Does a Functional Integral Really Need a Lagrangian?

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Abstract

Path integral formulation of quantum mechanics (and also other equivalent formulations) depends on a Lagrangian and/or Hamiltonian function that is chosen to describe the underlying classical system. The arbitrariness presented in this choice leads to a phenomenon called *Quantization ambiguity*. For example both $L_1 = \dot{q}^2$ and $L_2 = e^{\dot{q}}$ are suitable Lagrangians on a classical level ($\delta L_1 = \delta L_2$), but quantum mechanically they are diverse.

This paper presents a simple rearrangement of the path integral to a surface functional integral. It is shown that the surface functional integral formulation gives transition probability amplitude which is free of any Lagrangian/Hamiltonian and requires just the underlying classical equations of motion. A simple example examining the functionality of the proposed method is considered.

Dedicated to my friend and colleague Pavel Bóna.

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1 A standard path integral lore

According to Feynman [1], the probability amplitude of the transition of a system from the space-time configuration (q_0, t_0) to (q_1, t_1) is given as follows:

$$\mathbf{A}(q_1, t_1 \mid q_0, t_0) \propto \int [\mathcal{D}\tilde{\gamma}] \exp\Big\{\frac{i}{\hbar}\int\limits_{\tilde{\gamma}} p_a \mathrm{d}q^a - H\mathrm{d}t\Big\}. \qquad (1)$$

Here the path-summation is taken over all trajectories $\tilde{\gamma}(t) = (\tilde{q}(t), \tilde{p}(t), t)$ in the extended phase space which are constrained as follows:

$$\begin{aligned}\tilde{\gamma}(t_0) &= \big(\tilde{q}(t_0) = q_0, \tilde{p}(t_0) - \text{arbitrary}, t_0\big),\\ \tilde{\gamma}(t_1) &= \big(\tilde{q}(t_1) = q_1, \tilde{p}(t_1) - \text{arbitrary}, t_1\big).\end{aligned}$$

To obtain a proper normalization of the Feynman propagator, one requires:

$$\begin{aligned}&\delta(\tilde{q}_0 - q_0) = \int\limits_{\mathbf{R}^{n[q_1]}} \mathrm{d}q_1\, \mathbf{A}^*(q_1, t_1 \mid \tilde{q}_0, t_0)\, \mathbf{A}(q_1, t_1 \mid q_0, t_0),\\ &\delta(q_1 - q_0) = \lim_{t_1 \to t_0} \mathbf{A}(q_1, t_1 \mid q_0, t_0).\end{aligned}$$

The first equation asks for the conservation of the total probability and the second expresses the obvious fact that no evolution takes place whenever t_1 approaches t_0.

It is a miraculous consequence (not a requirement!) of the propagator definition (1) that it satisfies an evolutionary chain rule (Chapman-Kolmogorov equation)

$$\mathbf{A}(q_1, t_1 \mid q_0, t_0) = \int\limits_{\mathbf{R}^{n[q]}} \mathrm{d}q\; \mathbf{A}(q_1, t_1 \mid q, t)\, \mathbf{A}(q, t \mid q_0, t_0),$$

whose infinitesimal version is the celebrated Schrödinger equation.

It is a curious fact that Formula (1) was not originally discovered by Feynman. In his pioneering paper [2] he arrives at a functional integral in the configuration space only

$$\mathbf{A}(q_1, t_1 \mid q_0, t_0) \propto \int [\mathcal{D}q] \exp\Big\{\frac{i}{\hbar}\int\limits_{q(t)} L(q, \dot{q}, t)\mathrm{d}t\Big\}. \qquad (2)$$

Later, however, it was shown that this formula represents a very special case of the most general prescription (1). Formula (1) is at the heart of our further discussion.

2 DeHamiltonianization

A step beyond involves eliminating the Hamiltonian function H from Formula (1). The price to be paid for this will be to replace the path summation therein by a surface functional integration.

Our aim is the transition probability amplitude between (q_0, t_0) and (q_1, t_1). Let us suppose that there exists a unique classical trajectory in the extended phase space $\gamma_{cl}(t) = (q_{cl}(t), p_{cl}(t), t)$ which connects these points (locally this assumption is always satisfied). Then for any curve $\tilde{\gamma}(t) = (\tilde{q}(t), \tilde{p}(t), t)$ which

enters the path integration in (1), we can assign two auxiliary curves which we call $\lambda_0(s)$ and $\lambda_1(s)$. They are parameterized by $s \in [0,1]$ and specified as follows:

$$\begin{aligned}\lambda_0(s) &= (q_0, \pi_0(s), t_0) \text{ where}\\ \pi_0(0) &= p_{cl}(t_0),\ \pi_0(1) = \tilde{p}(t_0),\\ \lambda_1(s) &= (q_1, \pi_1(s), t_1) \text{ where}\\ \pi_1(0) &= p_{cl}(t_1),\ \pi_1(1) = \tilde{p}(t_1).\end{aligned}$$

Let us emphasize that neither $\lambda_0(s)$ nor $\lambda_1(s)$ varies with respect to the q and t coordinates in the extended phase space. They are allowed to evolve only with respect to the momentum variables. There are, of course, infinitely many of such curves, but as we will see nothing in the theory will be dependent on a particular choice of $\lambda_0(s)$ and $\lambda_1(s)$.

Using these curves one can write:

$$\begin{aligned}&\int_{\tilde{\gamma}} p_a \mathrm{d}q^a - H\mathrm{d}t =\\ &\int_{\gamma_{cl}} p_a \mathrm{d}q^a - H\mathrm{d}t + \oint_{\partial\Sigma} p_a \mathrm{d}q^a - H\mathrm{d}t\,, \qquad (3)\end{aligned}$$

where $\partial\Sigma = \tilde{\gamma} - \lambda_1 - \gamma_{cl} + \lambda_0$ is a contour spanned by four curves $\tilde{\gamma}(t)$, $\gamma_{cl}(t)$, $\lambda_0(s)$, $\lambda_1(s)$ counting their orientations.

The first integral on the right is the classical action $\mathbf{S}_{cl}(q_1, t_1 \mid q_0, t_0)$. While the contour integral in (3) can be rearranged to represent a surface integral:

$$\begin{aligned}&\oint_{\partial\Sigma} p_a \mathrm{d}q^a - H\mathrm{d}t =\\ &\int_{\Sigma} \mathrm{d}p_a \wedge \Big(\mathrm{d}q^a - \frac{\partial H}{\partial p_a}\mathrm{d}t\Big) - \frac{\partial H}{\partial q^a}\mathrm{d}q^a \wedge \mathrm{d}t =: \int_{\Sigma} \Omega\,. \qquad (4)\end{aligned}$$

Surface Σ spanning the contour $\partial\Sigma$ is understood here as a map from the parametric space $(t,s) \in [t_0, t_1] \times [0,1]$ to the extended phase space, i.e.

$$\Sigma : (t,s) \mapsto \big(q^a(t,s), p_a(t,s), t(t,s) = t\big)\,.$$

Partial derivatives of the initial Hamiltonian function can be substituted using the velocity-momentum relations and classical equations of motion:

$$\frac{\partial H}{\partial p_a} = T^{ab}p_b \big(= \dot{q}^a\big) \quad \text{and} \quad \frac{\partial H}{\partial q^a} = -F_a \big(= -\dot{p}_a\big)\,.$$

Here we consider the physically mostly relevant situation only. In this case the velocities and momenta become related linearly by the metric tensor $T_{ab}(q)$ (and its inverse) defined by the kinetic energy $T = \frac{1}{2}T_{ab}\dot{q}^a\dot{q}^b = \frac{1}{2}T^{ab}p_a p_b$ of the system, then

$$\Omega = \mathrm{d}p_a \wedge \mathrm{d}q^a - \Big(T^{ab}p_a \mathrm{d}p_b - F_a \mathrm{d}q^a\Big) \wedge \mathrm{d}t\,. \qquad (5)$$

This object represents a canonical two-form in the extended phase space. It is a straightforward generalization of the standard closed two-form $\mathrm{d}\theta = \mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t$ to the case when the forces are not potential-generated.

It is clear that for a given pair of trajectories $(\tilde{\gamma}, \gamma_{cl})$ there exists infinitely many Σ surfaces. They form a set which we call $\mathcal{U}_{\tilde{\gamma}}$. Since the surface integral $\int_{\Sigma} \Omega$ is only boundary dependent and Formulas (3) and (4) are satisfied, we can write:

$$\begin{aligned}&\exp\Big\{\frac{i}{\hbar}\int_{\tilde{\gamma}} p_a \mathrm{d}q^a - H\mathrm{d}t\Big\} =\\ &\frac{\mathrm{e}^{\frac{i}{\hbar}\mathbf{S}_{cl}}}{\infty_{\tilde{\gamma}}} \int_{\mathcal{U}_{\tilde{\gamma}}} [\mathcal{D}\Sigma] \exp\Big\{\frac{i}{\hbar}\int_{\Sigma}\Omega\Big\}.\end{aligned}$$

Here $\infty_{\tilde{\gamma}}$ stands for the number of elements pertaining to the corresponding stringy set $\mathcal{U}_{\tilde{\gamma}}$. Assuming no topological obstructions from the side of the extended phase space, $\infty_{\tilde{\gamma}}$ becomes an infinite constant independent of $\tilde{\gamma}$. Taking all of this into account we can rewrite (1) as follows:

$$\mathbf{A}(q_1, t_1 \mid q_0, t_0) \propto \mathrm{e}^{\frac{i}{\hbar}\mathbf{S}_{cl}} \int_{\mathcal{U}} [\mathcal{D}\Sigma] \exp\Big\{\frac{i}{\hbar}\int_{\Sigma}\Omega\Big\}\,. \quad (6)$$

In this formula the undetermined normalization constant ∞ was included into the integration measure $[\mathcal{D}\Sigma]$ and the path integral over $\tilde{\gamma}$'s was converted to the surface functional integral as was promised:

$$\int [\mathcal{D}\tilde{\gamma}] \int_{\mathcal{U}_{\tilde{\gamma}}} [\mathcal{D}\Sigma] \ldots = \int_{\bigcup_{\tilde{\gamma}} \mathcal{U}_{\tilde{\gamma}}} [\mathcal{D}\Sigma] \ldots =: \int_{\mathcal{U}} [\mathcal{D}\Sigma] \ldots$$

The set $\mathcal{U} = \bigcup_{\tilde{\gamma}} \mathcal{U}_{\tilde{\gamma}}$ over which the functional integration is carried out contains all extended phase space strings which are anchored to the given classical trajectory γ_{cl}.

To eliminate Hamiltonian H completely we need to express $\mathbf{S}_{cl}(q_1, t_1 \mid q_0, t_0)$ in terms of the force field. Such a quantity may not exist in general, however we will see that in special cases one can recover an appropriate analog of $\mathbf{S}_{cl}(q_1, t_1 \mid q_0, t_0)$ requiring a certain behavior of

$$\begin{aligned}&\mathbf{A}(q_1, t_1 \mid q_0, t_0) \propto\\ &\mathrm{e}^{\frac{i}{\hbar}\mathbf{S}_{cl}} \int_{\mathcal{U}} [\mathcal{D}\Sigma] \exp\Big\{\frac{i}{\hbar}\int_{\Sigma}\Omega\Big\}\,.\end{aligned}$$

in the limit $\hbar \to 0$.

3 Functionality

A major advantage of the surface functional integral formulation rests in its explicit independence on Hamiltonian H. From the point of view of classical physics, dynamical equations and the force fields entering them seem to be more fundamental than the Hamiltonian and/or Lagrangian function, which provide these equations in a relatively compact but ambiguous way, see [3]. Therefore from the conceptual point of view, Formula (6) gives us transition probability amplitude from a different and hopefully new perspective. It is clear that for the potential generated forces the surface functional integral formula (6) gives nothing new compared to (1), since in this case Ω is closed and can be represented as $\Omega = d(p_a \mathrm{d}q^a - H\mathrm{d}t)$. There are, of course, some hidden subtleties which we pass over either quickly or in silence, however, all of them are discussed in [4].

To show functionality we need to analyze either a strongly non-Lagrangian system [5] or a weakly non-Lagrangian one. For the sake of simplicity let us focus on the second case. To this end, let us consider a system consisting of a free particle with unit mass affected by friction:

$$\ddot{q} = -\kappa\dot{q} \Leftrightarrow \Omega = \mathrm{d}p \wedge (\mathrm{d}q - p\,\mathrm{d}t) - \kappa p\,\mathrm{d}q \wedge \mathrm{d}t\,.$$

In the considered example, the surface functional integral can be carried out explicitly (for details see [4]). At the end one arrives at the path integral in the configuration space with a surprisingly trivial result:

$$\int\limits_{\mathcal{U}} [\mathcal{D}\Sigma] \exp\Big\{\frac{i}{\hbar}\int\limits_{\Sigma}\Omega\Big\} \propto$$
$$\exp\Big\{-\frac{i}{\hbar}\int\limits_{t_0}^{t_1}\Big(\frac{1}{2}\dot{q}_{cl}^2 - \kappa p_{cl}q_{cl}\Big)\,\mathrm{d}t\Big\} \times$$
$$\int [\mathcal{D}q] \exp\Big\{\frac{i}{\hbar}\int\limits_{t_0}^{t_1}\Big(\frac{1}{2}\dot{q}^2 - \kappa p_{cl}q\Big)\,\mathrm{d}t\Big\}\,.$$

If we define $\mathbf{S}_{cl}$ to be

$$\mathbf{S}_{cl}(q_1, t_1 \mid q_0, t_0) = \int\limits_{t_0}^{t_1}\Big(\frac{1}{2}\dot{q}_{cl}^2 - \kappa p_{cl}q_{cl}\Big)\,\mathrm{d}t\,, \qquad (7)$$

then

$$\mathbf{A}(q_1, t_1 \mid q_0, t_0) \propto$$
$$\int [\mathcal{D}q] \exp\Big\{\frac{i}{\hbar}\int\limits_{t_0}^{t_1}\Big(\frac{1}{2}\dot{q}^2 - \kappa p_{cl}q\Big)\,\mathrm{d}t\Big\}$$

and in the classical limit $\hbar \to 0$ we arrive at the saddle point equation which is specified by the functional term in the exponent above:

$$\ddot{q} = -\kappa\dot{q}_{cl}\,.$$

This differential equation is different from the equation $\ddot{q} = -\kappa\dot{q}$ that we started with initially, but both of them coincide when a solution $q(t)$ satisfying $q(t_0) = q_0$ and $q(t_1) = q_1$ is looking for. In the present situation we gain:

$$\mathbf{S}_{cl} = \frac{\kappa}{4}(q_1 - q_0)\frac{(q_0 + 3q_1)\mathrm{e}^{-\kappa t_1} - (q_1 + 3q_0)\mathrm{e}^{-\kappa t_0}}{\mathrm{e}^{-\kappa t_0} - \mathrm{e}^{-\kappa t_1}}$$

and

$$\mathbf{A}(q_1, t_1 | q_0, t_0) = \sqrt{\frac{\kappa}{4\pi i\hbar \tanh\frac{\kappa}{2}(t_1 - t_0)}}\,\mathrm{e}^{\frac{i}{\hbar}\mathbf{S}_{cl}}\,. \quad (8)$$

Here we have already employed the normalization conditions specified in the first paragraph. One can immediately verify that in the frictionless limit ($\kappa \to 0$) the transition probability amplitude $\mathbf{A}(q_1, t_1 \mid q_0, t_0)$ matches the Schrödinger propagator for a single free particle.

4 Conclusion

Quantization of dissipative systems has been very attractive problem from the early days of quantum mechanics. It has been revived again and again across the decades. Many phenomenological techniques and effective methods have been suggested. References [6] and [7] provide a very basic list of papers dealing with this point.

We have developed here a new quantization method that generalizes the conventional path integral approach. We have focused only on the nonrelativistic quantum mechanics of spinless systems. However, the generalization to the field theory is straightforward.

Let us stress that the proposed method represents an alternative approach to [7] and possesses several qualitative advantages. For example, propagator (8) is invariant with respect to time translations, the same symmetry property which is possessed by the underlying equation of motion. Moreover, it is reasonable to expect that the "dissipative quantum evolution" will not remain Markovian. This fact is again confirmed, since the probability amplitude under consideration does not satisfy the memoryless Chapman-Kolmogorov equation mentioned in the first paragraph.

Finally, let us believe that the simple geometrical idea behind the surface functional integral quantization will fit within the Ludwig Faddeev dictum *quantization is not a science, quantization is an art*!

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$\mathcal{A}.\quad \mathcal{M}.\quad \mathcal{D}.\quad \mathcal{G}.$

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