Five-dimensional $\mathcal{N} = 4$ Supersymmetric Mechanics

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Abstract

We perform an $su(2)$ Hamiltonian reduction in the bosonic sector of the $su(2)$-invariant action for two free $(4,4,0)$ supermultiplets. As a result, we get the five dimensional $\mathcal{N} = 4$ supersymmetric mechanics describing the motion of an isospin carrying particle interacting with a Yang monopole. Some possible generalizations of the action to the cases of systems with a more general bosonic action constructed with the help of ordinary and twisted $\mathcal{N} = 4$ hypermultiplets are considered.

Keywords: supersymmetric mechanics, Hamiltonian reduction, non-Abelian gauge fields.

1 Introduction

The supersymmetric mechanics describing the motion of an isospin particle in background non-Abelian gauge fields has attracted a lot of attention in the last few years [1, 2, 3, 4, 5, 6, 7, 8, 9], especially due to its close relation with higher dimensional Hall effects and their extensions [10], as well as with the supersymmetric versions of various Hopf maps (see e.g. [1]). The key point of any possible construction is to find a proper realization for semi-dynamical isospin variables, which have to be invented for the description of monopole-type interactions in Lagrangian mechanics. In supersymmetric systems these isospin variables should belong to some supermultiplet, and the main question is what to do with additional fermionic components accompanying the isospin variables. In [3] fermions of such a kind, together with isospin variables, span an auxiliary $(4,4,0)$ multiplet with a Wess-Zumino type action possessing an extra $U(1)$ gauge symmetry[1]. In this framework, an off-shell Lagrangian formulation was constructed, with the harmonic superspace approach [11, 12], for a particular class of four-dimensional [3] and three-dimensional [5] $\mathcal{N} = 4$ mechanics, with a self-dual non-Abelian background. The same idea of coupling with an auxiliary semi-dynamical supermultiplet has also been elaborated in [6] within the standard $\mathcal{N} = 4$ superspace framework, and then it has been applied for the construction of Lagrangian and Hamiltonian formulations of the $\mathcal{N} = 4$ supersymmetric system describing the motion of the isospin particles in three [7] and four-dimensional [8] conformally flat manifolds, carrying the non-Abelian fields of the Wu-Yang monopole and the BPST instanton, respectively.

In both these approaches the additional fermions were completely auxiliary, and they were expressed through the physical ones on the mass shell. Another approach based on the direct use of the $SU(2)$ reduction, firstly considered on the Lagrangian level in the purely bosonic case in [1], has been used in the supersymmetric case in [9]. The key idea of this approach is to perform a direct $su(2)$ Hamiltonian reduction in the bosonic sector of the $\mathcal{N} = 4$ supersymmetric system, with the general $SU(2)$ invariant action for a self-coupled $(4,4,0)$ supermultiplet. No auxiliary superfields are needed within such an approach, and the procedure itself is remarkably simple and automatically successful.

As concerning the interaction with the non-Abelian background, the system considered in [9] was not too illuminating, due to its small number (only one) of physical bosons. In the present Letter, we extend the construction of [9] to the case of the $\mathcal{N} = 4$ supersymmetric system with five (and four in the special case) physical bosonic components. It is not, therefore, strange that the arising non-Abelian background coincides with the field of a Yang monopole (a BPST instanton field in the four dimensional case).

A very important preliminary step, discussed in details in Section 2, is to pass to new bosonic and fermionic variables, which are inert under the $SU(2)$ group, over which we perform the reduction. Thus, the $SU(2)$ group rotates only the three bosonic components, which enter the action through $SU(2)$ invariant currents. Just these bosonic fields become the isospin variables which the background field couples to. Due to the commutativity of $\mathcal{N} = 4$ supersymmetry with the reduction $SU(2)$ group, it survives upon reduction. In Section 3, we consider some possible generalizations, which include a system with more general bosonic action, a four-dimensional system which still includes eight fermionic components, and the variant of five-dimensional $\mathcal{N} = 4$ mechanics constructed with the help of ordinary and twisted $\mathcal{N} = 4$ hypermultiplets. Finally, in the Conclusion we discuss some unsolved problems and possible extensions of the present construction.

[1]Note that the first implementation of this idea was proposed in [4]

2 $(8_B, 8_F) \to (5_B, 8_F)$ reduction and the Yang monopole

As the first nontrivial example of the $SU(2)$ reduction in $\mathcal{N}=4$ supersymmetric mechanics we consider the reduction from the eight-dimensional bosonic manifold to the five dimensional one. To start with, let us choose our basic $\mathcal{N}=4$ superfields to be the two quartets of real $\mathcal{N}=4$ superfields $\mathcal{Q}^{i\hat\alpha}_A$ (with $i, \hat\alpha, A = 1, 2$) defined in the $\mathcal{N}=4$ superspace $\mathbb{R}^{(1|4)} = (t, \theta_{ia})$ and subjected to the constraints

$$D^{(ia}\mathcal{Q}^{j)\hat\alpha}_A = 0, \qquad \text{and} \qquad \left(\mathcal{Q}^{i\hat\alpha}_A\right)^\dagger = \mathcal{Q}_{i\hat\alpha A}, \quad (2.1)$$

where the corresponding covariant derivatives have the form

$$\begin{aligned} D^{ia} &= \frac{\partial}{\partial\theta_{ia}} + \mathrm{i}\theta^{ia}\partial_t, \qquad \text{so that} \\ \left\{D^{ia}, D^{jb}\right\} &= 2i\epsilon^{ij}\epsilon^{ab}\partial_t. \end{aligned} \quad (2.2)$$

These constrained superfields describe the ordinary $\mathcal{N}=4$ hypermultiplet with four bosonic and four fermionic variables off-shell [12, 13, 14, 15, 16, 17].

The most general action for $\mathcal{Q}^{i\hat\alpha}_A$ superfields is constructed by integrating an arbitrary superfunction $\mathcal{F}(\mathcal{Q}^{i\hat\alpha}_A)$ over the whole $\mathcal{N}=4$ superspace. Here, we restrict ourselves to the simplest prepotential of the form[2]

$$\begin{aligned} \mathcal{F}(\mathcal{Q}^{i\hat\alpha}_A) &= \mathcal{Q}^{i\hat\alpha}_A\mathcal{Q}_{i\hat\alpha A} \qquad \longrightarrow \\ S &= \int \mathrm{d}t\, \mathrm{d}^4\theta\, \mathcal{Q}^{i\hat\alpha}_A \mathcal{Q}_{i\hat\alpha A}. \end{aligned} \quad (2.3)$$

The rationale for this selection is, first of all, its manifest invariance under $su(2)$ transformations acting on the "$\hat\alpha$" index of $\mathcal{Q}^{i\hat\alpha}$. This is the symmetry over which we are going to perform the $su(2)$ reduction. Secondly, just this form of the prepotential guarantees $SO(5)$ symmetry in the bosonic sector after reduction. In terms of components, the action (2.3) reads

$$S = \int \mathrm{d}t \left[\dot Q^{i\hat\alpha}_A\, \dot Q_{i\hat\alpha A} - \frac{i}{8}\dot\Psi^{a\hat\alpha}_A \Psi_{a\hat\alpha A}\right] \quad (2.4)$$

where the bosonic and fermionic components are defined as

$$Q^{i\hat\alpha}_A = \mathcal{Q}^{i\hat\alpha}_A|\,, \qquad \Psi^{a\hat\alpha}_A = D^{ia}\mathcal{Q}^{\hat\alpha}_{iA}|\,, \quad (2.5)$$

and, as usually, $(\ldots)|$ denotes the $\theta_{ia} = 0$ limit. Thus, from the beginning we have just the sum of two independent non-interacting $(4, 4, 0)$ supermultiplets.

To proceed further, we introduce the following bosonic $q^{i\alpha}_A$ and fermionic $\psi^{a\alpha}_A$ fields

$$q^{i\alpha}_A \equiv Q^i_{A\hat\alpha}G^{\alpha\hat\alpha}, \qquad \psi^{a\alpha}_A \equiv \Psi^a_{\hat\alpha A}G^{\alpha\hat\alpha}, \quad (2.6)$$

where the bosonic variables $G^{\alpha\hat\alpha}$, subjected to $G^{\alpha\hat\alpha}G_{\alpha\hat\alpha} = 2$, are chosen as

$$\begin{aligned} G^{11} &= \frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}\phi}}{\sqrt{1+\Lambda\overline\Lambda}}\Lambda, \qquad G^{21} = -\frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}\phi}}{\sqrt{1+\Lambda\overline\Lambda}}, \\ G^{22} &= \left(G^{11}\right)^\dagger, \qquad G^{12} = -\left(G^{21}\right)^\dagger. \end{aligned} \quad (2.7)$$

The variables $G^{\alpha\hat\alpha}$ play the role of a bridge relating the two different $SU(2)$ groups realized on the indices α and $\hat\alpha$, respectively. In terms of the variables given above, the action (2.4) acquires the form

$$\begin{aligned} S = \int \mathrm{d}t\, \Big[&\dot q^{i\alpha}_A\, \dot q_{i\alpha A} - 2q^{i\alpha}_A\dot q^\beta_{iA}J_{\alpha\beta} + \\ &\frac{q^{i\alpha}_A q_{i\alpha A}}{2}J^{\beta\gamma}J_{\beta\gamma} - \frac{i}{8}\dot\psi^{a\alpha}_A\psi_{a\alpha A} + \\ &\frac{i}{8}\psi^{a\alpha}_A\psi^\beta_{aA}J_{\alpha\beta}\Big], \end{aligned} \quad (2.8)$$

where

$$J^{\alpha\beta} = J^{\beta\alpha} = G^{\alpha\hat\alpha}\dot G^\beta_{\hat\alpha}. \quad (2.9)$$

As follows from (2.6), the variables $q^{i\alpha}_A$ and $\psi^{a\alpha}_A$, which, clearly, contain five independent bosonic and eight fermionic components, are inert under $su(2)$ rotations acting on $\hat\alpha$ indices. Under these $su(2)$ rotations, realized now only on $G^{\alpha\hat\alpha}$ variables in a standard way

$$\delta G^{\alpha\hat\alpha} = \gamma^{(\hat\alpha\hat\beta)}G^\alpha_{\hat\beta}, \quad (2.10)$$

the fields $(\phi, \Lambda, \bar\Lambda)$ (2.7) transform as [17]

$$\begin{aligned} \delta\Lambda &= \gamma^{11}\mathrm{e}^{\mathrm{i}\phi}(1+\Lambda\overline\Lambda), \\ \delta\overline\Lambda &= \gamma^{22}\mathrm{e}^{-\mathrm{i}\phi}(1+\Lambda\overline\Lambda), \\ \delta\phi &= -2\mathrm{i}\gamma^{12} + \mathrm{i}\gamma^{22}\mathrm{e}^{-\mathrm{i}\phi}\Lambda - \mathrm{i}\gamma^{11}\mathrm{e}^{\mathrm{i}\phi}\overline\Lambda. \end{aligned} \quad (2.11)$$

It is easy to check that the forms $J^{\alpha\beta}$ (2.9), expressing in terms of the fields $(\phi, \Lambda, \bar\Lambda)$,

$$\begin{aligned} J^{11} &= -\frac{\dot\Lambda - i\Lambda\dot\phi}{1+\Lambda\overline\Lambda}, \qquad J^{22} = -\frac{\dot{\overline\Lambda} + i\overline\Lambda\dot\phi}{1+\Lambda\overline\Lambda}, \\ J^{12} &= -i\frac{1-\Lambda\overline\Lambda}{1+\Lambda\overline\Lambda}\dot\phi - \frac{\dot\Lambda\overline\Lambda - \Lambda\dot{\overline\Lambda}}{1+\Lambda\overline\Lambda} \end{aligned} \quad (2.12)$$

are invariant under (2.11). Hence, the action (2.8) is invariant under the transformations in (2.10).

Next, we introduce the standard Poisson brackets for bosonic fields

$$\{\pi, \Lambda\} = 1, \quad \left\{\bar\pi, \overline\Lambda\right\} = 1, \quad \{p_\phi, \phi\} = 1, \quad (2.13)$$

so that the generators of the transformations (2.11),

[2] We used the following definition of the superspace measure: $\mathrm{d}^4\theta \equiv -\frac{1}{96}D^{ia}D_{ib}D^{bj}D_{ja}$.

$$I_\phi = p_\phi, \qquad I = \mathrm{e}^{\mathrm{i}\phi}\big[(1+\Lambda\overline{\Lambda})\,\pi - \mathrm{i}\overline{\Lambda}\,p_\phi\big],$$
$$\bar{I} = \mathrm{e}^{-\mathrm{i}\phi}\big[(1+\Lambda\overline{\Lambda})\,\bar{\pi} + \mathrm{i}\Lambda\,p_\phi\big], \qquad (2.14)$$

will be the Noether constants of motion for the action (2.8). To perform the reduction over this SU(2) group we fix the Noether constants as (c.f. [1])

$$I_\phi = m \qquad \text{and} \qquad I = \bar{I} = 0, \qquad (2.15)$$

which yields

$$p_\phi = m \qquad \text{and} \qquad \pi = \frac{\mathrm{i}m\,\overline{\Lambda}}{1+\Lambda\overline{\Lambda}}, \quad \bar{\pi} = -\frac{\mathrm{i}m\,\Lambda}{1+\Lambda\overline{\Lambda}}. \qquad (2.16)$$

Performing a Routh transformation over the variables $(\Lambda, \overline{\Lambda}, \phi)$, we reduce the action (2.8) to

$$\widetilde{S} = S - \int \mathrm{d}t\, \left\{\pi\,\dot{\Lambda} + \bar{\pi}\,\dot{\overline{\Lambda}} + p_\phi\dot{\phi}\right\} \qquad (2.17)$$

and substitute the expressions (2.16) in $\tilde{S}$. At the final step, we have to choose the proper parametrization for bosonic components $q^{i\alpha}_A$ (2.6), taking into account that they contain only five independent variables. Following [1] we will choose these variables as

$$q^{i\alpha}_1 = \frac{1}{2}\epsilon^{i\alpha}\sqrt{r+z_5},$$
$$q^{i\alpha}_2 = \frac{1}{\sqrt{2(r+z_5)}}\left(x^{(i\alpha)} - \frac{1}{\sqrt{2}}\epsilon^{i\alpha}z_4\right), \qquad (2.18)$$

where

$$x^{12} = \frac{i}{\sqrt{2}}\,z_3, \quad x^{11} = \frac{1}{\sqrt{2}}\left(z_1 + iz_2\right),$$
$$x^{22} = \frac{1}{\sqrt{2}}\left(z_1 - iz_2\right), \quad r^2 = \sum_{i=1}^{5} z_i z_i\,, \qquad (2.19)$$

and now the five independent fields are z_m. Slightly lengthy but straightforward calculations lead to

$$\begin{aligned} S_{\rm red} = \int \mathrm{d}t\, \Big[& \frac{1}{4r}\dot{z}_m\dot{z}_m - \frac{i}{8}\dot{\psi}^{a\alpha}_A\psi_{a\alpha A} + \\ & \frac{i}{4r}H^{\alpha\beta}V_{\alpha\beta} + \frac{1}{128r}H^{\alpha\beta}H_{\alpha\beta} - \\ & \frac{m^2}{r} - \frac{m}{4r}v^\alpha\bar{v}^\beta H_{\alpha\beta} - \\ & \frac{4im}{r}v^\alpha\bar{v}^\beta V_{\alpha\beta} + im\left(\dot{v}^\alpha\bar{v}_\alpha - v^\alpha\dot{\bar{v}}_\alpha\right)\Big]. \end{aligned} \qquad (2.20)$$

Here

$$H^{\alpha\beta} = \psi^{a\alpha}_A\psi^\beta_{aA}, \quad v^\alpha = G^{\alpha 1},$$
$$\bar{v}^\alpha = G^{\alpha 2}, \quad v^\alpha\bar{v}_\alpha = 1, \qquad (2.21)$$

and to ensure that the reduction constraints (2.16) are satisfied we added Lagrange multiplier terms (the last two terms in (2.20)). Finally, the variables $V^{\alpha\beta}$ in the action (2.20) are defined in a rather symmetric way to be

$$V^{\alpha\beta} = \frac{1}{2}\left(q^{i\alpha}_A\dot{q}^\beta_{iA} + q^{i\beta}_A\dot{q}^\alpha_{iA}\right). \qquad (2.22)$$

To clarify the relations of these variables with the potential of the Yang monopole, one has to introduce the following isospin currents (which will form an $su(2)$ algebra upon quantization)

$$T^I = v^\alpha\left(\sigma^I\right)_\alpha^{\ \beta}\bar{v}_\beta, \qquad I = 1,2,3. \qquad (2.23)$$

Now, the $(v^\alpha\bar{v}^\beta)$-dependent terms in the action (2.20) can be rewritten as

$$\begin{aligned} & -\frac{m}{4r}v^\alpha\bar{v}^\beta H_{\alpha\beta} - \frac{4im}{r}v^\alpha\bar{v}^\beta V_{\alpha\beta} = \\ & m\,T^I\left(\frac{1}{r(r+z_5)}\eta^I_{\mu\nu}z_\mu\dot{z}_\nu + \frac{1}{8r}H^I\right), \\ & \mu,\nu = 1,2,3,4, \end{aligned} \qquad (2.24)$$

where

$$\eta^I_{\mu\nu} = \delta^I_\mu\delta_{\nu 4} - \delta^I_\nu\delta_{\mu 4} + \epsilon^A{}_{\mu\nu 4} \qquad (2.25)$$

is the self-dual t'Hooft symbol and the fermionic spin currents are introduced

$$H^I = H^\alpha_\beta\left(\sigma^I\right)_\alpha^{\ \beta}. \qquad (2.26)$$

Thus we conclude that the action (2.20) describes $\mathcal{N}=4$ supersymmetric five-dimensional isospin particles moving in the field of the Yang monopole

$$\mathcal{A}_\mu = -\frac{1}{r(r+z_5)}\eta^I_{\mu\nu}z_\nu T^I. \qquad (2.27)$$

We stress that the $su(2)$ reduction algebra, realized in (2.11), commutes with all (super)symmetries of the action (2.4). Therefore, all symmetry properties of the theory are preserved in our reduction and the final action (2.20) represents the $\mathcal{N}=4$ supersymmetric extension of the system presented in [1].

With this, we have completed the classical description of $\mathcal{N}=4$ five-dimensional supersymmetric mechanics describing the isospin particle interacting with a Yang monopole. Next, we analyze some possible extensions of the present system, together with some possible interesting special cases. In what follows we will concentrate on the bosonic sector only, while the full supersymmetric action could be easily reconstructed, if needed.

3 Generalizations, and cases of special interest

Let us consider more general systems with a more complicated structure in the bosonic sector. We will concentrate on the bosonic sector only, while the full supersymmetric action could be easily reconstructed.

3.1 $SO(4)$ invariant systems

Our first example is the most general system, which still possesses $SO(4)$ symmetry upon $SU(2)$ reduction. It is specified by the prepotential $\mathcal{F}$ (2.3) depending on two scalars X and Y

$$\begin{aligned} \mathcal{F} &= \mathcal{F}(X,Y), \\ X &= \mathcal{Q}_1^{i\hat{\alpha}}\mathcal{Q}_{1\,i\hat{\alpha}}, \\ Y &= \mathcal{Q}_2^{i\hat{\alpha}}\mathcal{Q}_{2\,i\hat{\alpha}}. \end{aligned} \tag{3.1}$$

Such a system is invariant under $SU(2)$ transformations realized on the "hatted" indices $\hat{\alpha}$ and thus the $SU(2)$ reduction that we discussed in the Section 2 goes in the same manner. In addition, the full $SU(2)\times SU(2)$ symmetry realized on the superfield $\mathcal{Q}_2^{i\hat{\alpha}}$ will survive in the reduction process. So we expected the final system to possess $SO(4)$ symmetry.

The bosonic sector of the system with prepotential (3.1) is described by the action

$$\begin{aligned} S = \int \mathrm{d}t \Bigg[& \left(F_x + \frac{1}{2}xF_{xx}\right)\dot{Q}_1^{i\hat{\alpha}}\dot{Q}_{1\,i\hat{\alpha}} + \\ & \left(F_y + \frac{1}{2}yF_{yy}\right)\dot{Q}_2^{i\hat{\alpha}}\dot{Q}_{2\,i\hat{\alpha}} + \\ & 2F_{xy}Q_2^{j\hat{\beta}}Q_{1\,j\hat{\alpha}}\dot{Q}_{2\,i\hat{\beta}}\dot{Q}_1^{i\hat{\alpha}}\Bigg]. \end{aligned} \tag{3.2}$$

Even with such a simple prepotential, the bosonic action (3.2) after reduction has a rather complicated form. A further, still meaningful simplification, could be achieved with the following prepotential

$$\mathcal{F} = \mathcal{F}(X,Y) = \mathcal{F}_1(X) + \mathcal{F}_2(Y), \tag{3.3}$$

where $\mathcal{F}_1(X)$ and $\mathcal{F}_2(Y)$ are arbitrary functions depending on X and Y, respectively. With such a prepotential the third term in the action (3.2) disappears and the action acquires a readable form. With our notations (2.18), (2.19) the reduced action reads

$$\begin{aligned} S = \int \mathrm{d}t \Bigg[& \frac{H_x H_y}{2\left((H_x-H_y)z_5+(H_x+H_y)r\right)}\dot{z}_\mu\dot{z}_\mu + \\ & \frac{(H_x-H_y)^2}{8r^2\left((H_x-H_y)z_5+(H_x+H_y)r\right)}\left(z_\mu\dot{z}_\mu\right)^2 + \\ & \frac{H_x-H_y}{4r^2}\left(z_\mu\dot{z}_\mu\right)\dot{z}_5 + \\ & \frac{1}{8}\left(\frac{H_x-H_y}{r^2}z_5+\frac{H_x+H_y}{r}\right)\dot{z}_5^2 + \\ & im\left(\dot{v}^\alpha\bar{v}_\alpha - v^\alpha\dot{\bar{v}}_\alpha\right) - \\ & \frac{2m^2}{(H_x+H_y)r+(H_x-H_y)z_5} - \\ & \frac{8imH_y}{(H_x+H_y)r+(H_x-H_y)z_5}v^\alpha\bar{v}^\beta V_{\alpha\beta}\Bigg], \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} H_x &= F_1'(x) + \frac{1}{2}xF_1''(x), \\ H_y &= F_2'(y) + \frac{1}{2}yF_2''(y), \end{aligned} \tag{3.5}$$

and

$$x = \frac{1}{2}\left(r+z_5\right), \qquad y = \frac{1}{2}\left(r-z_5\right). \tag{3.6}$$

Let us stress that the unique possibility to have an $SO(5)$ invariant bosonic sector is to choose $H_x = H_y = const$. This is just the case we considered in Section 2. With arbitrary potentials H_x and H_y we have a more general system with the action (3.4), describing the motion of the $\mathcal{N}=4$ supersymmetric particle in five dimensions and interacting with a Yang monopole and some specific potential.

3.2 Non-linear supermultiplet

It has been known for a long time that in some special cases one could reduce the action for hypermultiplets to the action containing one fewer physical bosonic components — to the action of a so-called non-linear supermultiplet [12, 17, 18]. The main idea of such a reduction is to replace of the time derivative of the "radial" bosonic component of hypermultiplet $Log(q^{ia}q_{ia})$ by an auxiliary component B without breaking the $\mathcal{N}=4$ supersymmetry [19]:

$$\frac{\mathrm{d}}{\mathrm{d}t}Log(q^{ia}q_{ia}) \to B. \tag{3.7}$$

Clearly, to perform such a replacement in some action the "radial" bosonic component has to enter this action only with a time derivative. This condition strictly constraints the variety of the possible hypermultiplet actions in which this reduction works.

For performing the reduction from a hypermultiplet to the non-linear one, parametrization (2.18) is not very useful. Instead, we choose the following parameterizations for independent components of two hypermultiplets $q_1^{i\alpha}$ and $q_2^{i\alpha}$

$$q_1^{i\alpha} = \frac{1}{\sqrt{2}}\epsilon^{i\alpha}e^{\frac{1}{2}u}, \quad q_2^{i\alpha} = x^{(i\alpha)} - \frac{1}{\sqrt{2}}\epsilon^{i\alpha}z_4, \tag{3.8}$$

where

$$\begin{aligned} x^{12} &= \frac{i}{\sqrt{2}}z_3, \quad x^{11} = \frac{1}{\sqrt{2}}\left(z_1+iz_2\right), \\ x^{22} &= \frac{1}{\sqrt{2}}\left(z_1-iz_2\right). \end{aligned} \tag{3.9}$$

Thus, the five independent components are u and z_μ ($\mu = 1,\ldots,4$), and

$$x = q_1^2 = e^u, \qquad y = q_2^2 = \sum_{\mu=1}^{4} z_\mu z_\mu \equiv r_4^2. \tag{3.10}$$

With this parametrization the action (3.4) acquires the form

$$S = \int \mathrm{d}t \left[\frac{G_1 G_2 e^u}{e^u G_1 + G_2 r_4^2} \dot{z}_\mu \dot{z}_\mu + \frac{G_2^2}{e^u G_1 + G_2 r_4^2} (z_\mu \dot{z}_\mu)^2 + \frac{1}{4} G_1 e^u \dot{u}^2 + im\left(\dot{v}^\alpha \bar{v}_\alpha - v^\alpha \dot{\bar{v}}_\alpha\right) - \frac{m^2}{e^u G_1 + G_2 r_4^2} - \frac{4imG_2}{e^u G_1 + G_2 r_4^2} v^\alpha \bar{v}^\beta V_{\alpha\beta} \right], \qquad (3.11)$$

where

$$G_1 = G_1(u) = F_1'(x) + \frac{1}{2} x F_1''(x),$$
$$G_2 = G_2(r_4) = F_2'(y) + \frac{1}{2} y F_2''(y). \qquad (3.12)$$

If we choose $G_1 = e^{-u}$, then the "radial" bosonic component u will enter the action (3.11) only through the kinetic term $\sim \dot{u}^2$. Thus, performing replacement (3.7) and excluding the auxiliary field B by its equation of motion we will finish with the action

$$S = \int \mathrm{d}t \left[\frac{G_2}{1 + G_2 r_4^2} \left(\dot{z}_\mu \dot{z}_\mu + G_2 (z_\mu \dot{z}_\mu)^2 \right) + im\left(\dot{v}^\alpha \bar{v}_\alpha - v^\alpha \dot{\bar{v}}_\alpha\right) - \frac{m^2}{1 + G_2 r_4^2} - \frac{4imG_2}{1 + G_2 r_4^2} v^\alpha \bar{v}^\beta V_{\alpha\beta} \right]. \qquad (3.13)$$

The action (3.13) describes the motion of an isospin particle on a four-manifold with $SO(4)$ isometry carrying the non-Abelian field of a BPST instanton and some special potential. Our action is rather similar to those recently constructed in [3, 8, 2], but it contains twice more physical fermions.

3.3 Ordinary and twisted hypermultiplets

One more way to generalize the results presented in the previous Section is to consider simultaneously the ordinary hypermultiplet $\mathcal{Q}^{j\hat{\alpha}}$ obeying (2.1) together with twisted hypermultiplet $\mathcal{V}^{a\hat{\alpha}}$ — a quartet of $\mathcal{N} = 4$ superfields subjected to constraints [17]

$$D^{i(a}\mathcal{V}^{b)\hat{\alpha}} = 0, \qquad \text{and} \qquad \left(\mathcal{V}^{a\hat{\alpha}}\right)^\dagger = \mathcal{V}_{a\hat{\alpha}}. \qquad (3.14)$$

The most general system which is explicitly invariant under $SU(2)$ transformations realized on the "hatted" indices is defined, similarly to (3.1), by the superspace action depending on two scalars X, Y

$$S = \int \mathrm{d}t\, \mathrm{d}^4\theta \mathcal{F}(X, Y),$$

$$X = \mathcal{Q}^{i\hat{\alpha}} \mathcal{Q}_{i\hat{\alpha}}, \qquad Y = \mathcal{V}^{a\hat{\alpha}} \mathcal{V}_{a\hat{\alpha}}. \qquad (3.15)$$

The bosonic sector of the action (3.15) is a rather simple

$$S = \int \mathrm{d}t \left[\left(F_x + \frac{1}{2} x F_{xx} \right) \dot{Q}^{i\hat{\alpha}} \dot{Q}_{i\hat{\alpha}} - \left(F_y + \frac{1}{2} y F_{yy} \right) \dot{V}^{a\hat{\alpha}} \dot{V}_{a\hat{\alpha}} \right]. \qquad (3.16)$$

Thus, we see that the term causes the most complicated structure of the action with two hypermultiplets, which disappear in the case of ordinary and twisted hypermultiplets. Clearly, the bosonic action after $SU(2)$ reduction will have the same form (3.4), but with

$$H_x = F_x + \frac{1}{2} x F_{xx},$$
$$H_y = -\left(F_y + \frac{1}{2} y F_{yy} \right). \qquad (3.17)$$

Here $F = F(x, y)$ is still a function of two variables x and y. The mostly symmetric situation again corresponds to the choice

$$H_x = H_y \equiv h(x, y) \qquad (3.18)$$

with the action

$$S = \int \mathrm{d}t \left[\frac{h}{4r} \dot{z}_m \dot{z}_m + im\left(\dot{v}^\alpha \bar{v}_\alpha - v^\alpha \dot{\bar{v}}_\alpha\right) - \frac{m^2}{h\, r} - \frac{4im}{r} v^\alpha \bar{v}^\beta V_{\alpha\beta} \right]. \qquad (3.19)$$

Unfortunately, due to definition (3.17), (3.18) the metric $h(x, y)$ cannot be chosen fully arbitrarily. For example, looking for an $SO(5)$ invariant model with $h = h(x + y)$ we could find only two solutions[3]

$$h_1 = const., \qquad h_2 = 1/(x + y)^3. \qquad (3.20)$$

Both solutions describe a cone-like geometry in the bosonic sector, while the most interesting case of the sphere S^5 cannot be treated within the present approach.

Finally, we would like to draw attention to the fact that with $h = \text{const.}$ the bosonic sectors of systems with two hypermultiplets and with one ordinary and one twisted hypermultiplets coincide. This is just one more justification for the claim that "almost free" systems can be supersymmetrized in various ways.

[3]The same metric has been considered in [20].

4 Conclusion

In this paper, starting with the non-interacting system of two $\mathcal{N} = 4$ hypermultiplets, we perform a reduction over the $SU(2)$ group which commutes with supersymmetry. The resulting system describes the motion of an isospin carrying particle on a conformally flat five-dimensional manifold in the non-Abelian field of a Yang monopole and in some scalar potential. The most important step for this construction passes to new bosonic and fermionic variables, which are inert under the $SU(2)$ group over which we perform the reduction. Thus, the $SU(2)$ group rotates only three bosonic components, which enter the action through $SU(2)$ invariant currents. Just these bosonic fields become the isospin variables which the background field couples to. Due to the commutativity of $\mathcal{N} = 4$ supersymmetry with the reduction $SU(2)$ group, it survives upon reduction. We considered some possible generalizations of the action to the cases of systems with a more general bosonic action, a four-dimensional system which still includes eight fermionic components, and a variant of five-dimensional $\mathcal{N} = 4$ mechanics constructed with the help of ordinary and twisted $\mathcal{N} = 4$ hypermultiplets. The main advantage of the proposed approach is its applicability to any system which possesses $SU(2)$ invariance. If, in addition, this $SU(2)$ commutes with supersymmetry, then the resulting system will automatically be supersymmetric.

Possible direct applications of our construction include a reduction in the cases of systems with non-linear $\mathcal{N} = 4$ supermultiplets [21], systems with more than two (non-linear) hypermultiplets, in systems with bigger supersymmetry, say for example $\mathcal{N} = 8$, etc. However, the most important case, which is still missing within our approach, is the construction of the $\mathcal{N} = 4$ supersymmetric particle on the sphere S^5 in the field of a Yang monopole. Unfortunately, the use of standard linear hypermultiplets makes the solution of this task impossible, because the resulting bosonic manifolds have a different structure (conical geometry) to include S^5.

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