A Finite Liouville Dress for c< 1 Boundary Degenerate Matter

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Abstract

We review the derivation of a general formula for the Liouville dressing factor in the boundary 3-point tachyon correlator with $c < 1$ degenerate matter.

Keywords: non-critical string, tachyon correlators, boundary conditions.

1 Introduction

The simplest example of a non-critical string theory is 2d Liouville gravity induced by $c_M < 1$ matter [1]. It combines two Virasoro theories with central charges parametrised by a generically real number b, $c_M = 13 - 6(b^2 + 1/b^2) < 1$ and $c_L = 26 - c_M > 25$, so that when we add a pair of reparametrisation ghosts of central charge -26 the total conformal anomaly vanishes. The D-brane dynamics in an open non-critical string is determined by the boundary correlation functions (numbers) of the physical fields of ghost number one, "massless tachyons", see e.g. [2, 3, 4, 5, 6, 7] for more recent discussions.

The full boundary tachyon field factorises into a matter times a Liouville "dressing" vertex operator, producing a similar factorisation of the full 3-point function. In this work we address our attention to the pure Liouville factor of it in the case where the matter factor corresponds to degenerate Virasoro representations. The matter fields are vertex operators of the scaling dimension $\Delta M(e) = e(e - 1/b + b)$ labelled by degenerate $c_M < 1$ Virasoro representations. This implies that the charges $\{\beta_i\}$ of the dressing Liouville boundary vertex operators ${}^{\sigma_i}B^{\sigma_i}_{\beta_i}$, of scaling dimensions

$$\Delta_L(\beta) = \beta(Q - \beta) = 1 - \Delta_M(e),$$

take the values

$$\beta_i = b + m_i b - \frac{ni}{b}, \quad 2m_i, 2n_i \in \mathbb{Z}_{\geq 0} \tag{1.1}$$

or their reflected $\beta \to Q - \beta$ counterparts ($Q = b + 1/b$), so that without loss of generality we shall work with the values in (1.1). The range of the boundary parameters σ_i is generically parametrised by the continuous Liouville spectrum $2\sigma - Q$ – pure imaginary but also admits continuation to real values. These Liouville boundary fields correspond to the FZZ branes [8].

The matter factor of the 3-point boundary tachyon correlator is a straightforward generalisation of the factor in the rational b^2-case. It is alternatively reproduced by an analytic continuation of a residuum of the integral Ponsot-Teschner (PT) formula [9] at points corresponding to $c > 25$ degenerate Virasoro representations. The same analytic continuation applies to fusing matrices, which differ from the boundary field crossing matrices (3-point boundary correlators) by a renormalisation of the three boundary vertices. Thus the formulae in [9, 10] for the quantum 3j and 6j symbols, designed generically for the continuous $c > 25$ spectrum, are in a sense universal, since we can reproduce from them the Coulomb gas quantities in both $c < 1$ and $c > 25$ Virasoro regions. However this integral formula is not very explicit, and its main characteristics are not immediately visible when applied to the spectrum of representations (1.1). Another alternative is to solve the pentagon equations recursively. The final result is a meromorphic expression in the boundary cosmological parameters, the derivation of which we review here, see [11] for more details. It generalises a special (thermal) case result of [6] and partial results in the microscopic approach in [5].

2 Boundary 3-point Liouville constant

The matter fusion rules impose restrictions on the values in (1.1), namely all $m^k_{ij} := m_i + m_j - m_k$, $n^k_{ij} = n_i + n_j - n_k$ are non-negative integers, so that

$$2m_{123} = \sum_{i=1}^{3} 2m_i = 0 \bmod 2.$$

The 3-point boundary Liouville functions that we are interested in are related to the boundary field crossing matrices

$$\mathcal{C}_{\sigma_2,Q-\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix} = \langle {}^{\sigma_1}B_{\beta_3}{}^{\sigma_3}B_{\beta_2}{}^{\sigma_2}B_{\beta_1}{}^{\sigma_1}\rangle = \mathcal{C}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} = S(\sigma_1,\beta_3,\sigma_3)\mathcal{C}^{\sigma_3,\sigma_2,\sigma_1}_{Q-\beta_3,\beta_2,\beta_1}, \tag{2.1}$$

where $S(\sigma_1,\beta_3,\sigma_3)$ is the reflection amplitude [8].

The associativity condition for OPE of boundary fields, together with the fusion transformation relating the s and t channels, lead to an integral pentagon-like equation for the boundary field 3-point functions

$$\int \mathrm{d}\beta_S \mathcal{C}^{\sigma_4,\sigma_3,\sigma_1}_{Q-\beta_3,\beta_2,\beta_s}\mathcal{C}^{\sigma_3,\sigma_2,\sigma_1}_{Q-\beta_s,\beta,\beta_1}\mathcal{F}_{\beta_s,\beta_t}\begin{bmatrix}\beta_2 & \beta\\ \beta_3 & \beta_1\end{bmatrix} = \mathcal{C}^{\sigma_4,\sigma_2,\sigma_1}_{Q-\beta_3,\beta_t,\beta_1}\mathcal{C}^{\sigma_4,\sigma_3,\sigma_2}_{Q-\beta_t,\beta_2,\beta}, \tag{2.2}$$

where $\mathcal{F}_{\beta_s,\beta_t}$ is the fusing matrix computed in [10]. The boundary 3-point functions $\mathcal{C}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}$ are meromorphic with respect to the variables β_1,β_2,β_3 [9], while the fusion coeficients $\mathcal{F}_{\beta_s,\beta_t}\begin{bmatrix}\beta_2 & \beta\\ \beta_3 & \beta_1\end{bmatrix}$ are meromorphic in all six variables and invariant under the reflections $\beta_i \to Q-\beta_i$.

When one of the operators corresponds to a degenerate representation, the $\mathcal{F}_{\beta_s,\beta_t}$ and $\mathcal{C}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}$ coeficients develop singularities such that the integral in (2.2) gives rise to a finite sum over representations in accordance with the fusion rules [9]. In particular, for $\beta=-b/2$, equation (2.2) becomes (see e.g. [5]):

$$\begin{aligned}
&C_{\sigma_3,\beta_2-t\frac{b}{2}}\begin{bmatrix}\beta_2 & -\frac{b}{2}\\ \sigma_4 & \sigma_2\end{bmatrix}\mathcal{C}_{\sigma_2=\sigma_3\pm\frac{b}{2},\beta_3}\begin{bmatrix}\beta_2-t\frac{b}{2} & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix} = \\
&F_{+t}\begin{bmatrix}\beta_2 & -\frac{b}{2}\\ \beta_3 & \beta_1\end{bmatrix}C_{\sigma_3\pm\frac{b}{2}\beta_1-\frac{b}{2}}\begin{bmatrix}-\frac{b}{2} & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix}\mathcal{C}_{\sigma_3,\beta_3}\begin{bmatrix}\beta_2 & \beta_1-\frac{b}{2}\\ \sigma_4 & \sigma_1\end{bmatrix}+\\
&F_{-t}\begin{bmatrix}\beta_2 & -\frac{b}{2}\\ \beta_3 & \beta_1\end{bmatrix}C_{\sigma_3\pm\frac{b}{2}\beta_1+\frac{b}{2}}\begin{bmatrix}-\frac{b}{2} & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix}\mathcal{C}_{\sigma_3,\beta_3}\begin{bmatrix}\beta_2 & \beta_1+\frac{b}{2}\\ \sigma_4 & \sigma_1\end{bmatrix},\\
&t=\pm 1
\end{aligned} \tag{2.3}$$

where C and F are the appropriate residues of $\mathcal{C}$ and $\mathcal{F}$. For $t=1$ it becomes

$$\begin{aligned}
\mathcal{C}^{\sigma_3,\sigma_2\pm\frac{b}{2},\sigma_1}_{\beta_3,\beta_2,\beta_1} = &\frac{\Gamma(1-2\beta_2 b)\Gamma((2\beta_1-b)b)}{\Gamma(1-b(\beta_2+\beta_3-\beta_1))\Gamma(b(\beta_1+\beta_3-\beta_2-b))}\cdot\mathcal{C}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2+\frac{b}{2},\beta_1-\frac{b}{2}}+\\
&\frac{b^2\sqrt{\lambda_L}\Gamma(1-2\beta_2 b)\Gamma(1-2\beta_1 b)g\mp(\sigma_2,\beta_1,\sigma_1)}{2\sin\pi b(2\beta_1-Q)\Gamma(1-b(\beta_1+\beta_2+\beta_3-Q))\Gamma(1-b(\beta_1+\beta_2-\beta_3))}\cdot\mathcal{C}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2+\frac{b}{2},\beta_1+\frac{b}{2}},
\end{aligned} \tag{2.4}$$

where $\lambda_L=\pi\mu\gamma(b^2)$ is the (normalised) cosmological constant and

$$g_-(\sigma_2,\beta_1,\sigma_1) = g_+(\sigma_2,Q-\beta_1,\sigma_1) = -4\sin\pi b\left(\beta_1-\sigma_1-\sigma_2+\frac{b}{2}\right)\sin\pi b\left(\beta_1-\sigma_2+\sigma_1-\frac{b}{2}\right). \tag{2.5}$$

There is a second equation with a shift $\beta_2-b/2$ on the r.h.s. as well as two dual equations for $1/b\to b/2$. The derivation of (2.4) is standard and the coeficients in front of the correlators are given by the products of fusing matrix elements and 3-point boundary functions containing a fundamental field. The latter are computed by free field Coulomb gas methods [8], assuming that for degenerate representations the Cardy multiplicity coincides with the Verlinde multiplicity. In the case above, this means that the two boundaries of the field ${}^{\sigma_2}B^{\sigma_1}_{-\frac{b}{2}}$ satisfy $\sigma_2=\sigma_1\pm b/2$.

2.1 The simplest correlator

We start with the derivation of the simplest correlator with three identical charges equal to b, i.e., the correlator of three cosmological operators, or boundary Liouville screening charges. It is reproduced by the second term on the r.h.s. of the equality (2.4) choosing $\beta_1=\dfrac{b}{2}=\beta_2$, $\beta_3=b$. For this choice the equation needs regularisation since the coeficient in front of the correlator becomes divergent. The remaining two correlators are represented as reflections (2.1) with respect to β_3 (the l.h.s.) and β_2 (the first term on the r.h.s.) of correlators which also diverge, if we assume that they are given by the integral PT formula. Indeed they satisfy the charge conservation conditions $(Q-\beta_3)+\beta_2+\beta_1=Q$ and $\beta_3+(Q-\beta_2-b/2)+(\beta_1-b/2)=Q$, respectively, and their residua equal $1/2\pi$ (to agree with the normalisation in [9]). Thus, in a proper regularisation of (2.4), these two correlators are replaced by the corresponding reflection amplitudes, which appear as the initial data in the equation. We recall their general expression computed in [8],

$$\begin{aligned} S(\sigma_2,\beta,\sigma_1) &= \frac{2\pi}{b\Gamma(1+\frac{1}{b}(Q-2\beta))\Gamma(b(Q-2\beta))} G_2(\sigma_2,\beta,\sigma_1), \\ G_2(\sigma_2,\beta,\sigma_1) &= \frac{\lambda_L^{\frac{1}{2b}(Q-2\beta)} S_b(2\beta-Q)}{\prod_{s=\pm} S_b(\beta+s(\sigma_2+\sigma_1-Q))S_b(\beta+s(\sigma_2-\sigma_1))}\,, \end{aligned} \tag{2.6}$$

where $S_b(\alpha)=\Gamma_b(\alpha)/\Gamma_b(Q-\alpha)=2\sin\pi b(\alpha-b)S_b(\alpha-b)$ and $\Gamma_b(x)$ is the double gamma function; $S_b(b)=b$. In the case under consideration here $\beta=b$ and inserting in (2.4) we reproduce the cyclically symmetric expression proposed in the microscopic approach [5],

$$\begin{aligned} C^{\sigma_3,\sigma_2,\sigma_1}_{b,b,b} &= \frac{2\pi\sqrt{\lambda_l}^{-1}}{(\Gamma(1-b^2))^2\Gamma(\frac{1}{b^2}-1)} \cdot \frac{G_2(\sigma_3,b,\sigma_1)-G_2(\sigma_3,b,\sigma_2)}{g_-(\sigma_1,\frac{b}{2},\sigma_2)} = \\ & \frac{2\pi\lambda_L^{\frac{Q-3b}{2b}}}{S_b(\frac{2}{b})(\Gamma(1-b^2))^2\Gamma(\frac{1}{b^2}-1)} \cdot \frac{(\tilde c_1(c_2-c_3)+\tilde c_2(c_3-c_1)+\tilde c_3(c_1-c_2)}{(c_2-c_1)(c_1-c_3)(c_3-c_2)} \end{aligned} \tag{2.7}$$

where the boundary cosmological constants $\sim c_i$ and their dual appear,

$$c_i=2\cos\pi b(b-2\sigma_i), \tilde c_i=2\cos\pi\frac{1}{b}\left(\frac{1}{b}-2\sigma_i\right). \tag{2.8}$$

Similar regularised versions of (2.4) arise for other values of the charges corresponding to reflections of Coulomb gas correlators.

2.2 One parameter correlators, cyclic symmetry

We shall use Eq. (2.4) as a recursion relation, starting from the explicit expression (2.7). Let us first introduce some general notation:

$$G^{(-)}(\sigma_2,\beta,\sigma_1) := S_b(-\beta+\sigma_2+\sigma_1)S_b(Q-\beta+\sigma_2-\sigma_1) = g_-\left(\sigma_2,\beta+\frac{b}{2},\sigma_1\right) G^{(-)}(\sigma_2,\beta+b,\sigma_1). \tag{2.9}$$

For a non-negative integer k and an integer n of parity $p(n)$ denote

$$B(\sigma_2,\sigma_1)^{(k;p(n))} := \frac{G^{(-)}(\sigma_2,-\frac{kb}{2}-\frac{n}{2b},\sigma_1)}{G^{(-)}(\sigma_2,b+\frac{kb}{2}-\frac{n}{2b},\sigma_1)} = (-1)^{(k+1)(n+1)}B(\sigma_1,\sigma_2)^{(k;p(n))}. \tag{2.10}$$

Applying (2.9), the ratio (2.10) is expressed as a $k+1$ order polynomial in $\{c_i\}$ using that for $k\neq 0$

$$g_-\left(\sigma_2,\frac{b}{2}-k\frac{b}{2}+\frac{n}{2b},\sigma_1\right) g_-\left(\sigma_2,\frac{b}{2}+k\frac{b}{2}+\frac{n}{2b},\sigma_1\right) = c_1^2+c_2^2-c_1c_2(-1)^n 2\cos\pi kb^2-(2\sin\pi kb^2)^2 \tag{2.11}$$

while $B(\sigma_2,\sigma_1)^{(0;p(n))}=(-1)^n c_2-c_1$. Similarly, we define the dual $\tilde B(\sigma_2,\sigma_1)^{(n;p(k))}$

$$\tilde B(\sigma_2,\sigma_1)^{(n;p(k))} := \frac{G^{(-)}(\sigma_2,-\frac{n}{2b}-\frac{kb}{2},\sigma_1)}{G^{(-)}(\sigma_2,\frac{1}{b}+\frac{n}{2b}-\frac{kb}{2},\sigma_1)} = (-1)^{(k+1)(n+1)}\tilde B(\sigma_1,\sigma_2)^{(n;p(k))} \tag{2.12}$$

so that the reflection amplitude is expressed as the ratio of polynomials

$$\frac{\lambda_L^{\frac{2\beta_2-Q}{2b}} G_2(\sigma_2,\beta_2=b+m_2b-\frac{n_2}{b},\sigma_1)}{S_b(2\beta_2-Q)} = \frac{G^{(-)}(\sigma_2,\beta_2,\sigma_1)}{G^{(-)}(\sigma_2,Q-\beta_2,\sigma_1)} = \frac{\tilde B(\sigma_2,\sigma_1)^{(2n_2;p(2m_2))}}{B(\sigma_2,\sigma_1)^{(2m_2;p(2n_2))}}. \tag{2.13}$$

Finally we introduce

$$
\begin{aligned}
P_2 \;\equiv\; P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} &:= (-1)^{m^3_{12}+2m_2}\lambda_L^{-\frac{m^3_{12}}{2}}\frac{S_b((2m_1+1)b)S_b((2m_2+1)b)}{S_b(b)}\,\cdot \\
&\sum_{p=0}^{m^3_{12}}\frac{S_b((m^3_{12}+1)b)}{S_b((p+1)b)S_b((m^3_{12}+1-p)b)}\times\frac{G_2(\sigma_2+p\frac{b}{2},\beta_2-p\frac{b}{2},\sigma_3)}{G_2(\sigma_2,\beta_2,\sigma_3)} \\
&\frac{G_2(\sigma_2-(m^3_{12}-p)\frac{b}{2},\beta_1-(m^3_{12}-p)\frac{b}{2},\sigma_1)}{G_2(\sigma_2,\beta_1,\sigma_1)}
\end{aligned}
\tag{2.14}
$$

and similarly P_1 and P_3, which are obtained from (2.14) by cyclic permutations. The finite sum (2.14) is proportional to a truncated basic hypergeometric function ${}_4\phi_3(\ldots;q,q)$. It can be expanded as a polynomial in the variables $\{c_i\}$ (a special case of Askey-Wilson polynomials).

We begin with the "thermal" case with all $n_i = 0$ in (1.1). We first use such a regularised equation in which the first term on the r.h.s. of (2.4) reduces to a 2-point function in order to obtain recursively the most general correlator with $m^2_{13} = 0$. Then using the analog of the general equation (2.4) for shifts of the pair (β_3, β_2), we obtain

$$
\begin{aligned}
C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} &= -\frac{\lambda_L^{\frac{Q-\beta_{123}}{2b}}\prod(\beta_3,\beta_2,\beta_1)}{B(\sigma_1,\sigma_2)^{(2m_1;0)}B(\sigma_2,\sigma_3)^{(2m_2;0)}B(\sigma_3,\sigma_1)^{(2m_3;0)}}F^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}, \\
F^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} &= (-1)^{2m_1}((-1)^{2m_2}\tilde{c}_2-\tilde{c}_3)B(\sigma_3,\sigma_1)^{(2m_3;0)}P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} - \\
&\quad (-1)^{2m_2}((-1)^{2m_3}\tilde{c}_3-\tilde{c}_1)B(\sigma_2,\sigma_3)^{(2m_2;0)}P^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3} = \\
&\quad -\Big(\tilde{c}_1B(\sigma_3,\sigma_2)^{(2m_2;0)}P^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3}+\tilde{c}_2B(\sigma_1,\sigma_3)^{(2m_3;0)}P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}+\tilde{c}_3B(\sigma_2,\sigma_1)^{(2m_1;0)}P^{\sigma_1,\sigma_3,\sigma_2}_{\beta_1,\beta_3,\beta_2}\Big)
\end{aligned}
\tag{2.15}
$$

where

$$
\prod(\beta_3,\beta_2,\beta_1) = \frac{b^{e_0(Q-\beta_{123})}\Gamma_b(2q-\beta_{123})\Gamma_b(Q-\beta^1_{23})\Gamma_b(Q-\beta^2_{13})\Gamma_b(Q-\beta^3_{12})}{S_b(\frac{1}{b})S_b(\frac{2}{b})\Gamma_b(Q)\Gamma_b(Q-2\beta_1)\Gamma_b(Q-2\beta_2)\Gamma_b(Q-2\beta_3)}. \tag{2.16}
$$

In the last equality of (2.15) we have exploited (2.10) and the relation.

$$
B(\sigma_3,\sigma_1)^{(2m_3;0)}P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} + \text{cyclic permutations} = 0 \tag{2.17}
$$

which is equivalent to the cyclic symmetry of the correlator, now explicit in (2.15). Symmetry is ensured by the fact that the expression given by the first equality satisfies all the equations related by cyclic permutations.

The composition of the reflection of all three fields with the reflection amplitude as in (2.1) and the duality transformation $b \to 1/b$ (changing notation $m_i \to n_i$) gives the correlator in the other thermal case, when all $m_i = 0$ in (1.1). In this case the product of $B^{(0;p(2n_i))}$ replaces the denominator in (2.15) and the formula confirms the structure suggested in the microscopic approach of [5]. The dual polynomial $\tilde{P}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}$ is defined by changing in (2.14) $\beta_i \to Q-\beta_i$, $b \to 1/b$, $m_i \to n_i$. With the help of some identities for the basic hypergeometric functions one reproduces the formula in [6] for the case $\{m_i = 0, n_i$ – integers$\}$. The expression in [6] is however not explicitly symmetric under cyclic permutations, rather this symmetry is checked to hold on examples.

2.3 The general correlator

To obtain the Liouville correlator defined for general values (1.1), we can either use the dual pentagon equations, or we can start from the correlator with all $m_i = 0$. In one of the steps, the pentagon equation (2.4) is regularised again so that the second term on the r.h.s. is given by G_2 times a non-trivial Coulomb gas Liouville correlator. The final result is an expression generalising the first line in (2.15),

$$
\begin{aligned}
C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} &= \frac{\lambda_L^{\frac{Q-\beta_{123}}{2b}}\prod'(\beta_3,\beta_2,\beta_1)}{B(\sigma_1,\sigma_2)^{(2m_1;p(2n_1))}B(\sigma_2,\sigma_3)^{(2m_2;p(2n_2))}B(\sigma_3,\sigma_1)^{(2m_3;p(2n_3))}}\times \\
&(-1)^{2m_22n_1}\Big((-1)^{2m_1+2n_2}\tilde{B}(\sigma_2,\sigma_3)^{(2n_2;p(2m_2))}\tilde{P}^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3}B(\sigma_3,\sigma_1)^{(2m_3;p(2n_3))}P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} - \\
&(-1)^{2m_2+2n_1}\tilde{B}(\sigma_3,\sigma_1)^{(2n_3;p(2m_3))}\tilde{P}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}B(\sigma_2,\sigma_3)^{(2m_2;p(2n_2))}P^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3}\Big)
\end{aligned}
\tag{2.18}
$$

with the prefactor

$$
\prod{}'(\beta_3,\beta_2,\beta_1) = \frac{(-1)^{m_{123}n_{123}}\prod(\beta_3,\beta_2,\beta_1)S^3_b(\frac{1}{b})S_b(\frac{2}{b}-b)}{S_b(\frac{n^3_{12}+1}{b})S_b(\frac{n^1_{23}+1}{b})S_b(\frac{n^2_{13}+1}{b})S_b(\frac{n_{123}+2}{b}-b)}. \tag{2.19}
$$

Here, say, the polynomial P_2 is given by the first formula (2.14), where now all β_i are given by (1.1), with only the sign in front of (2.14) modified to $(-1)^{m_{12}^3(1+2n_3)+2m_3 2n_3+2m_2} = (-1)^{m_{123}(1+2n_3)+2m_1}$. Let us also write down the expression for one of the dual polynomials

$$\tilde{P}_1 \equiv \tilde{P}^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3} \frac{(-1)^{n_{123}(1+2m_2)+2n_3}\lambda_L^{-n_{13}^2/2} S_b(\frac{2n_1+1}{b}) S_b(\frac{2n_3+1}{b})}{S_b(\frac{1}{b})} \sum_{u=0}^{n_{13}^2} \frac{S_b(\frac{n_{13}^2+1}{b})}{S_b(\frac{1+u}{b}) S_b(\frac{n_{13}^2+1-u}{b})} \times \tag{2.20}$$
$$\frac{G_2(\sigma_1+\frac{u}{2b}, Q-\beta_1-\frac{u}{2b},\sigma_2)}{G_2(\sigma_1,Q-\beta_1,\sigma_2)} \frac{G_2(\sigma_1-\frac{n_{13}^2-u}{2b}, Q-\beta_3-\frac{n_{13}^2-u}{2b}\sigma_3)}{G_2(\sigma_1,Q-\beta_3,\sigma_3)}.$$

The cyclic symmetry of the full correlator is ensured by construction and is equivalent to a relation generalising (2.17),

$$(-1)^{2n_2(2m_2+1)} B(\sigma_3,\sigma_1)^{(2m_3;p(2n_3))} P_2 + \text{cyclic permutations} = 0 \tag{2.21}$$

and its dual with the dual polynomials and $m_i \leftrightarrow n_i$. In particular, when all $m_i = 0$ the dual relation reproduces the cyclic identity satisfied by the first order dual polynomials $\tilde{B}(\sigma_2,\sigma_3)^{(0;p(2m_2))} = (-1)^{2m_2}\tilde{c}_2 - \tilde{c}_3$, etc., which appear in the numerator in (2.15). The composition of the duality transformation $b \to 1/b$, $m_i \leftrightarrow n_i$ with reflection of all three fields keeps (2.18) invariant.

3 Summary and discussion

We have obtained the general Liouville dressing factor in the tachyon 3-point boundary correlator with degenerate $c < 1$ representations. Formula (2.18) represents the Liouville correlator as a ratio of polynomials of the boundary cosmological parameters $c_i, \tilde{c}_i$ generalising the partial results in [5, 6]. This solution of the Liouville pentagon equations extends to the minimal gravity theory with rational b^2, in which case there may appear further truncations of the sums. The general 3-point boundary tachyon correlator is a product of (2.18) and the matter 3-point boundary correlator, satisfying a 4-term equation, see [11] for an explicit formula and further discussion.

A possible extension of our result would allow us to describe also the 3-point boundary tachyon correlators corresponding to the ZZ branes. For this purpose, the roles of the matter and Liouville spectra and the corresponding correlators are essentially inverted: the Coulomb gas Liouville correlator for degenerate $c > 25$ representations describing both the charges and the boundaries should be combined with a matter factor obtained by analytic continuation of the solution (2.18). Note that the corresponding discrete $c < 1$ spectrum parametrises the irreducible representations embedded as submodules of the reducible Virasoro modules. The analogous characteristics of the $c > 25$ spectrum (1.1) have been exploited in the construction of the 4-point bulk tachyon correlators [12].

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