# The velocity of Rayleigh waves along a prestressed semi-infinite medium assuming a two-dimensional anisotropy

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## 1. Introduction.

It is probable that certain features of symmetry in geological structures can have some effect comparable to anisotropy on the propagation of seismic waves, when that symmetry is significant on an average, at the scale of the wave length. Take for instance the layered structure due to sedimentary formation. When the layers are horizontal (Fig. 1), it is known that a modification of elastic constants with depth induces velocity dispersion in the Rayleigh surface waves, and also makes possible the appearance of the (dispersive) Love waves. Discarding here any variation with depth, we may consider only the effect of a transverse anisotropy which could be displayed by such a structure. If the latter extends on sufficiently large areas, as compared to the wave length of the traversing waves, with small irregularities which would not

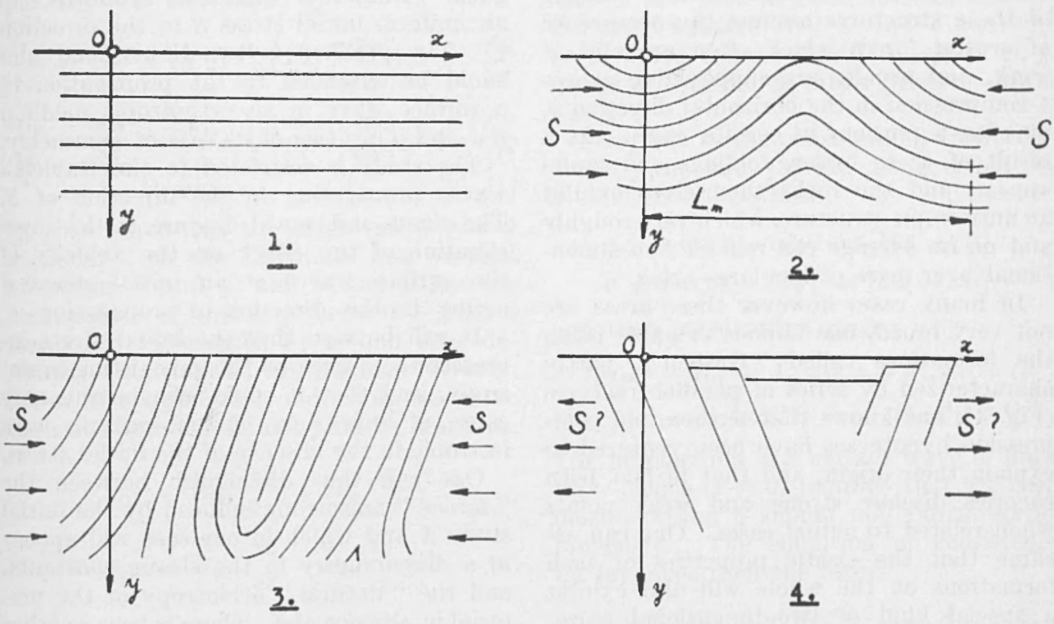


Fig. 1 to 4. – Different geological formations which could, on the average, display anisotropy, with or without initial crustal forces $S$.

impair the general symmetry, one must expect that some effect will be detected on the surface wave.

On the other hand, if such layered structures have been highly compressed and have folded in a given horizontal direction (Fig. 2 and 3), the same assumption of average homogeneity could be made for the waves propagating along the surface in the direction perpendicular to the folds. The " modulation which appears especially in Fig. 2 for the elastic constants could be neglected if its wave-length $L_m$ is small compared to the wave-length of the waves. In that case, only average values of the constants will have to be retained.

Accordingly certain parts of the world display geological structures which may roughly be considered as two-dimensional, and as such can be discribed in a vertical cross-section $(x, y)$. The tectonic theories of these structures assume the occurrence of crustal forces which often are still at work, and usually are supposed to exerce a compression in the horizontal direction $x$, or even a tension, in certain cases. As a result of these forces, foldings and faults appear, and the rocks themselves exhibit an anisotropic structure, which very roughly and on an average can remain two-dimensional over more or less large areas.

In many cases however these areas are not very broad, one famous example being the formations called " Graben ", partly characterized by series of parallel fractures (Fig. 4); one knows that tension and compression hypotheses have been ventured to explain their origin, and that in fact both theories disclose strong and weak points when related to actual cases. One can assume that the elastic properties of such formations on the whole will also exhibit a special kind of two-dimensional anisotropy, and the purpose of this paper is to study its effect on the propagation of surface waves.

It should be noted here that the propagation of elastic waves in infinite unstressed aeolotropic media have been the subject of several studies [(2) to (5) and (10) to (12)]. Besides the well-known existence of the principal velocities of wave propagation, (ref. first chapter of Love's Treatise on the Mathematical Theory of Elasticity), one must point out that the distinction between dilatational and distortional waves tends to disappear in the case of aeolotropy. The propagation of *surface* waves in an unstressed semi-infinite medium endowed with transverse isotropy, (or hexagonal symmetry) about the vertical axis was studied by Stoneley (7) and Satô (8); there is then of course no privileged direction of propagation for Rayleigh waves, due to the symmetry of revolution of the system. Such characteristic directions arise in cases of higher degrees of anisotropy, as has been shown by Stoneley for cubic crystals (11), and more generally by Synge (12).

In short, we consider now the propagation of surface waves in the direction $x$, in a semi-infinite $(y \geqslant 0)$ homogeneous, transverse isotropic (aeolotropic with hexagonal symmetry) medium, submitted to an uniform initial stress $S$ in the direction $x$. The problem is two-dimensional and could be extended to the propagation of a surface wave in an orthotropic medium if $x$ and $y$ are two of its axis of symmetry.

The study is restricted to the Rayleigh waves propagating in the direction of $S$. The main and novel feature is the investigation of the effect on the velocity of the surface waves of an initial stress $S$ acting in the direction of propagation.

It will be seen that the initial stress increases the degree of the preexisting aeolotropy, and that it could also be the only cause of anisotropy, if the medium were isotropic in the absence of the initial stress.

One can thus distinguish between the " forced " anisotropy induced by the initial stress $S$ and which in our case will appear in a dissymmetry in the elastic constants, and the " natural " aeolotropy of the material in absence of $S$. There is here another reason to assume the preexistence of natural aeolotropy: the fact that in most natural media the presence of an initial stress causes a slow change of the elastic constants, change which in general will consist in an increase of the " natural " aeolotropy.

The assumption of homogeneity allows an easy solution of the partial differential equations of the wave propagation, the coefficients remaining constant. Such equations, written down for the general case of anisotropic elasticity have been given previously by Biot in an important paper on "The influence of initial stress on elastic waves" (1). In this reference, Rayleigh waves are considered from the point of view of the gravity effect, and Bromwich's result is restated for an incompressible medium. The influence of the gravity has been recognized to be small and needs not be reconsidered here.

## 2. Equations of the Rayleigh wave.

With Biot's notations for the components of the stress tensor $s_{ij}$, the equation of oscillation in two dimensions $(x, y)$ are, (see ref. (1), p. 529, equ. 34, where we have put $S_{22} = 0$ and $S_{11} = S$)

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + (S/2)\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \varrho \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + (S/2)\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \varrho \cdot \frac{\partial^2 v}{\partial t^2} \qquad [1]$$

In these equations the subscripts 1 and 2 are related to the horizontal and vertical directions $x$ and $y$, and $u$ and $v$ are the components of the displacement. Inertia forces are represented by second time derivatives multiplied by $\varrho$, the mass density.

For the assumed two-dimensional wave the stress-strain relations reduces it self to:

$$s_{11} = B_{11}\frac{\partial u}{\partial x} + B_{12}\frac{\partial v}{\partial y}$$

$$s_{22} = B_{21}\frac{\partial u}{\partial x} + B_{22}\frac{\partial v}{\partial y} \qquad [2]$$

$$s_{12} = (B_{33}/2)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

with $B_{21} = B_{12} + S$. Thus, for $S \neq 0$, the elastic coefficients are no more symmetric.

Substituting [2] in [1], one obtains:

$$B_{11}\frac{\partial^2 u}{\partial x^2} + A_{12}\frac{\partial^2 v}{\partial x \partial y} + A_{13}\frac{\partial^2 u}{\partial y^2} = \varrho \cdot \frac{\partial^2 u}{\partial t^2}$$

$$A_{32}\frac{\partial^2 v}{\partial x^2} + A_{12}\frac{\partial^2 u}{\partial x \partial y} + B_{22}\frac{\partial^2 v}{\partial y^2} = \varrho \cdot \frac{\partial^2 v}{\partial t^2} \qquad [3]$$

with the notation:

$$A_{13} = (B_{33} - S)/2 \quad , \quad A_{32} = (B_{33} + S)/2$$

and

$$A_{12} = B_{12} + A_{32} = B_{21} + A_{13} = (B_{12} + B_{21} + B_{33})/2.$$

The boundary conditions are, for the free surface $y = 0$, (horizontal plane boundary),

$$s_{12} \sim \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = O \ ,$$

$$s_{22} - B_{21}\frac{\partial u}{\partial x} + B_{22}\frac{\partial v}{\partial y} = O \ . \qquad [4]$$

For geophysical applications this would mean that varying external forces on the surface are neglected, and this indeed is permissible, for instance, at the surface of the continents.

Furthermore, there is, of course, in the case of the semi-infinite medium, the additional condition that $u$ and $v$ converge to zero for infinite positive values of $y$, whose orientation is taken positive downwards.

Now, due to the constancy of their coefficients, the equ. [3] can be satisfied by a solution of the form:

$$u - a \exp(-\alpha k y) \sin[k(x - ct)]$$

$$v = b \exp(-\alpha k y) \cos[k(x - ct)] \qquad [5]$$

and their substitution gives:

$$k^2\left[a(\varrho c^2 - B_{11} + A_{13}\alpha^2) + b A_{12}\alpha\right] = O$$

$$k^2\left[-a A_{12}\alpha + b(\varrho c^2 - A_{32} + B_{22}\alpha^2)\right] = O. \qquad [6]$$

The propagation of a surface wave of finite wave-length $(k \neq 0)$ requires that the coefficients $a$ and $b$ should not vanish simultaneously. The condition of consistency for [6] takes then the form:

$$\alpha^4 B_{22} A_{13} + \alpha^2\left[(\varrho c^2 - A_{32}) A_{13} + (\varrho c^2 - B_{11}) B_{22} + A^2_{12}\right] + (\varrho c^2 - B_{11})(\varrho c^2 - A_{32}) = O \ . \qquad [7]$$

Let us call $\alpha^2_1$ and $\alpha^2_2$ the roots of [7], and let us assume that they are real but unequal (see Appendix I for the case of equal roots).

When the roots $\alpha^2_1$ and $\alpha^2_2$ are both positive, one can always take for $k\alpha$ the sign which makes it positive in [5], in order to insure the vanishing of the solutions for increasing values of $y$.

The alternate signs with the corresponding complementary terms and boundary conditions, should intervene in the case where waves are propagating in a plate bounded by two parallel planes, instead of a semi-infinite medium.

There are two such values for $a$, and as a consequence of the linearity of the equ. [3], a complete solution for $u$ and $v$ can be put in the form:

$$\begin{aligned} u &= [a' \exp(-a'ky) + \\ &\quad + a'' \exp(-a''ky)] \sin [k(x-ct)] \\ v &= [\beta' a' \exp(-a'ky) + \\ &\quad + \beta'' a'' \exp(-a''ky)] \cos [k(x-ct)] \end{aligned} \qquad [8]$$

with only two constants $a'$ and $a''$ to be determined by the boundary conditions [4], for $y = 0$; the $\beta's$ are the previously defined values of the ratio $b/a$. The other constants associated with the values $-a'$ and $-a''$ of $a$ disappear due to the condition at $y - \infty$. The case where $a^2_1$ and $a^2_2$ are conjugate complex always yields two conjugate values for $ka$ with positive real parts; the coefficients $a'$ and $a''$ in [8] are then also conjugate complex, and the functions of $y$ between brackets in [8] would reduce to decreasing exponentials multiplied by sinusoidal functions (see Appendix III).

(The case where one at least of the roots $a^2$ is real non-positive is considered in Appendix II).

### 3. The velocity equation.

In expression [8], $a'$ and $a''$ are thus functions of the elastic constants, of the initial stress $S$, and also of the wave velocity $c$, which appears in equ. [7]. This velocity will now be determined by the boundary conditions [4].

Substituting $u$ and $v$ in these conditions, one finds:

$$\begin{aligned} a'(a' + \beta') + a''(a'' + \beta'') &= 0 \\ a'(B_{21} - B_{22} a' \beta') + a''(B_{21} - B_{22} a'' \beta'') &= 0 \end{aligned} \qquad [9]$$

whence the condition of consistency, which is also symmetrical in $a$ and $\beta$:

$$\begin{aligned} &(a' - a'')(B_{21} + B_{22} \beta' \beta'') + \\ &+ (\beta' - \beta'')(B_{21} + B_{22} a' a'') = 0 \end{aligned} \qquad [10]$$

The $\beta'$ and $\beta''$ are to be deduced from the system [6], which has been made consistent by [7]. We find:

$$\begin{aligned} \beta' - \beta'' &= -(B_{11} - \varrho c^2)(a' - a'') / \\ &\quad / A_{12} a' a'' - A_{13}(a' - a'') / A_{12} \\ \beta' \beta'' &= (B_{11} - \varrho c^2)^2 / \\ &\quad / A^2_{12} a' a'' - A_{13}(B_{11} - \varrho c^2)(a'^2 + a''^2) / \\ &\quad / A^2_{12} a' a'' + A^2_{13} a' a'' / A^2_{12} \end{aligned} \qquad [11]$$

and substituting, we note that the assumption $a' = a''$ allows the removal of the factor $(a' - a'')$ from [10]. We have then:

$$\begin{aligned} &a'^2 a''^2 B_{22} A_{13}(A_{13} - A_{12}) - \\ &- a' a'' A_{12}[B_{21}(A_{13} - A_{12}) + B_{22}(B_{11} - \varrho c^2)] - \\ &- (a'^2 + a''^2) B_{22} A_{13}(B_{11} - \varrho c^2) + \\ &+ (B_{11} - \varrho c^2)[B_{22}(B_{11} - \varrho c^2) - B_{21} A_{12}] = 0 \end{aligned} \qquad [12]$$

expression which contains only the sum and products of the roots $a'^2$ and $a''^2$ of [7]. These can be easily formulated in terms of the coefficients of [7], and their substitution in [12] yields, after rationalisation, the deceptively simple equation:

$$\begin{aligned} &B_{22}(\varrho c^2 - A_{32})(\varrho c^2 - B_{11} + B^2_{21} / B_{22})^2 - \\ &- A_{13}(\varrho c^2 - B_{11})(\varrho c^2 - S)^2 = 0 . \end{aligned} \qquad [13]$$

Putting $S = 0$ (absence of initial stress) one falls back on the velocity equation given by Stoneley [7], which, in the present notation takes the form:

$$\begin{aligned} &B_{22}(\varrho c^2 - B_{33} / 2)(\varrho c^2 - B_{11} + B^2_{12} / \\ &/ B_{22})^2 - \varrho^2 c^4 (\varrho c^2 - B_{11}) B_{33} / 2 = 0 . \end{aligned}$$

It is worth noting that the speed $c$ given by [13] under an initial stress $S$ could also be computed starting from a ficticious unstressed medium, of characteristics $\varrho'$, $B'_{11}$, $B'_{22}$, $B'_{12} = B'_{21}$ and $B'_{33}$, by taking:

$$\varrho c^2 - S = \varrho' c^2 \qquad B_{11} - S = B'_{11}$$

$$B_{33} - S = B'_{33} \qquad B_{12} + S = B'_{12} = B'_{21} = B_{21}$$

and $\qquad B_{22} = B'_{22}$

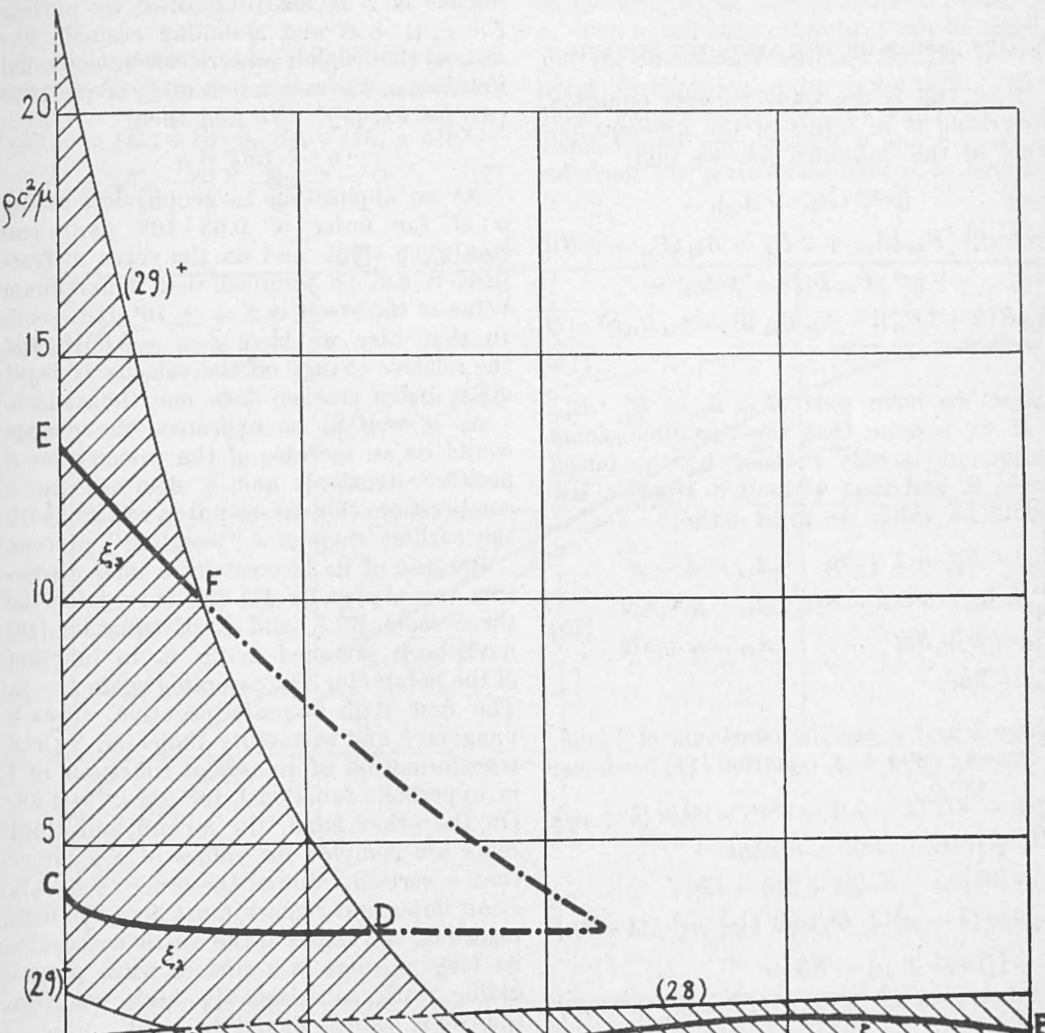


Fig. 5. – Roots $\xi_i$ of the velocity equation in function of $S/\mu$, for a prestressed medium, with $\lambda = \mu$. Permissible areas are shaded.

Having then determined the roots $\varrho' c^2$, one gets $c = \sqrt{(\varrho' c^2 + S)/\varrho}$.

## 4. Discussion of the velocity equation.

Equ. [13] is the wave velocity equation. Rewriting it in terms of the different powers of the unknown $\varrho c^2$, we find:

$$\begin{gathered}(\varrho c^2)^3 (B_{22} - A_{13}) - \\ - (\varrho c^2)^2 \left[B_{22} (A_{32} + 2 B) - A_{13} (B_{11} + 2 S)\right] \\ + \varrho c^2 \left[B_{22} B(B + 2 A_{32}) - \right. \\ \left. A_{13} S (S + 2 B_{11})\right] - A_{32} B_{22} B^2 + A_{13} B_{11} S^2 = O\end{gathered} \quad [14]$$

where we have put: $B = B_{11} - B^2_{21} / B_{22}$.

If we assume that the two-dimensional anisotropy is only induced by the initial stress $S$, and that without it Hooke's law would be valid, we must write:

$$\begin{aligned} B_{11} &= B_{22} = \lambda + 2\mu & A_{12} &= \lambda + \mu \\ B_{12} &= B_{21} - S = \lambda - S/2 & A_{13} &= \mu - S/2 \\ B_{21} &= \lambda + S/2 & A_{32} &= \mu + S/2 \\ B_{33} &= 2\mu, \end{aligned} \quad [15]$$

where $\lambda$ and $\mu$ are the constants of Lamé.

Putting $\varrho c^2/\mu = \xi$, equation [14] becomes:

$$\begin{gathered}\xi^3 \left[1 + S/2 (\lambda + \mu)\right] - \xi^2 \left[8 - S\lambda/\mu (\lambda + \mu)\right] \\ + \xi \left[(24\lambda + 32\mu - 8 S\lambda/\mu - \right. \\ - S^2 \left[(2\mu - S)(3\lambda + 2\mu) - 4\lambda^2\right] / \\ \left. / 4\mu^2 (\lambda + \mu) + S^4/16\mu^2 (\lambda + \mu)\right] / (\lambda + 2\mu) \\ - \left[(16 (\lambda + \mu) + 8S - \right. \\ - S^2 (4\lambda + 6\mu)/\mu^2 + S^3 (2\lambda^2 + 2\mu^2 + 3\lambda\mu) / \\ / 2\mu^3 (\lambda + \mu) + S^4 (4\lambda + \mu) / \\ \left. / 16\mu^3(\lambda + \mu) + S^5/32\mu^3(\lambda + \mu)\right] / (\lambda + 2\mu) = O.\end{gathered} \quad [16]$$

It is readily verified that this expression reduces itself to the known equation for the Rayleigh waves, when we take $S = O$. Its roots give the values of the velocity in a medium naturally isotropic, where the anisotropy is induced by the initial stress $S$. One sees at once that the effect of this stress on the value of the roots as deduced for the isotropic case $S = O$ is small at the same time as $S/\mu$.

If $\xi_o$ is the value of a root for the unstressed medium, an estimate of the influence of $S$ is easily obtained by putting $\xi = \xi_o (1 + \varepsilon)$ and assuming $\varepsilon$ small, like $S/\mu$, so that higher powers can be neglected For $\lambda = \mu$, the smallest root $\xi_o$ is $(0.9194)^1$ (see for ex. [9]). We find then:

$$\varepsilon = 1.01 \; S/\mu \, .$$

As an application to geophysics, taking $\mu$ of the order of $0.63 \cdot 10^{12}$ dynes/cm$^2$ (see [9], p. 153), and on the verge of fracture, it can be assumed that a maximum value of the stress is $S = \pm 10^9$ dynes/cm$^2$. In that case we have $\varepsilon = \pm 0.0016$ and the relative change on the velocity is negligible, being smaller than one thousandth.

As it was to be expected, the change would be an increase of the velocity for $S$ positive (tension), and a decrease for a compression; this can be put in relation with the earliest stage of a "buckling" process.

Because of its interest for a later discussion (see Appendix II) the curves for the three roots $\xi_1$, $\xi_2$ and $\xi_3$ of equation [16] have been sketched in fig. 5, in function of the parameter $S/\mu$, assuming again $\lambda = \mu$. The first root becomes negative, (thus $c$ imaginary and instability indicated by the transformation of sinusoidal functions of $t$ in hyperbolic functions), for $S/\mu < - 1.33$. On the other hand, the second and third roots are complex for values of $S/\mu$ larger than a certain value of the order of $+ 0.2$.

An important remark must be made here regarding the values of the elastic constants. As long as there is a surface wave propagating with a reasonably high velocity, isentropic values should be used.

On the other hand it is obvious that a standing wave with zero frequency, which is the buckling case, calls for isothermal constants, which imply lower values of the critical compressive stress ($- S$). To be strict the study of very slow waves should be thermodynamical in nature and involve the equation of heat conduction.

## 5. Critical values of the initial stress.

We have seen that the wave velocity can vanish for some so-called critical values of $S$, it is easy to show that at least one such value lies in the following interval:

$$- B_{33} < S_{cr} < O .$$

It must satisfy equ. [14] when $c = O$, and therefore is a real root of the equation:

$$f(S) = -(B_{33} + S)\,[B_{11} B_{22} - (B_{12} + S^2)]^2 + + (B_{33} - S)\, B_{11}\, B_{22}\, S^2 = O \qquad [17]$$

rubber, where $S$ can assume values of the same order of magnitude as $\mu$, which must be determined at the considered value of $S$. The initial strain due to $S$ can be large, and the anisotropy induced by the initial stress distribution quite important. The linear equations [2] remain valid for the displacement components $u$ and $v$ measured from the predeformed state, as long as

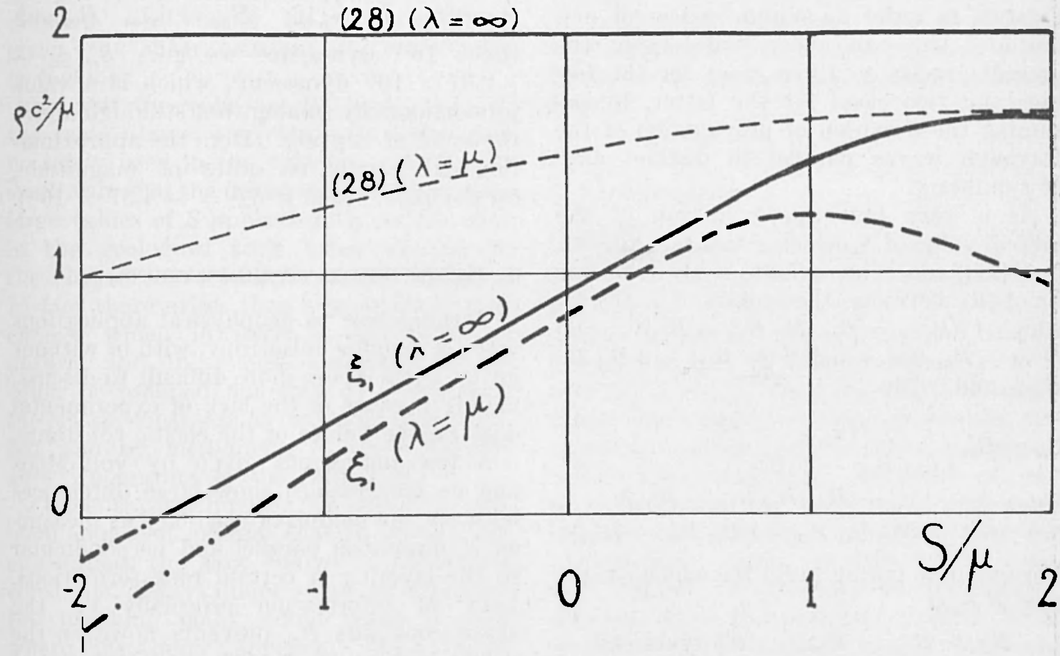


Fig. 6

obtained from [14] by replacing the $A$'s by their expression in the $B$'s and $S$. All the $B$'s in this equation being positive, except possibly $B_{12}$, one sees immediately that $f(o)$ is negative, and $f(-B_{33})$ positive. Further $f(S)$ remains positive for $S < -B_{33}$ and negative for $S > S_{33}$.

In normal circumstances elastic instability would only be observed if the compression $-S$ could, without rupture of the material, reach values which are not too small compared to the elastic constants, which everytime must be determined in the prestressed state.

This is of course exceptionnal for most materials, but there is a case in point for these components are small as it is assumed here for the surface waves.

Now rubber is to be considered as a nearly incompressible material, and we will assume the limit value 0,5 for Poisson's ratio $\nu$. We have then $\lambda = \infty$, and the equation [16] takes then the form:

$$\xi^3 - \xi^2 (8 - S/\mu) + \xi (24 - 8\, S/\mu + S^2/\mu^2) - - 16 + 4\, S^2/\mu^2 - S^3/\mu^3 = O\,. \qquad [18]$$

Fig. 6 gives the variation of the first root $\xi_1$, for $-2\mu \leqslant S \leqslant +2\mu$. The two other roots are complex in this range.

The root $\xi_1$ becomes negative, and there appears thus instability, for $S > -1{,}679\ \mu$.

This is the only real value of $S$ for which $\xi$ vanishes, for

$$S^3/\mu^3 - 4\, S^2/\mu^2 + 16 = (S/\mu + 1{,}679) \cdot \left[S/\mu - (2{,}8395 + s.\ 1{,}22)\right] \cdot \left[S/\mu - (2{,}8395 - s.\ 1{,}22)\right].$$

In the same drawing the curve of $\xi_1$ for $\lambda = \mu$ has been reproduced, in dotted lines, for comparison.

Taking new orthorhombic or hexagonal crystals in order to obtain orders of magnitude, one can every time apply the present results to three cases for the former and two cases for the latter, by assuming the direction of propagation of the Rayleigh waves parallel to distinct axes of symmetry.

As a very first approximation of the critical value of $S$, which satisfies $f(S) = O$, (see [17]), one can assume a straight line for $f(S)$ between the points $S = O$ [for which $f(O) = -B_{33}(B_{11}B_{22} - B_{12})^2$], and $S = -B_{33}$ (for which $f(-B_{33}) = 2\, B_{33} B_{11} B_{22}$), and write

$$S_{cr} \cong B_{33} \frac{f(O)}{f(-B_{33}) - f(O)} = \frac{-B_{33}(B_{11}B_{22} - B^2_{12})^2}{2\, B^2_{33} B_{11} B_{22} + (B_{11}B_{22} - B^2_{12})^2}$$

For example taking beryl for which, in the case $S \cong O$

$$B_{33} = 2C_{14} = 13{,}2 \times 10^{11}\ \text{dynes/cm}^2$$
$$B_{11} = C_{11} = 28{,}5\ \text{»}\ \text{»}$$
$$B_{22} = C_{33} = 25{,}0\ \text{»}\ \text{»}$$
$$B_{12} = C_{13} = 7{,}0$$

$$S_{cr} \cong 8{,}43 \times 10^{11}\ \text{dynes/cm}^2$$

and assuming here these values unchanged with $S$ (for lack of further information), we find:

$$S_{cr} \cong -8{,}43 \times 10^{11}\ \text{dynes/cm}^2$$

which is not too different from the exact value: $-9{,}925 \times 10^{11}$, but in every case very much above the breaking point.

For zinc

$B_{33} = 8$, $B_{11} = 16$, $B_{22} = 6{,}1$, $B_{12} = 5$ times $10^{11}$ dynes/cm² we get: $S_{cr} \cong -2{,}37 \times 10^{11}$ dynes/cm², which is a value proportionnally smaller, but still high above the limit of rupture. (Here the approximation gives only on ordes of magnitude, for the exact value is $S_{cr} = -4{,}87 \times 10^{11}$).

## 6. Geophysical considerations.

Turning now to geophysical applications cases of genuine anisotropy, with or without initial stress $S$, are more difficult to discuss, mainly because of the lack of experimental data on the values of the elastic constants.

A few indications given by von Moos and de Quervain [6] show large differences between the moduli of elasticity as measured in directions parallel and perpendicular to the layering of certain rock formations. Lack of information especially on the shear constants $B_{33}$ prevents however the determination of the elastic constants of equ. [13].

For instance, these authors report the following data,

| | Modulus of elasticity (kg/cm²) $E$ (par.) | $E$ (perp.) | Inverse Poisson's ratio, $m$ |
|---|---|---|---|
| layered granite . . . . . . . . . | 400.000 | 250.000 | — |
| layered biotite . . . . . . . . . | 280.000 | 80.000 | — |
| layered sandstone | | | |
| fine grained . . . . . . . | 80.000 to 150.000 | 30.000 to 50.000 | 6 to 10 |
| thick » . . . . . . . | 30.000 to 50.000 | 15.000 to 30.000 | 6 to 10 |

In this table $m$ is the ratio between longitudinal and transverse deformations. Of course its value should depend on the direction of the tension, and when this last is not perpendicular to the layers, on the transverse direction. This must explain at least partly the dispersion of the values reported in ([6]).

The relations which bind the $E's$ and the $m's$, to the $B's$ are given in Appendix IV. It is shown that even when the different $m's$ should be more precisely given, the shear constant $B_{33}$ would still be lacking.

Even if the value of $B_{33}$ were given in the cases above, the constants could only be safely used in the velocity equation for small values of the initial stress $S$. Moreover large values of $S$ applied for a certain time in the geological scale bring changes in the elastic characteristics of the material. In fact there arises then a problem of slow change which in itself is rheological in nature, and in a first approximation it could be handled not only by considering the elastic constants as functions of $S$ and $t$, but also by introducing in these " constants " operators of the form $d/dt$.

Concentrating now on the stability problem, one can assume that in certain circumstances the slow change in the elastic constants, always under values of $S$ beneath the breaking point, could reach a point where instability appears. Just before buckling the time operators $d/dt$ and $d^2/dt^2$ would play a negligible role, letting elastic criteria determine the onset of buckling, by an equation of the form [17], where now all the constants $B$ are to be considered as functions of $S$.

This type of buckling, where the wave length must be kept small with respect to the thickness of the terrestrial crust, should be put in contrast with the one considered by Jeffreys ([9]). which involves large scale bending of the terrestrial shell.

One of the shortcomings of the present theory lies in the assumption of homogeneity; neglecting the change of the rheological characteristics of the material with depth should restrict any application to small scale orogeny, with reasonably short wave length.

The indeterminacy of the wave length, (there is no length scale in a semi infinite medium), allows a Fourier superposition to give shape of wave. In particular, a standing wave solution could always be adapted to a plane strip of finite extension.

The presence of initial irregularities of the surface has not been taken in account and would not doubt influence the buckling process. There is however a difference with the role of bilateral initial deformation in compressed columns. It is to be suspected that the largest irregularities would influence the forthcoming pattern of buckling.

## 7. Conclusion.

The simplest problem involving the effect of initial stresses on the propagation of Rayleigh waves in a case of two-dimensional symmetry with transverse anisotropy, has led to the solution [5], with equ. [7] and [13] to determine the coefficients $Re(a)$ of exponential amplitude decrease with depth, and the wave velocity $c$.

The two-dimensional character of our problem avoids the complication met more generally by Stoneley ([11]), who already for homogenious cubic crystals found that, with respect to directions of propagation different from the principal axes or their bissectrices, the surface waves either do not exist, or exhibit vertical oscillation planes which are inclined to the direction of propagation.

As it had already been shown by Stoneley ([11]) and Synge ([12]) for unstressed media, we find that the distinction between compressional and distorsional waves does not exist any more in an aeolotropic medium, and further that the purely exponential decay of the Rayleigh wave with depth must be generalized to an exponentially damped sinusoidal oscillation. Such a generalization does not occur however in the case of the naturally isotropic medium, for normal values of the initial stress $S$.

The direct effect of a compressional initial stress $S$ on the velocity $c$ is a decrease, which for practical purposes is usually negligible. A $S$-dependant variation of the elastic constants lies presently outside the scope of experimental data, but may bring interesting results in the field of small scale orogeny, when near-buckling processes could be involved.

On the other hand, besides purely physical applications, the theory may also offer some technical interest, for most materials from which machine parts are made of, not only may be prestressed, but can also exhibit some kind of polycrystalline anisotropy near the surfaces. Accordingly the behaviour of surfaces waves with very high ultrasonic frequencies could in certain cases shed some light on the state of the material near the surfaces.

From the theoretical point of view, one effect of initial stresses is to set up a higher degree of anisotropy by destroying the symmetry of cross elastic constants. It would be interesting to extend the general theory developed by Synge ([12]) for the 21-constants aeolotropic, but unstressed medium, to the case of initial stress. Keeping the assumption of uniformity, this would bring five new parameters in an in infinite medium (the five independant components of the stress deviator), and three only for a semi-infinite medium, due to the boundary conditions.

*Appendix I.*

The case where roots $a^2_1$ and $a^2_2$ of equ. [7] are equal does not introduce any substantial change in the preceding results. The discriminant of [7] must vanish, and by eliminating $\varrho c^2$ between this relation and the equation [13], which of course holds in the limit, one finds a relation which must be satisfied by the elastic coefficients of the medium.

The limiting process whereby the elastic coefficients are varied in order to make the roots $a^2$ coalesce, gives us immediately the correct forms of the solution [8]. These expressions can be replaced by the more convenient linear combinations:

$$u = \left\{ (a_1/2) \left[\exp(-a'ky) + \exp(-a''ky)\right] + \right.$$
$$\left. + \frac{a_2}{a' - a''} \left[\exp(-a'ky) - \exp(-a''ky)\right] \right\}$$
$$\cdot \sin\left[k(x - ct)\right]$$

$$v = \left\{ (a_1/2 \left[\beta' \exp(-a'ky) + \beta'' \exp(-a''ky)\right] + \right.$$
$$\left. + \frac{a_2}{a' - a''} \left[\beta' \exp(-a'ky) - \beta'' \exp(-a''ky)\right] \right\}$$
$$\cdot \cos\left[k(x - ct)\right] \qquad [19]$$

so that in the limit, for $a' = a'' = a$:

$$u = (a_1 - a_2 ky) \exp(-aky) \sin\left[k(x - ct)\right]$$
$$v = \left[a_1\beta + a_2 (d\beta/da) - a_2\beta ky\right] \exp(-aky)$$
$$\cos\left[k(x - ct)\right] \qquad [20]$$

the values of $\beta$ and $(d\beta/da)$ being given by [11]:

$$(d\beta/da) = -(B_{11} - \varrho c^2)/A_{12}\, a^2 - A_{13}/A_{12}$$
$$\beta^2 = (B_{11} - \varrho c^2)^2/A^2_{12}\, a^2 - 2\, A_{13}\, (B_{11} - \varrho c^2) /$$
$$/ A^2_{12} + a^2 A^2_{13} / A^2_{12} \qquad [21]$$

Assuming the discriminant equal to zero, equ. [7] yields:

$$a^4 = (\varrho c^2 - B_{11})\, (\varrho c^2 - A_{32}) \,/\, B_{22}\, A_{13}$$

and, due to [13] this becomes

$$a^2 = \pm\, (\varrho c^2 - B_{11})\, (\varrho c^2 - S) \,/\, B_{22}$$
$$(\varrho c^2 - B_{11} + B^2_{21} \,/\, B_{22}) \qquad [22]$$

with the proper sign for which the discriminant vanishes. If $a^2$ is positive, it is its positive real square root which is introduced in [20]. Now if $a^2$ is negative, its roots are pure imaginary conjugates. This case belongs to Appendix II.

*Appendix II.*

Let us suppose that one at least of the roots of equ. [7], say $a_1{}^2$, is real negative. We have then two conjugate pure imaginary values, $a_1$ and $-a_1 = \bar{a}_1$, and their substitution in expressions of the form [8] would give oscillatory non-decreasing functions of $y$ in the semi-infinite medium, so that physically the corresponding coefficient must vanish, (at least for a propagating wave of finite energy). If both roots $a_i{}^2$ are negative, no Rayleigh wave of finite energy could propagate through the semi-infinite medium, at least at the considered velocity.

Now it is easy to see that if the roots $\alpha^2_i$ are real, both are of the same sign, for their product is:

$$P = (\varrho c^2 - B_{11})\,(\varrho c^2 - A_{32}) \,/\, B_{22}\, A_{13}$$

and following [13] this is:

$$P = (\varrho c^2 - B_{11})^2\,(\varrho c^2 - S)^2 \,/\, B^2_{22}\,(\varrho c^2 - B_{11} + B^2_{21} \,/\, B_{22})^2 \quad [22]$$

which is positive. This supposes of course $c$ real, which is the condition for a wave of bounded amplitude.

Thus there is no wave whose velocity would be such that the roots $\alpha^2_i$ be real (discriminant of [7] positive), and their sum negative, that is for which:

$$[A_{13}\,(\varrho c^2 - A_{32}) + B_{22}\,(\varrho c^2 - B_{11}) + A^2_{12}] \,/\, B_{22}\, A_{13} > O$$

The condition on the discriminant becomes then:

$$(\varrho c^2 - A_{32})/B_{22} + (\varrho c^2 - B_{11})/A_{13} + A^2_{12}/B_{22}A_{13} > 2\sqrt{[\varrho c^2 - B_{11}) \,/\, A_{13}]\,[\varrho c^2 - A_{32})/B_{22}]} \quad [23]$$

the square-rooted quantity being always positive, following [13]; the brackets have thus the same sign. If they are both positive, the "admissibility" condition becomes, by reversing inequality [23]:

$$[\sqrt{(\varrho c^2 - A_{32}) \,/\, B_{22}} - \sqrt{(\varrho c^2 - B_{11}) \,/\, A_{13}}]^2 + A^2_{12} \,/\, B_{22}\, A_{13} < O \quad [24]$$

and if they are both negative:

$$A^2_{12} \,/\, B_{22}\, A_{13} < [\sqrt{-(\varrho c^2 - A_{32})/B_{22}} + \sqrt{-(\varrho c^2 - B_{11})/A_{13}}]^2 \quad [25]$$

Let us consider the normal case: $B_{22} A_{13} > O$.

Then [24] is impossible; thus for a passing wave, both bracketts must be negative and [25] should be true. The constant $B_{22}$ being positive, we assume $A_{13} > O$, or $S < B_{33}$. The conclusion is then that $\varrho c^2$ must be smaller than the smallest of $C_{32}$ and $B_{11}$. If $B_{33} < B_{11}$, (in an isotropic medium this demands $\lambda > O$), the velocity of a passing wave satisfies the condition:

$$\varrho c^2 < (B_{33} + S) \,/\, 2 < B_{33} \quad [26]$$

plus the condition [25].

For a pre-stressed isotropic medium, the assumption $C_{13} > O$ gives $S < 2\mu$, and condition [25] becomes:

$$S^2/4 - 2\mu\,(\lambda + 2\mu) + \varrho c^2\,(\lambda + 3\mu - S/2) < 2 \cdot \sqrt{[\mu^2 - S^2/4 - \varrho c^2\,(\mu - S/2)]\,[(\lambda + 2\mu)^2 - \varrho c^2\,(\lambda + 2\mu)]} \quad [27]$$

This is satisfied if the first member is negative, that is

$$\varrho c^2 > [2\mu\,(\lambda + 2\mu) - S^2/4]/(\lambda + 3\mu - S/2) \quad [28]$$

For $S = O$ and $\lambda = \mu$, the last member reduces to $3\mu/2$, which indeed is large than the first root already mentioned: ($\varrho c^2/\mu = (0.9194)^2$).

Would [28] *not* be effective, then both members of [27] can be squared, and $\varrho c^2$, being larger than [28], should be included between the two values

$$\{S^2\,(\lambda + \mu + S/2) - 4\,S\,(\lambda + \mu)\,(\lambda + 2\mu) \pm \pm 2\,S\sqrt{2\,(\lambda+\mu)\,(\lambda+2\mu)\,[(\lambda+\mu)\,(\lambda+3\mu) + S^2/4]}\} \cdot \cdot [4\,(\lambda + \mu + S/2)^2]^{-1} \quad [29]$$

In fig. 5 the curves of the expressions [28], $[29]^+$ and $[29]^-$ have been plotted in function of $S/\mu$, after dividing them by $\mu$; in this case $\lambda$ has been taken equal to $\mu$. The possible velocities for progressive waves are given by the shaded areas, and we see that if the whole range of the lowest root, from $A$ to $B$, is valid, the second and the third roots would only be permissible respectively for the values of $S/\mu$ on the left of $D$ and $F$.

In the case $\lambda/\mu = \infty$ already considered, the only real solution $\xi_1$ of fig. 6 remains below the limit prescribed by [28], which is $2\mu$. The condition [29] are then irrelevant in the range $-2 < \dfrac{S}{\mu} < 2$ of fig. 6.

*Appendix III.*

The fact that the roots $\alpha^2_i$ of equ. [7] are real or complex has an important incidence on the character of the surface wave; in the latter case the classical exponential decay of the Rayleigh wave amplitude with depth must be replaced by an exponentially damped sinusoidal oscillation. This seems to have been first met by Stoneley ([6]) in a study of surface waves in an unstressed cubic crystal, and more generally to have been recognized as a generalization of the Rayleigh type of surface wave by Synge ([7]).

The deciding criterium is the sign of the discriminant of [7]:

$$D = [(\varrho c^2-A_{32})A_{13} + (\varrho c^2 - B_{11})B_{22} + A^2{}_{12}]^2 - 4\, B_{22}\, A_{13}\, (\varrho c^2 - B_{11})\, (\varrho c^2 - A_{32})$$

which also writes:

$$D = [(\varrho c^2-A_{32})A_{13}-(\varrho c^2-B_{11})B_{22}+A^2{}_{12}]^2 + 4\, A^2{}_{12}\, B_{22}\, (\varrho c^2 - B_{11}). \qquad [30]$$

The value of $\varrho c^2$ should be extracted from [13] to allow a discussion of the influence of the elastic constants and also of the initial stress $S$, contained in the $A's$. It seems simpler to discuss the boundary case $D = O$, by extracting $\varrho c^2$ from this equation and substituting it in [13]. But the complication of the formulas is such that the general procedure seems unrewarding.

The special case of a naturally isotropic medium, (i. e. the anisotropy is only induced by the initial stress $S$), is simples enough; the discriminant $D$ can be reduce to the form: (see form. [15]),

$$D = (\varrho c^2)^2\,(\lambda+\mu+S/2)^2 + 2\,\varrho c^2 \{4\,(\lambda+\mu + S/2)\,[(\lambda+\mu)\,(\lambda+2\mu) + S^2/16] - (\lambda+\mu)\,(\lambda+2\mu)\,S\} + S^2\,[(\lambda+\mu)\,(\lambda+2\mu) + S^2/16]. \qquad [31]$$

One sees that if $|S| < 2\,(\lambda+\mu)$, $D$ is certainly positive, and thus the decay with depth of the Rayleigh wave remains purely exponential. It is interesting to note in case of further discussion that the discriminant of $D$ takes itself a simple form:

$$(\lambda+\mu)^2(\lambda+2\mu)[16(\lambda+\mu+S/2)(\lambda+\mu)(\lambda+2\mu) + S^2\,(\lambda+\mu+S/2) + S^2\,(\lambda+2\mu)]. \qquad [32]$$

Let us close here by noting that for an unstressed ($S = O$) medium with a vertical axis of tranverse isotropy, Synge ([7]) has shown that for one of one of the velocities $c$, corresponding to a spheroidal sheet of the slowness surface, the Rayleigh wave retains its classical exponential profile.

*Appendix IV.*

Let us designate by the suffix 3 the direction perpendicular to the strata of a layered formation. It is also the axis of a symmetry of revolution in the elastic properties, so that the relations between normal stress and displacement in the absence of initial stresses are given by:

$$\begin{aligned} \sigma_{11} &= C_{11}\,u_{1,1} + C_{12}\,u_{2,2} + C_{13}\,u_{3,3} \\ \sigma_{22} &= C_{12}\,u_{1,1} + C_{11}\,u_{2,2} + C_{13}\,u_{3,3} \\ \sigma_{33} &= C_{13}\,u_{1,1} + C_{13}\,u_{2,2} + C_{33}\,u_{3,3} \end{aligned} \qquad [33]$$

If an effort is exerced on a specimen in the direction 3, without stress in direction 1 and 2 ($\sigma_{11} = \sigma_{22} = O$), then

$$E_{\text{perp}} = \sigma_{33}/u_{3,3} = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{11} & C_{13} \\ C_{13} & C_{13} & C_{33} \end{vmatrix} : \begin{vmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{11} & 0 \\ C_{13} & C_{13} & 1 \end{vmatrix} = C_{33} - 2\,C^2{}_{13}\,/\,(C_{11} + C_{12}) \qquad [34]$$

$$m_{\text{perp}} = -\,(u_{3,3}\,/\,u_{1,1}) = -\,(u_{3,3}\,/\,u_{2,2}) = \begin{vmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{11} & 0 \\ C_{13} & C_{13} & 1 \end{vmatrix} : \begin{vmatrix} 0 & C_{12} & C_{13} \\ 0 & C_{11} & C_{13} \\ 1 & C_{13} & C_{33} \end{vmatrix} = (C_{11} + C_{12})\,/\,C_{13}\,. \qquad [35]$$

If the effort is exerced in the direction 1 (or 2), without stress in the direction 3 and in the direction 2 (or 1), thus for example $\sigma_{11} \neq O$ $\sigma_{22} = \sigma_{33} = O$, one has:

$$E_{\text{par}} = \sigma_{11}\,/\,u_{1,1} = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{11} & C_{13} \\ C_{13} & C_{13} & C_{33} \end{vmatrix} : \begin{vmatrix} 1 & C_{12} & C_{13} \\ 0 & C_{11} & C_{13} \\ 0 & C_{13} & C_{33} \end{vmatrix} = C_{11} - [C_{12}\,(C_{12}\,C_{33} - C^2{}_{13}) + C^2{}_{13}\,(C_{11} - C_{12})]\,/\,(C_{11}\,C_{33} - C^2{}_{13}) \qquad [36]$$

$$m'_{\text{par}} = -\,u_{1,1}\,/\,u_{3,3} = -\begin{vmatrix} 1 & C_{12} & C_{13} \\ 0 & C_{11} & C_{13} \\ 0 & C_{13} & C_{33} \end{vmatrix} : \begin{vmatrix} C_{11} & C_{12} & 1 \\ C_{12} & C_{11} & 0 \\ C_{13} & C_{13} & 0 \end{vmatrix} = (C_{11}\,C_{33} - C^2{}_{13})\,/\,(C_{11} - C_{12})\,C_{13} \qquad [37]$$

$$m''_{par} = - u_{1,1} / u_{2,2} =$$

$$= - \begin{vmatrix} 1 & C_{12} & C_{13} \\ 0 & C_{11} & C_{13} \\ 0 & C_{13} & C_{33} \end{vmatrix} : \begin{vmatrix} C_{11} & 1 & C_{13} \\ C_{12} & 0 & C_{13} \\ C_{13} & 0 & C_{33} \end{vmatrix} =$$

$$= (C_{11} C_{33} - C^2_{13}) / (C_{12} C_{33} - C^2_{13}) \quad [38]$$

Had we taken $\sigma_{22} \neq O$, and $\sigma_{11} = \sigma_{33} = O$, we should find the same values for:

$$E_{par} = \sigma_{22} / u_{2,2} \ , \ m_{par} = - u_{2,2} / u_{3,3}$$

and $\quad m''_{par} = - u_{2,2} / u_{1,1}$ .

The following expression emerge:

$$E_{perp} = C_{33} - 2\, C_{13} / m_{perp} \quad [39]$$

$$E_{par} = C_{11} - C_{13} / m'_{par} - C_{12} / m''_{par} \quad [40]$$

and its is also clear that there is a relation between the five $E's$ and $m's$ for they can be expressed in function of the four coefficients $C_{11}$, $C_{12}$, $C_{13}$ and $C_{33}$.

Their expressions in terms of the $B's$ used in this text are found, in the case $S = O$,

*a*) if the axis 3 is vertical, by writing: $C_{33} = B_{22}$, $C_{13} = B_{12}$, $C_{11} = B_{11}$ (and $C_{12} = B_{13}$);

*b*) if the axis 3 is horizontal, in the direction of the wave by putting $C_{33} = B_{11}$ $C_{13} = B_{12}$, $C_{11} = B_{22}$, (and $C_{12} = B_{23}$).

In both cases $B_{33} = 2\, C_{44}$ does not appear, and this constant being here independant, it cannot be determined by the considered pure tension test, unlike thus the isotropic case.

## RIASSUNTO

*Si deriva l'equazione di velocità per le onde superficiali del tipo Rayleigh nel caso di un mezzo idealizzato di uniforme anisotropia bi-dimensionale; si presume che le costanti elastiche siano ovunque le stesse, ma differiscano a seconda che si tratti di direzioni orizzontali e verticali; è inclusa, inoltre, la possibile interferenza di sollecitazioni iniziali uniformi (tensioni orizzontali o compressioni).*

*Si debbono, quindi, considerare cinque costanti elastiche indipendenti invece di quattro, al fine di studiare la propagazione di un'onda superficiale tridimensionale, poichè l'effetto di una sollecitazione trasversale iniziale è quello di distruggere la simmetria delle costanti elastiche trasverse.*

*Questo genere speciale di anisotropia potrebbe apparire in media in determinate porzioni della superficie terrestre, e lo scopo del presente lavoro è di determinare il suo effetto specifico sulla velocità delle onde di Rayleigh, confrontato con il caso isotropico.*

*È importante introdurre l'anisotropia puramente elastica, quando si studia l'influenza delle sollecitazioni iniziali, dato che spesso un effetto di tali sollecitazioni è quello di modificare, a lungo andare, le caratteristiche elastiche del materiale.*

*Sebbene sia dimostrato che, generalmente, le sollecitazioni iniziali influiscano direttamente molto poco sul valore della velocità dell'onda, la modificazione apportata alle costanti elastiche, potrebbe avere una significativa influenza.*

*Nel caso presente è difficile un'applicazione speciale ai problemi geofisici, per la mancanza di dati sperimentali sulle costanti elastiche nelle formazioni geologiche anisotrope, per non parlare, poi, delle loro modificazioni sotto forti sollecitazioni elastiche iniziali.*

*Viene discusso il manifestarsi di instabilità statica dovuto a sollecitazioni di compressione; viene considerata, in particolare, la sua possibile incidenza su problemi orogenici di scarso rilievo.*

## ABSTRACT

*The velocity equation for the surface waves of the Rayleigh type is derived in the case of an idealized medium of uniform two-dimensional anisotropy: the elastic constants are assumed to be everywhere the same but different as far as vertical and horizontal directions are concerned, and moreover the possible inference of uniform initial stresses, (horizontal tensions or compressions) is included. There are then five independant elastic constants instead of four to be taken in account, in order to study the propagation of a two-dimensional surface-wave, the effect of a transverse initial stress being to destroy the symmetry of the cross elastic constants.*

*This special kind of anisotropy may appear on an average in certain portions of the earth surface, and the purpose of the paper is to determine its specific effect on the velocity of the Rayleigh wave, as compared to the isotropic case.*

*It is important to include purely elastic anisotropy when investigating the influence of initial stresses, for often one effect of such stresses is to modify on the long run the elastic characteristics of the material.*

*Although it is shown that initial stresses affect usually very little in a direct way the value of the wave velocity, the modification brought to the elastic constants could have a significant influence.*

*A special application to geophysical problems is difficult in the present case, because of the lack of experimental data on the elastic constants in anisotropic geological formations, not to mention their modification under large initial stresses.*

*The occurrence of static instability under compressive stresses is discussed; in particular its possible incidence on small scale orogenic processes is considered.*

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