$N = 4$, $d = 1$ Supersymmetric Hyper-Kähler Sigma Models and Non-Abelian Monopole Background

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Abstract

We construct a Lagrangian formulation of $\mathcal{N} = 4$ supersymmetric mechanics with hyper-Kähler sigma models in a bosonic sector in a non-Abelian background gauge field. The resulting action includes a wide class of $\mathcal{N} = 4$ supersymmetric mechanics describing the motion of an isospin-carrying particle over spaces with non-trivial geometry. In two examples that we discuss in details, the background fields are identified with the field of BPST instantons in flat and Taub-NUT spaces.

Keywords: supersymmetric mechanics, Hyper-Kähler spaces, non-Abelian gauge fields.

1 Introduction

$\mathcal{N} = 4$ supersymmetric mechanics provides a nice framework for the study of many interesting features of higher dimensional theories. At the same time, the existence of a variety of off-shell $\mathcal{N} = 4$ irreducible linear supermultiplets in $d = 1$ [1, 2, 3, 4, 5] makes the situation in one dimension even more interesting, and this is what prompted us to investigate such supersymmetric models themselves, without reference to higher dimensional counterparts. Being a supersymmetric invariant theory, $\mathcal{N} = 4$ mechanics admits a natural formulation in terms of superfields living in a standard and/or in a harmonic superspace [6], adapted to one dimension [7]. In any case, the preferred approach for discussing supersymmetric mechanics is the Lagrangian one. Being quite useful, the Lagrangian approach has one subtle point, when we try to describe the system in an arbitrary gauge background. While the inclusion of the Abelian gauge background can be done straightforwardly [7], the non-Abelian background reqiures new ingredients — isospin variables — which have to be included in the description [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. These isospin variables become purely internal degrees of freedom after quantization and form an auxiliary $\mathcal{N} = 4$ supermultiplet, together with the auxiliary fermions.

There are various approaches for introducing such auxiliary superfields and couplings with them, but until now all constructed models have been restricted to have conformally flat sigma models in the bosonic sector. This restriction has an evident source — it has been known for a long time that all linear $\mathcal{N} = 4$ supermultiplets can be obtained through a dualization procedure from the $\mathcal{N} = 4$ "root" supermultiplet — the $\mathcal{N} = 4$ hypermultiplet [18, 19, 20, 21, 22, 23], while the bosonic part of the general hypermultiplet action is conformal to the flat one. The only way to escape this flatness situation is to use nonlinear supermultiplets [24, 25, 26], instead of linear ones.

The main aim of this paper is to construct the Lagrangian formulation of $\mathcal{N} = 4$ supersymmetric mechanics on conformal to hyper-Kähler spaces in non-Abelian background gauge fields. To achieve this goal we combine two ideas

- We introduce the coupling of matter supermultiplets with an auxiliary fermionic supermultiplet $\Psi^{\hat\alpha}$ containing on-shell four physical fermions and four auxiliary bosons playing the role of isospin variables. The very specific coupling results in a component action which contains only time derivatives of the fermionic components present in $\Psi^{\hat\alpha}$. Then, we dualize these fermions into auxiliary ones, ending up with the proper action for matter fields and isospin variables. This procedure was developed in [11].
- As the next step, starting from the action for the $\mathcal{N} = 4$ tensor supermultiplet [27, 28] coupled with the superfield $\Psi^{\hat\alpha}$, following [24], we dualize the auxiliary component A into a fourth physical boson, finishing with the action having a geometry conformal to the hyper-Kähler one in the bosonic sector.

The resulting action contains a wide class of $\mathcal{N} = 4$ supersymmetric mechanics describing the motion of an isospin-carrying particle over spaces with non-trivial geometry and in the presence of a non-Abelian background gauge field. In two examples that we discuss in details, these backgrounds correspond to the field of the BPST instanton in the flat and Taub-NUT spaces. In order to make our presentation self-sufficient, we include in Section 2 a sketchy description of our construction applied to the linear tensor and hypermultiplet. We also discuss the relation between these supermultiplets in the context of our approach (Section 3), which immediately leads to the generalized procedure presented in Section 4.

2 Isospin particles in conformally flat spaces

One way to incorporate the isospin-like variables in the Lagrangian of supersymmetric mechanics is to couple the basic superfields with auxiliary fermionic superfields $\Psi^{\hat\alpha}, \bar\Psi_{\hat\alpha}$, which contain these isospin variables [11]. Such a coupling, being written in a standard $\mathcal{N}=4$ superspace, has to be rather special, in order to provide a kinetic term of the first order in time derivatives for the isospin variables and to describe the auxiliary fermionic components present in $\Psi^{\hat\alpha}, \bar\Psi_{\hat\alpha}$. Following [11], we introduce the coupling of auxiliary Ψ superfields with some arbitrary, for the time being, $\mathcal{N}=4$ supermultiplet X as

$$S_c = -\frac{1}{32}\int \mathrm{d}t\mathrm{d}^4\theta\,(X+g)\,\Psi^{\hat\alpha}\bar\Psi_{\hat\alpha}, \quad g=\text{const.} \tag{2.1}$$

The Ψ supermultiplet is subjected to the irreducible conditions [5]

$$D^i\Psi^1=0,\quad D^i\Psi^2+\overline{D}^i\Psi^1=0,\quad \overline{D}_i\Psi^2=0, \tag{2.2}$$

and thus it contains four fermionic and four bosonic components

$$\psi^{\hat\alpha}=\Psi^{\hat\alpha}|,\quad u^i=-D^i\bar\Psi^2|,\quad \bar u_i=\overline{D}_i\Psi^1|, \tag{2.3}$$

where the symbol $|$ denotes the $\theta=\bar\theta=0$ limit and $\mathcal{N}=4$ covariant derivatives obey standard relations

$$\left\{D^i,\overline{D}_j\right\}=2\mathrm{i}\delta^i_j\partial_t. \tag{2.4}$$

It has been demonstrated in [11] that if the $\mathcal{N}=4$ superfield X is subjected to the constraints [29, 5]

$$D^iD_iX=0,\quad \overline{D}_i\overline{D}^iX=0,\quad \left[D^i,\overline{D}_i\right]X=0, \tag{2.5}$$

then the component action which follows from (2.1) can be written as

$$\begin{aligned} S_c \;=\; \int \mathrm{d}t\,\Big[&-(x+g)\left(\rho^1\bar\rho^2-\rho^2\bar\rho^1\right) - \\ &\frac{i}{4}(x+g)\left(\dot u^i\bar u_i-u^i\dot{\bar u}_i\right)+\frac14 A_{ij}u^i\bar u^j + \\ &\frac12\eta_i\left(\bar u^i\bar\rho^2+u^i\rho^2\right)+\frac12\bar\eta^i\left(u_i\rho^1+\bar u_i\bar\rho^1\right)\Big], \end{aligned} \tag{2.6}$$

where the new fermionic components $\rho^{\hat\alpha},\bar\rho_{\hat\alpha}$ are defined as

$$\rho^{\hat\alpha}=\dot\psi^{\hat\alpha},\quad \bar\rho_{\hat\alpha}=\dot\psi_{\hat\alpha}. \tag{2.7}$$

The components of the superfield X entering the action (2.6) have been introduced as

$$\begin{aligned} &x=X|,\quad A_{ij}=A_{(ij)}=\frac12\left[D_i,\overline{D}_j\right]X|,\\ &\eta^i=-iD^iX|,\quad \bar\eta_i=-i\overline{D}_iX|. \end{aligned} \tag{2.8}$$

What makes the action (2.6) interesting is that, despite the non-local definition of the spinors $\rho^{\hat\alpha},\bar\rho_{\hat\alpha}$ (2.7), the action is invariant under the following $\mathcal{N}=4$ supersymmetry transformations:

$$\begin{aligned} &\delta\rho^1=-\bar\epsilon^i\dot{\bar u}_i,\quad \delta\rho^2=\epsilon_i\dot{\bar u}^i,\\ &\delta u^i=-2i\epsilon^i\bar\rho^1+2i\bar\epsilon^i\bar\rho^2,\quad \delta\bar u_i=-2i\epsilon_i\rho^1+2i\bar\epsilon_i\rho^2,\\ &\delta x=-i\epsilon_i\eta^i-i\bar\epsilon^i\bar\eta_i,\quad \delta\eta^i=-\bar\epsilon^i\dot x-i\bar\epsilon^jA^i_j,\\ &\delta\bar\eta_i=-\epsilon_i\dot x+i\epsilon_jA^j_i,\quad \delta A_{ij}=-\epsilon_{(i}\dot\eta_{j)}+\bar\epsilon_{(i}\dot{\bar\eta}_{j)}. \end{aligned} \tag{2.9}$$

In the action (2.6) the fermionic fields $\rho^{\hat\alpha},\bar\rho_{\hat\alpha}$ are auxiliary ones, and thus they can be eliminated by their equations of motion

$$\rho^1=\frac{1}{2(x+g)}\eta_i\bar u^i,\qquad \rho^2=-\frac{1}{2(x+g)}\bar\eta^i\bar u_i. \tag{2.10}$$

Finally, the action describing the interaction of Ψ and X supermultiplets acquires a very simple form

$$\begin{aligned} S_c \;=\; \frac14\int \mathrm{d}t\,\Big[&-i(x+g)\left(\dot u^i\bar u_i-u^i\dot{\bar u}_i\right)+ \\ &A_{ij}u^i\bar u^j+\frac{1}{x+g}\eta_i\bar\eta_j\left(u^i\bar u^j+u^j\bar u^i\right)\Big]. \end{aligned} \tag{2.11}$$

Thus, in the fermionic superfields Ψ only the bosonic components $u^i,\bar u_i$, entering the action with a kinetic term linear in time-derivatives, survive. After quantization, these variables become purely internal degrees of freedom.

In order to be meaningful, action (2.1) has to be extended by the action for the supermultiplet X itself. If the superfield X obeying (2.5) is considered as an independent superfield, then the most general action reads

$$S=S_x+S_c=-\frac{1}{32}\int \mathrm{d}t\,\mathrm{d}^4\theta\mathcal{F}(X)+S_c, \tag{2.12}$$

where $\mathcal{F}(X)$ is an arbitrary function of X. In this case the components A_{ij} (2.8) are auxiliary ones, and they have to be eliminated by their equations of motion. The resulting action describes $\mathcal{N}=4$ supersymmetric mechanics with one physical boson x and four physical fermions $\eta^i,\bar\eta_j$ interacting with isospin variables $u^i,\bar u_i$. This is the system that has been considered in [8, 10, 11].

It is clear that treating the scalar bosonic superfield X as an independent one is too restrictive, because the constraints (2.5) leave in this supermultiplet only one physical bosonic component x, which is not enough to describe the isospin particle. In the present approach, a way to overcome this limitation was proposed in [14, 17]. The key point is to treat superfield X as a composite one, constructed from $\mathcal{N}=4$ supermultiplets with a larger number of physical bosons. The two reasonable superfields from which it is possible to construct superfield X are $\mathcal{N}=4$ tensor supermultiplet $\mathcal{V}^{ij}$ [27, 28] and a one-dimensional hypermultiplet $Q^{i\alpha}$ [18, 19, 20, 21, 7].

Tensor supermultiplet

The $\mathcal{N}=4$ tensor supermultiplet is described by the triplet of bosonic $\mathcal{N}=4$ superfields $\mathcal{V}^{ij}=\mathcal{V}^{ij}$ subjected to the constraints

$$D^{(i}\mathcal{V}^{jk)}=\overline{D}^{(i}\mathcal{V}^{jk)}=0,\qquad \left(\mathcal{V}^{ij}\right)^{\dagger}=\mathcal{V}_{ij}, \quad (2.13)$$

which leave in $\mathcal{V}^{ij}$ the following independent components:

$$\begin{aligned} v^a &= -\frac{\mathrm{i}}{2}\left(\sigma^a\right)_i{}^j\mathcal{V}^i_j|,\quad \lambda^i=\frac{1}{3}D^j\mathcal{V}^i_j|, \quad (2.14)\\ \bar\lambda_i &= \frac{1}{3}\overline{D}_j\mathcal{V}^j_i|,\quad A=\frac{\mathrm{i}}{6}D^i\overline{D}^j\mathcal{V}_{ij}|. \end{aligned}$$

Thus, its off-shell component field content is $(3,4,1)$, i.e. three physical v^a and one auxiliary A bosons and four fermions $\lambda^i,\bar\lambda_i$ [27, 28]. Under $\mathcal{N}=4$ supersymmetry these components transform as follows:

$$\begin{aligned} \delta v^a &= \mathrm{i}\epsilon^i(\sigma^a)_i^j\bar\lambda_j-\mathrm{i}\lambda^i(\sigma^a)_i^j\bar\epsilon_j,\\ \delta A &= \bar\epsilon_i\dot\lambda^i-\epsilon^i\dot{\bar\lambda}_i,\\ \delta\lambda^i &= \mathrm{i}\epsilon^iA+\epsilon^j(\sigma^a)_j^i\dot v^a, \quad (2.15)\\ \delta\bar\lambda_i &= -\mathrm{i}\bar\epsilon_iA+(\sigma^a)_i^j\bar\epsilon_j\dot v^a. \end{aligned}$$

Now one may check that the composite superfield

$$X=\frac{1}{|\mathcal{V}|}\equiv\frac{1}{\sqrt{\mathcal{V}^a\mathcal{V}^a}}\,, \quad (2.16)$$

where $\mathcal{V}^a=-\frac{\mathrm{i}}{2}\left(\sigma^a\right)_i{}^j\mathcal{V}^i_j$, obeys (2.5) in virtue of (2.13). Clearly, now all components of the X superfield, i.e. the physical boson x, fermions $\eta^i,\bar\eta_i$ and auxiliary fields A^{ij} (2.8) are expressed through the components of the $\mathcal{V}^{ij}$ supermultiplet (2.14) as

$$\begin{aligned} x &= \frac{1}{|v|},\qquad \eta^i=\frac{v^a}{|v|^3}(\lambda\sigma^a)^i,\quad \bar\eta_i=\frac{v^a}{|v|^3}(\sigma^a\bar\lambda)_i,\\ A^i_j &= -3\frac{v^av^b}{|v|^5}(\lambda\sigma^a)^i(\sigma^b\bar\lambda)_j-\frac{v^a(\sigma^a)^i_j}{|v|^3}A+ \quad (2.17)\\ &\quad \frac{1}{|v|^3}\epsilon^{abc}v^a\dot v^b(\sigma^c)^i_j+\frac{1}{|v|^3}\left(\delta^i_j\lambda^k\bar\lambda_k-\lambda_j\bar\lambda^i\right). \end{aligned}$$

In what follows, we will also need the expression for A^i_j components (2.17) in terms of $\eta^i,\bar\eta_i$ fermions (2.8), which reads

$$\begin{aligned} A^i_j &= -\frac{v^a(\sigma^a)^i_j}{|v|^3}A+\frac{1}{|v|^3}\epsilon^{abc}v^a\dot v^b(\sigma^c)^i_j- \quad (2.18)\\ &\quad |v|\left(\eta^i\bar\eta_j+\eta_j\bar\eta^i\right)-\frac{1}{|v|}v^a(\eta\sigma^a\bar\eta)\,v^b(\sigma^b)^i_j. \end{aligned}$$

Finally, one should note that, while dealing with the tensor supermultiplet $\mathcal{V}^{ij}$, one may generalize the S_x action (2.12) to have the full action in the form

$$S=S_v+S_c=-\frac{1}{32}\int \mathrm{d}t\,\mathrm{d}^4\theta\mathcal{F}(\mathcal{V})+S_c, \quad (2.19)$$

where $\mathcal{F}(\mathcal{V})$ is now an arbitrary function of $\mathcal{V}^{ij}$. After eliminating the auxiliary component A in the component form of (2.19) we will obtain the action describing the $\mathcal{N}=4$ supersymmetric three-dimensional isospin particle moving in the magnetic field of a Wu-Yang monopole and in some specific scalar potential [14][1].

Hypermultiplet

Similarly to the tensor supermultiplet one may construct the superfield X starting from the $\mathcal{N}=4$ hypermultiplet. The $\mathcal{N}=4$, $d=1$ hypermultiplet is described in $\mathcal{N}=4$ superspace by the quartet of real $\mathcal{N}=4$ superfields $\mathcal{Q}^{i\alpha}$ subjected to the constraints

$$\begin{aligned} D^{(i}\mathcal{Q}^{j)\alpha} &= 0,\quad \overline{D}^{(i}\mathcal{Q}^{j)\alpha}=0,\\ \left(\mathcal{Q}^{i\alpha}\right)^{\dagger} &= \mathcal{Q}_{i\alpha}. \quad (2.20) \end{aligned}$$

This supermultiplet describes four physical bosonic and four physical fermionic variables

$$\begin{aligned} q^{i\alpha} &= \mathcal{Q}^{i\alpha}|,\quad \eta^i=-iD^i\left(\frac{2}{\mathcal{Q}^{j\alpha}\mathcal{Q}_{j\alpha}}\right)|,\\ \bar\eta_i &= -i\overline{D}_i\left(\frac{2}{\mathcal{Q}^{j\alpha}\mathcal{Q}_{j\alpha}}\right)|, \quad (2.21) \end{aligned}$$

and it does not contain any auxiliary components [7, 18, 19, 20, 21].

One may easily check that if we define the composite superfield X as

$$X=\frac{2}{\mathcal{Q}^{i\alpha}\mathcal{Q}_{i\alpha}}, \quad (2.22)$$

then it will obey (2.5) in virtue of (2.20) [5]. For the hypermultiplet $\mathcal{Q}^{i\alpha}$ we defined the fermionic components to coincide with those present in the X superfield (2.8), while the former auxiliary components A_{ij} are now expressed via the components of $\mathcal{Q}^{i\alpha}$ as

$$\begin{aligned} A_{ij} &= -\frac{4\mathrm{i}}{(q^{k\beta}q_{k\beta})^2}\left(\dot q^{\alpha}_i q_{j\alpha}+\dot q^{\alpha}_j q_{i\alpha}\right)-\\ &\quad \frac{(q^{k\beta}q_{k\beta})}{2}\left(\eta_i\bar\eta_j+\eta_j\bar\eta_i\right). \quad (2.23) \end{aligned}$$

As in the case of the tensor supermultiplet, one may write the full action with the hypermultiplet self-interacting part S_q added as

$$S=S_q+S_c=-\frac{1}{32}\int \mathrm{d}t\,\mathrm{d}^4\theta\mathcal{F}(\mathcal{Q})+S_c, \quad (2.24)$$

[1] An alternative description of the same system has recently been constructed in [16]

where now $\mathcal{F}(\mathcal{Q})$ is an arbitrary function of $\mathcal{Q}^{i\alpha}$. The action (2.24) describes the motion of an isospin particle on a conformally flat four-manifold carrying the non-Abelian field of a BPST instanton [17]. This system has recently been obtained in different frameworks in [13, 15].

To close this Section one should mention that, while dealing with the tensor supermultiplet $\mathcal{V}^{ij}$ and the hypermultiplet $\mathcal{Q}^{i\alpha}$, the structure of the action S_c (2.1) can be generalized to be [17]

$$S_c = -\frac{1}{32}\int \mathrm{d}t\,\mathrm{d}^4\theta\; Y\Psi^{\hat\alpha}\bar\Psi_{\hat\alpha}, \tag{2.25}$$

with Y obeying

$$\begin{aligned}\triangle_3 Y = 0 &\quad \text{in case of the tensor supermultiplet}\\ \triangle_4 Y = 0 &\quad \text{in case of the hypermultiplet.}\end{aligned} \tag{2.26}$$

Here $\triangle_n$ denotes the Laplace operator in a flat Euclidean n-dimensional space. Clearly, our choice $Y = X + g$ with X defined in (2.16), (2.22) corresponds to spherically-symmetric solutions of (2.26).

3 From the hypermultiplet to the tensor supermultiplet and back

One of the most attractive features of our approach is the unified structure of the action S_c (2.1) which has the same form for any type of supermultiplets that we are using to construct a composite superfield X. This is what opens the way to relate the different systems via duality transformations. Indeed, it has been known for a long time [1, 2, 3, 4, 22, 23] that in one dimension one may switch between supermultiplets with a different number of physical bosons, by expressing the auxiliary components through the time derivative of physical bosons, and vice versa. Here we will use this mechanism to obtain the action of the tensor multiplet (2.19) from the hypermultiplet (2.24) action and then, alternatively, action (2.24) (with some restrictions) from (2.19). In what follows, to make some expressions more transparent, we will use, sometimes, the following stereographic coordinates for the bosonic components of hypermultiplet (2.21) and tensor supermultiplet (2.14):

$$\begin{aligned} q^{11} &= \frac{e^{\frac12(u-\mathrm{i}\phi)}}{\sqrt{1+\Lambda\overline{\Lambda}}}\Lambda, \quad q^{21} = -\frac{e^{\frac12(u-\mathrm{i}\phi)}}{\sqrt{1+\Lambda\overline{\Lambda}}},\\ q^{22} &= \left(q^{11}\right)^\dagger, \quad q^{21} = -\left(q^{12}\right)^\dagger, \end{aligned} \tag{3.1}$$

$$\begin{aligned} V^{11} &= 2\mathrm{i}\frac{e^u}{1+\Lambda\overline{\Lambda}}\Lambda, \quad V^{22} = -2\mathrm{i}\frac{e^u}{1+\Lambda\overline{\Lambda}}\overline{\Lambda},\\ V^{12} &= -\mathrm{i}e^u\left(\frac{1-\Lambda\overline{\Lambda}}{1+\Lambda\overline{\Lambda}}\right). \end{aligned} \tag{3.2}$$

One may easily check that these definitions are compatible with (2.16) and (2.22).

From hypermultiplet to tensor supermultiplet

The main ingredient for getting the tensor supermultiplet action from the hypermultiplet one is provided by the expression for "auxiliary" components A^{ij} in terms of the components of superfields $\mathcal{V}^{ij}$ and $\mathcal{Q}^{i\alpha}$ in (2.18) and (2.23), respectively. Identifying the right hand sides of (2.18) and (2.23), one may find the expression of the auxiliary component A present in the superfield $\mathcal{V}^{ij}$ in terms of components of $\mathcal{Q}^{i\alpha}$:

$$\begin{aligned} A \;=\; &\mathrm{i}\left(\dot q^{i1}q_i^2 + \dot q^{i2}q_i^1\right) + \\ &\frac14 (q^{k\alpha}q_{k\alpha})^2\, q^{i1}q^{j2}\left(\eta_i\bar\eta_j + \eta_j\bar\eta_i\right). \end{aligned} \tag{3.3}$$

Another way, and probably the easiest one, to check the validity of (3.3) is to use the following superfield representation for the tensor supermultiplet [7]:

$$\mathcal{V}^{ij} = \mathrm{i}\left(\mathcal{Q}^{i1}\mathcal{Q}^{j2} + \mathcal{Q}^{j1}\mathcal{Q}^{i2}\right). \tag{3.4}$$

This "composite" superfield $\mathcal{V}^{ij}$ automatically obeys (2.13) as a consequence of (2.20).

Being partially rewritten in terms of components (3.1), expression (3.3) reads

$$\begin{aligned} \dot\phi \;=\; &e^{-u}A - \mathrm{i}\frac{\dot\Lambda\overline{\Lambda} - \Lambda\dot{\overline{\Lambda}}}{1+\Lambda\overline{\Lambda}} - \\ &\frac14 e^{-u}(q^{k\alpha}q_{k\alpha})^2\, q^{i1}q^{j2}\left(\eta_i\bar\eta_j + \eta_j\bar\eta_i\right). \end{aligned} \tag{3.5}$$

Thus, we see that, in order to get the action for the tensor supermultiplet, one has to replace, in the component action for the hypermultiplet, the time derivative of field ϕ by the combination on the r.h.s. of (3.5), which includes the new auxiliary field A. An additional restriction comes from the S_q part of the action (2.24), which now has to depend only on the "composite" superfield $\mathcal{V}^{ij}$ (3.4). If it is so, then in the full action (2.24) component ϕ will enter only through $\dot\phi$, and the discussed replacement will be valid.

From the tensor supermultiplet to the hypermultiplet

It is clear that the backward procedure also exists. Indeed, from (2.17) and (2.23) one may get the following expression for A:

$$A = \frac1f\left[\dot\phi + \frac{\partial}{\partial v_a}f(\lambda\sigma^a\bar\lambda) - B_a\dot v_a\right], \tag{3.6}$$

where

$$\begin{aligned} f &= \frac{1}{|v|}, \quad \text{and}\;, \quad B_1 = -\frac{v_2(v_3+|v|)}{(v_1^2+v_2^2)|v|},\\ B_2 &= \frac{v_1(v_3+|v|)}{(v_1^2+v_2^2)|v|}, \quad B_3 = 0. \end{aligned} \tag{3.7}$$

It is easy to check that in the coordinates (2.14), (3.2) we have

$$B_a \dot{v}_a = -\mathrm{i}\frac{\dot{\Lambda}\bar{\Lambda} - \Lambda\dot{\bar{\Lambda}}}{1 + \Lambda\bar{\Lambda}} \quad \text{and} \quad |v| = e^u \tag{3.8}$$

in full agreement with (3.5). Thus, to get the hypermultiplet action (2.24) from that for the tensor supermultiplet (2.19), one has to dualize the auxiliary component A into a new physical boson ϕ using (3.6). Of course, we do not expect to get the most general action for the hypermultiplet interacting with the isospin-containing supermultiplet Ψ, because the S_v part in (2.19) depends only on the $\mathcal{V}^{ij}$ supermultiplet. But we will surely get a particular class of hypermultiplet actions with one isometry, with the Killing vector ∂_ϕ.

4 Hyper-Kähler sigma model with isospin variables

The consideration we carried out in the previous Section has one subtle point. Indeed, if we rewrite (3.7) as

$$\dot{\phi} = B_a \dot{v}_a - f_{,a}(\lambda\sigma^a\bar{\lambda}) + fA, \qquad f_{,a} \equiv \frac{\partial}{\partial v_a} f, \tag{4.1}$$

then the r.h.s. of (4.1) has to transform as a full time derivative under supersymmetry transformations (2.15). One may check that it is so, if f and B_a are chosen as in (3.7). However, this choice is not unique. It has been proved in [24] that the r.h.s. of (4.1) transforms as a full time derivative, if the functions f and B_a satisfy the equations

$$\triangle_3 f \equiv f_{,aa} = 0, \qquad f_{,a} = \epsilon_{abc} B_{c,b}. \tag{4.2}$$

Thus, one may construct a more general action for four-dimensional $\mathcal{N} = 4$ supersymmetric mechanics using the component action for the tensor supermultiplet and substituting there the new dualized version of the auxiliary component A (4.1).

Integrating over theta's in (2.19) and eliminating the auxiliary fermions $\rho^{\hat{\alpha}}$ (2.10), (2.17), we will get the following component action for the tensor supermultiplet:

$$\begin{aligned} S = \frac{1}{8}\int \mathrm{d}t \Bigg[& F\left(\dot{v}_a\dot{v}_a + A^2\right) + \mathrm{i}\left(\dot{\xi}^i\bar{\xi}_i - \xi^i\dot{\bar{\xi}}_i\right) + \\ & \mathrm{i}\epsilon_{abc}\frac{F_{,a}}{F}\dot{v}_b\Sigma_c - \mathrm{i}\frac{F_{,a}}{F}\Sigma_a A - \frac{1}{6}\frac{\triangle_3 F}{F^2}\Sigma_a\Sigma_a - \\ & 2\mathrm{i}\left(\dot{w}^i\bar{w}_i - w^i\dot{\bar{w}}_i\right) + \\ & 4\frac{1+3g|v|}{F(1+g|v|)^2|v|^4}\left(v_a I_a\right)\left(v_b\Sigma_b\right) - \\ & 4\frac{g}{F(1+g|v|)^2|v|}\left(I_a\Sigma_a\right) - \\ & \frac{4\mathrm{i}}{(1+g|v|)|v|^2}\left(v_a I_a\right)\, A + \\ & \frac{4\mathrm{i}}{(1+g|v|)|v|^2}\epsilon_{abc}v_a\dot{v}_b I_c \Bigg], \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} F &= \triangle_3\, \mathcal{F}(\mathcal{V})|, \\ I^a &= \frac{\mathrm{i}}{2}\left(w\sigma^a\bar{w}\right), \\ \Sigma^a &= -\mathrm{i}\left(\xi\sigma^a\bar{\xi}\right), \end{aligned} \tag{4.4}$$

and the re-scaled fermions and isospin variables are chosen to be

$$\xi^i = \sqrt{F}\,\lambda^i, \qquad w^i = \sqrt{g + \frac{1}{|v|}}\, u^i. \tag{4.5}$$

Substituting (4.1) into (4.3), we obtain the resulting action

$$\begin{aligned} S = \frac{1}{8}\int \mathrm{d}t \Bigg[& F\left(\dot{v}_a\dot{v}_a + \frac{1}{f^2}\left(\dot{\phi} - B_a\dot{v}_a\right)^2\right) + \\ & \mathrm{i}\left(\dot{\xi}^i\bar{\xi}_i - \xi^i\dot{\bar{\xi}}_i\right) - 2\mathrm{i}\left(\dot{w}^i\bar{w}_i - w^i\dot{\bar{w}}_i\right) - \\ & \mathrm{i}\left[\frac{1}{f}\delta_{ab}\left(\dot{\phi} - B_c\dot{v}_c\right) + \epsilon_{abc}\dot{v}_c\right] \cdot \\ & \left(\frac{F_{,a}}{F}\Sigma_b + \frac{4}{(1+g|v|)|v|^2}v_a I_b\right) + \\ & \frac{4}{F}\frac{1+3g|v|}{(1+g|v|)^2|v|^4}\left(v_a I_a\right)\left(v_b\Sigma_b\right) - \\ & \frac{1}{F}\frac{4g}{(1+g|v|)^2|v|}\left(I_a\Sigma_a\right) + \\ & \frac{1}{3F^2}\left(\frac{F_{,a}f_{,a}}{f} - \frac{F f_{,a}f_{,a}}{f^2} - \frac{1}{2}\triangle_3 F\right)\Sigma_b\Sigma_b \Bigg]. \end{aligned} \tag{4.6}$$

Action (4.6) is our main result. It describes the motion of a $\mathcal{N} = 4$ supersymmetric four-dimensional isospin carrying particle in a non-Abelian field of some monopole. The metric of this four-dimensional space is defined in terms of two functions: the bosonic part of our pre-potential F (4.4) and the harmonic function f (4.2). The supersymmetric version of the coupling with the monopole (second line in action (4.6)) is defined by the same harmonic function f and the coupling constant g. In the more general case (2.25), we will have two harmonic functions — f and Y, besides the pre-potential F.

Among all possible systems with action (4.6) there is a very interesting sub-class which corresponds to hyper-Kähler sigma models in the bosonic sector. This case is distinguished by the condition

$$F = f. \tag{4.7}$$

Clearly, in this case the bosonic kinetic term of action (4.6) acquires the familiar form of the one dimensional version of the general Hawking-Gibbons solution for four-dimensional hyper-Kähler metrics with one triholomorphic isometry [30]:

$$S_{kin} = \frac{1}{8}\int \mathrm{d}t \left[f\dot{v}_a\dot{v}_a + \frac{1}{f}\left(\dot{\phi} - B_a\dot{v}_a\right)^2\right],$$
$$\triangle_3 f = 0, \quad \mathrm{rot}\,\vec{B} = \vec{\nabla} f. \tag{4.8}$$

It is worth noting that the bosonic part of $\mathcal{N}=4$ supersymmetric four dimensional sigma models in one dimension does not necessarily have to be a hyper-Kähler one. This fact is reflected in the arbitrariness of the pre-potential F in action (4.6). Only under the choice $F = f$ is the bosonic kinetic term reduced to the Gibbons-Hawking form (4.8). Let us note that for hyper-Kähler cases the four-fermionic term in action (4.6) disappears. This fact has been previously established in [24]. Now we can see that the additional interaction with the background non-Abelian gauge field does not destroy these nice properties.

Among all possible bosonic metrics one may easily find the following interesting ones.

Conformally flat spaces

There are two choices for the function f which correspond to the conformally flat metrics in the bosonic sector.

The first choice is realized by

$$f = \frac{1}{|v|}. \tag{4.9}$$

This is just the case we have considered in Section 2. The gauge field in this case is the field of BPST instanton [17].

Next, an almost trivial solution, also corresponding to the flat metrics in the bosonic sector, is selected by the condition

$$f = \text{const.}, \qquad B_a = 0. \tag{4.10}$$

Note that the relation with the tensor supermultiplet, in this case, is achieved through the following "composite" construction of $\mathcal{V}^{ij}$ [31]

$$\mathcal{V}^{ij} = \mathcal{Q}^{(i\alpha)}. \tag{4.11}$$

One may check that the constraints on $\mathcal{V}^{ij}$ (2.13) follow directly from (4.11) and (2.20).

Let us recall that in both these cases we have not specified the pre-potential F yet. Therefore, the full metrics in the bosonic sector is defined up to this function.

Taub-NUT space

One should stress that the previous two cases are unique, because only for these choices of f can the resulting action (4.6) be obtained directly from the hypermultiplet action (2.24). With other solutions for f we come to the theory with the nonlinear $\mathcal{N}=4$ hypermultiplet [24, 25]. Among the possible solutions for f which belong to this new situation, the simplest one corresponds to one center Taub-NUT metrics with

$$f = p_1 + \frac{p_2}{|v|}, \qquad p_1, p_2 = \text{const}. \tag{4.12}$$

In order to achieve the maximally symmetric case, we will choose these constants as

$$p_1 = g, \quad p_2 = 1 \quad \rightarrow \quad f = g + \frac{1}{|v|}. \tag{4.13}$$

With such a definition, f coincides with the function $Y = g + \frac{1}{|v|}$ (2.25) entering in our basic action S_c in (2.1), (2.16). To get the Taub-NUT metrics, one has also to fix the pre-potential F to be equal to f. The resulting action which describes the $\mathcal{N}=4$ supersymmetric isospin carrying particle moving in a Taub-NUT space reads

$$\begin{aligned}
S_{Taub-NUT} = {} & \frac{1}{8}\int \mathrm{d}t \Bigg[\left(g + \frac{1}{|v|}\right)\dot{v}_a\dot{v}_a + \\
& \frac{1}{\left(g+\frac{1}{|v|}\right)}\left(\dot{\phi} - B_a\dot{v}_a\right)^2 + \mathrm{i}\left(\dot{\xi}^i\bar{\xi}_i - \xi^i\dot{\bar{\xi}}_i\right) - \\
& 2\mathrm{i}\left(\dot{w}^i\bar{w}_i - w^i\dot{\bar{w}}_i\right) + \frac{\mathrm{i}}{(1+g|v|)|v|^2}\cdot \\
& \left[\frac{v_a}{\left(g+\frac{1}{|v|}\right)}\left(\dot{\phi} - B_c\dot{v}_c\right) - \epsilon_{abc}v_b\dot{v}_c\right](\Sigma_a - 4I_a) + \\
& \frac{4(1+3g|v|)}{(1+g|v|)^3|v|^3}(v_aI_a)(v_b\Sigma_b) - \\
& \frac{4g}{(1+g|v|)^3}(I_a\Sigma_a)\Bigg].
\end{aligned} \tag{4.14}$$

The bosonic term in the second line of this action can be rewritten as

$$\begin{aligned}
\mathcal{A}_aI_a = {} & \frac{\mathrm{i}}{2}\left[\frac{1}{f}\frac{\partial \log f}{\partial v_a}\left(\dot{\phi} - B_a\dot{v}_a\right) - \right. \\
& \left. \epsilon_{abc}\frac{\partial \log f}{\partial v_b}\dot{v}_c\right] I_a,
\end{aligned} \tag{4.15}$$

where f is defined in (4.13). In this form the vector potential $\mathcal{A}_a$ coincides with the potential of a Yang-Mills $SU(2)$ instanton in the Taub-NUT space [32, 33], if we may view I_a, as defined in (4.4), as proper isospin matrices. The remaining terms in the second and third lines of (4.14) provide a $\mathcal{N}=4$ supersymmetric extension of the instanton.

Finally, to close this Section, let us note that more general non-Abelian backgrounds can be obtained from the multi-centered solutions of the equation for the harmonic function Y (2.26), which defined the coupling of the tensor supermultiplet with auxiliary fermionic ones. Thus, the variety models we constructed are defined through three functions: prepotential $\mathcal{F}$ (2.19) which is an arbitrary function, 3D harmonic function Y (2.25), (2.26) defining the coupling with isospin variables and, through the again 3D harmonic function f (4.1), (4.2), which appeared during the dualization of the auxiliary component of the tensor supermultiplet. It is clear that we can always redefine F to be $F = \tilde{F} f$. Thus, all our models are conformal to hyper-Kähler sigma models with $\mathcal{N} = 4$ supersymmetry describing the motion of a particle in the background non-Abelian field of the corresponding instantons.

5 Conclusion

In this paper we have constructed the Lagrangian formulation of $\mathcal{N} = 4$ supersymmetric mechanics with hyper-Kähler sigma models in the bosonic sector in the non-Abelian background gauge field. The resulting action includes the wide class of $\mathcal{N} = 4$ supersymmetric mechanics describing the motion of an isospin-carrying particle over spaces with non-trivial geometry. In two examples that we discussed in detail, the background fields are identified with the field of BPST instantons in the flat and Taub-NUT spaces.

The approach used in the paper has utilized two ideas: (i) the coupling of matter supermultiplets with an auxiliary fermionic supermultiplet $\Psi^{\hat{\alpha}}$ containing on-shell four physical fermions and four auxiliary bosons playing the role of isospin variables, and (ii) the dualization of the auxiliary component A of the tensor supermultiplet into a fourth physical boson. The final action that we constructed contains three arbitrary functions: the pre-potential $\mathcal{F}$, a 3D harmonic function Y which defines the coupling with isospin variables and, again 3D harmonic, a function f which appeared during the dualization of the auxiliary component of the tensor supermultiplet. The usefulness of the proposed approach is demonstrated by the explicit example of the simplest system with non-trivial geometry — the $\mathcal{N} = 4$ supersymmetric action for one-center Taub-NUT metrics. We identified the background gauge field in this case, which appears automatically in our framework, with the field of the BPST instanton in the Taub-NUT space. Thus, one may hope that the other actions will possess the same structure.

Of course, the presented results are just preliminary in the quest for full understanding of $\mathcal{N} = 4$ supersymmetric hyper-Kähler sigma models in non-Abelian backgrounds. Interesting questions that remain unanswered include, in particular:

- The full analysis of the general coupling with an arbitrary harmonic function Y has yet to be carried out.
- The structure of the background gauge field has to be further clarified: is this really the field of some monopole (instanton) for some hyper-Kähler metrics?
- The Hamiltonian construction is really needed. Let us note that the Supercharges have to be very specific, because the four-fermions coupling is absent in the case of HK metrics!
- It is quite interesting to check the existence of the conserved Runge-Lenz vector in the fully supersymmetric version.
- Explicit examples of other hyper-Kähler metrics (say, multi-centered Eguchi-Hanson and Taub-NUT ones) would be very useful.
- Questions of quantization and analysis of the spectra, at least in the cases of well known, simplest hyper-Kähler metrics, are doubtless urgent tasks.

Finally, let us stress that our construction is restricted to the case of hyper-Kähler metrics with one translational (triholomorphic) isometry. It will be very nice to find a similar construction applicable to the case of geometries with rotational isometry. We hope this may be done within the approach discussed in [34].

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