cos^2\theta}\right)\right] \quad (8)$$

where $S=s_b-s_a$.

The factor in front of the path integral reached from the Jacobian can be a symmetrization according to points a and b, as follows

$$\frac{1}{\sin\theta_b\cos\theta_b} = \frac{2}{\sqrt{\sin 2\theta_a\sin 2\theta_b}}\cdot$$
$$\exp\left(i\int_0^S \mathrm{d}s\,(-i)\,\frac{\cos 2\theta}{\sin 2\theta}\dot{\theta}\right) \quad (9)$$

Thus Eq. (6) happens

$$K(x_b,x_a,T) = \int_0^\infty \mathrm{d}Se^{iS/2\mu}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}e^{iET}\cdot$$
$$\frac{4iaq}{\sqrt{\sin 2\theta_a\cos 2\theta_b}}K(\theta_b,\theta_a;S) \quad (10)$$

where

$$K(\theta_b,\theta_a;S)=\int D\theta Dp_\theta\cdot$$
$$\exp\left\{i\int_0^S \mathrm{d}s\left[p_\theta\dot{\theta}-\frac{p_\theta^2}{2\mu}-\right.\right. \quad (11)$$
$$\left.\left.\frac{1}{2\mu}\left(\frac{K(K-1)}{\sin^2\theta}+\frac{\lambda(\lambda-1)}{\cos^2\theta}\right)-\frac{ip_\theta\cos 2\theta}{2\mu\sin 2\theta}\right]\right\}$$

and K and λ are

$$K = \frac{1}{2}\left[1+\sqrt{32\mu a^2(V_0+E)}\right]$$
$$\lambda = \frac{1}{2}\left[1+\sqrt{32\mu a^2E}\right] \quad (12)$$

if the factor contribution to the Jacobian is symmetrized as [11] the contributions to the kernel become

$$\dot{\theta}_j\longrightarrow\dot{\theta}_j\pm\frac{i\cos\theta_j}{2\mu\sin\theta_j} \quad (13)$$

So the problem is transformed into the path integral for Pöschl-Teller potential, for which an exact solution is known [11]. $K(\theta_b,\theta_a;S)$ can be obtained as

$$K(\theta_b,\theta_a;S)=\int D\theta Dp_\theta\cdot$$
$$\exp\left\{i\int_0^S \mathrm{d}s\left[p_\theta\dot{\theta}-\frac{p_\theta^2}{2\mu}-\right.\right. \quad (14)$$
$$\left.\left.\frac{1}{2\mu}\left(\frac{K(K-1)}{\sin^2\theta}+\frac{\lambda(\lambda-1)}{\cos^2\theta}\right)\right]\right\}$$

The kernel can be obtained in the form

$$K(\theta_b,\theta_a;S)= \quad (15)$$
$$\sum_{n=0}^{\infty}\exp\left[-i\,(S/2\mu)\,(K+\lambda+2n)^2\right]\psi_n(\theta_a)\,\psi_n^*(\theta_b)$$

where

$$\psi_n(\theta) = \sqrt{2\,(K+\lambda+2n)}\cdot$$
$$\sqrt{\frac{\Gamma(n+1)\,\Gamma(K+\lambda+n)}{\Gamma\left(\lambda+n+\frac{1}{2}\right)\Gamma\left(K+n+\frac{1}{2}\right)}}\times \quad (16)$$
$$(\cos\theta)^\lambda\,(\sin\theta)^K\,P_n^{(K-1/2,\lambda-1/2)}\cdot$$
$$\left(1-2\sin^2\theta\right)$$

With integrating over $\mathrm{d}S$, the Green's function for the Hulthen potential can be obtained as

$$G(x_b,x_a;E)=\frac{8\mu aq}{\sqrt{\sin 2\theta_a\cos 2\theta_b}}\cdot \quad (17)$$
$$\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}\frac{e^{iET}}{(K+\lambda+2n)^2-1}\psi_n(\theta_a)\,\psi_n^*(\theta_b)$$

Therefore, the kernel of the physical system is rewritten as

$$K(x_b,x_a;E) = \sum_{n=0}^{\infty}e^{-iE_nT}\varphi_n(x_a)\varphi_n^*(x_b)=$$
$$\sum_{n=0}^{\infty}\exp\left\{\left[-\frac{1}{8\mu aq\,(n+1)^2}\cdot\right.\right.$$
$$\left.\left.\left[(n+1)^2+2\mu a^2V_0\right]^2\right]T\right\}\cdot$$
$$\phi_n(u_b)\phi_n^*(u_a). \quad (18)$$

Integrating over dE, we can get the energy eigenvalues

$$E_n=\frac{1}{8\mu a^2\,(n+1)^2}\left[2\mu a^2\frac{V_0}{q}-(n+1)^2\right]^2 \quad (19)$$

and the normalized wave functions in terms of Jacobi polynomials are

$$\phi(x) = \frac{1}{2\sqrt{2}\sqrt{n+1}}\sqrt{4\,(n+1)^2 - (\lambda_n - K_n)^2}\cdot$$
$$\sqrt{\frac{\Gamma\,(n+1)\,\Gamma\,(K_n+\lambda_n+n)}{\Gamma\left(\lambda_n+n+\frac{1}{2}\right)\Gamma\left(K_n+n+\frac{1}{2}\right)}}\times$$
$$\frac{\exp\left[(K_n-1/2)\,x/2a\right]}{\left(1+e^{-x/a}\right)^{(K_n+\lambda_n-1/2)}}P_n^{(K_n-1/2,\lambda-1/2)}\cdot$$
$$\left(-\frac{1+e^{-ix/a}}{1-e^{-ix/a}}\right) \tag{20}$$

where we got

$$\begin{aligned} K_n &= \frac{1}{8\mu a^2\,(n+1)^2}\left[(n+1)^2 - 2\mu a^2\frac{V_0}{q}\right]; \\ \lambda_n &= \frac{1}{2} + \frac{1}{n+1}\left[(n+1)^2 + 2\mu a^2\frac{V_0}{q}\right] \end{aligned} \tag{21}$$

Here we see that the PT Symmetric and Non-Hermitian Hulthen potential has real energy spectra.

3 Non- PT-symmetric and non-Hermitian Hulthen Potential

The non PT-symmetric and non Hermitian Hulthen potential is determined by taking $\frac{1}{a} \to \frac{i}{a}$, $V_0 \to A+iB$ and $q \to iq$ as

$$V(x) = -\frac{iV_o e^{-ix/a}}{1-iqe^{-ix/a}} \tag{22}$$

We will follow the same steps for getting the wave function and the energy spectrum. A suitable coordinate transformation kernel is obtained as

$$K(x_b,T;x_a,0) = \frac{q}{2a}\sin\theta_b\cos\theta_b\int D\theta Dp_\theta\cdot \tag{23}$$
$$\exp\left[i\int \mathrm{d}t\,(p_\theta\dot{\theta} - \frac{\sin^2\theta\cos^2\theta}{4\alpha^2}\frac{p_\theta^2}{2\mu} - V_0\cos^2\theta)\right].$$

If we follow the steps in sec. (2), we will obtain energy eigenvalues

$$\begin{aligned} E_n &= \frac{1}{8\mu a^2\,(n+1)^2}\cdot \\ &\quad \left[(n+1)^2 + 2\mu a^2\frac{(iA-B)}{q}\right]^2 \end{aligned} \tag{24}$$

and the normalized wave functions in terms of Jacobi polynomials are

$$\phi(x) = \frac{1}{2\sqrt{2}\sqrt{n+1}}\sqrt{4\,(n+1)^2 - (\lambda_n - K_n)^2}\cdot$$
$$\sqrt{\frac{\Gamma\,(n+1)\,\Gamma\,(K_n+\lambda_n+n)}{\Gamma\left(\lambda_n+n+\frac{1}{2}\right)\Gamma\left(K_n+n+\frac{1}{2}\right)}}\times$$
$$\frac{\exp\left[(K_n-1/2)\,x/2a\right]}{\left(1+e^{x/a}\right)^{(K_n+\lambda_n-1/2)}}\cdot$$
$$P_n^{(K_n-1/2,\lambda-1/2)}\left(\frac{1-e^{-x/a}}{1+e^{-x/a}}\right) \tag{25}$$

K_n and λ_n are the same in Eq. (21). It is clear that the energy spectra are real only if $Re(V_0) = 0$.

4 Conclusion

We have calculated the energy eigenvalues and the corresponding wave functions for the PT-/non-PT Symmetric and non-Hermitian Deformed Hulthen Potential. We obtained that PT-/non-PT Symmetric and non-Hermitian forms of potentials have real energy spectra by restricting the potential parameters.

References

[1] Bender, C. M.: *Reports on Progress In Physics* **70**(6), 947–1018 (2007); Bender, C. M., Darg, D. W.: *Journal of Mathematical Physics* **48**(4), 042703 (2007); Bender, C. M., Boetcher, S.: *Pys. Lett.* **80**, 5243 (1998); Bender, C. M., Boetcher, S., Meisenger, P. N.: *J. Math. Phys.* **40**, 2201 (1999); Bender, C. M., Dunne, G. V., Meisenger, P. N.: *Phys. Lett. A*, 252, 272 (1999).

[2] Yesiltas, O., Simsek, M., Sever, R., Tezcan, C.: *Phys. Scripta*, T67, 472 (2003).

[3] Berkdemir, C., Berkdemir, A., Sever, R.: *Phys. Rev.* C72, 027001 (2005).

[4] Faridfathi, G., Sever, R., Aktas, M.: *J. Math. Chem.* **38**, 533 (2005).

[5] Egrifes, H., Sever, R.: *Int. J. Theo. Phys.* **46**, 935 (2007).

[6] Ikhdair, S. M., Sever, R.: *J. of Math. Chem.* **42**, 461 (2007).

[7] Ikhdair, S. M., Sever, R.: *J. Mod. Phys. E* **17**, 1107 (2008).

[8] Kandirmaz, N., Sever, R.: *Chinese J. Phys.* **47**, 47 (2009).

[9] Kandirmaz, N., Sever, R.: *Physica Scripta*, **87**, 3, (2010).

[10] Feynmann, R., Hibbs, A.: *Quantum Mechanics and Path Integrals.* McGraw Hill, New York, (1965).

[11] Duru, I. H.: *Phys. Rev. D*, **28**, 2689 (1983); Duru, I. H., Kleinert, H.: *Phys. Lett.* **B84**, 185 (1979); *Fortschr. Phys.* **30**, 401 (1982); Duru, I. H.: *Phys.Rev. D*, **30**, 2121 (1984); Duru, I. H.: *Phys. Lett A*, **119**, 163 (1986).

[12] Peak, D., Inomata, A.: *J. Math. Phys.* **10**, 1422 (1969).

[13] Erkoc, S., Sever, R.: *Phys. Rev D*, **30**, 2117 (1984).

[14] Erkoc, S., Sever, R.: *Phys. Rev A*, **37**, 2687 (1988).

[15] Pak, N. K., Sokmen, I.: *Phys. Lett. 100A*, **327** (1984).

[16] Gradshteyn, I. S., Ryzhic, I. M.: *Table of Integrals, Series, and Products*, 2nd ed., Academic Press, New York, (1981).

Nalan Kandirmaz
E-mail: nkandirmaz@mersin.edu.tr
Mersin University
Department of Physics
Mersin, Turkey

Ramazan Sever
Middle East Technical University
Department of Physics
Ankara, Turkey