On the General Theory of Thermal and Gravitational Excitation of Atmospheric Oscillations

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Introduction – In another paper (1), that we here indicate as *I*, we have considered the effects of the height variation of the Earth's radius vector R and of the gravity acceleration g on atmospheric tides. We have reconsidered the classical theory on the presumption that, since the atmospheric tides are a resonance phenomenon, the effects of such height variations may be rather considerable. On the other hand, the tidal phenomenon extends to a rather high atmospheric level; for example the values of R and g at a height of 100 km above the ground are respectively 1.5% greater and 3% less than the corresponding values at ground level.

In *I* we have worked only the theory of the gravitational oscillations and have quantitatively considered a very simple case of an atmosphere whose scale height H does not depend upon the height. Although this model is very idealized, however, it is able to give some interesting information on the effective importance of the corrective terms introduced in the classical equations: in some respects (for example in the amplitude of the pressure oscillation and the height of the eventual nodal points), the difference between the generalized and the classical results may be considerable, both as a percentage and as actual values.

In the present paper, we consider the theory for both gravitational and thermal oscillations in our more general formulation and study some numerical cases representing the effective physical conditions of the atmosphere.

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1. – *The mathematical theory.*

1.1. – The notation is that used by previous authors ([2,3,4]):

z, ϑ, Φ	=	height on the ground, colatitude and longitude of a point in the atmosphere
R	=	radius of the earth
$g(z) = GR^2/(R+z)^2$	=	acceleration of gravity; G its value at ground
ω	=	angular velocity of the earth
u, v, w	=	southward, eastward and upward components of air velocity at (z, ϑ, Φ)
p_0, ϱ_0, T_0	=	static pressure, density and temperature (functions of z only)
p, ϱ, T	=	departures of pressure, density and temperature from static values p_0, ϱ_0, T_0
p, ϱ, T	=	total pressure, density and temperature
$\mathbf{R}$	=	gas constant for unit mass of gas
γ	=	the ratio of specific heats c_p and c_v, at constant pressure and at constant volume
$H = \mathbf{R}T_0/g$	=	scale height of atmosphere (function of z only)
$2\pi/\sigma$	=	period of tidal oscillation
f	=	$\sigma/(2\omega)$
$\Omega(z, \vartheta, \Phi)$	=	tide-producing potential

The only noticeable differences from the previous authors lie in writing the radius rector as $R + z$ and the acceleration gravity as $g(z) = GR^2/(R+z)^2$. In our new hypotheses the divergence of velocity, χ, assumes the following expression, containing a corrective term $2w/(R+z)$.

$$\chi(z,\vartheta,\Phi) = \frac{1}{(R+z)\,\text{sen}\,\vartheta}\,\frac{\partial}{\partial\vartheta}(u\,\text{sen}\,\vartheta) + \frac{1}{(R+z)\,\text{sen}\,\vartheta}\,\frac{\partial v}{\partial\Phi} + \frac{\partial w}{\partial z} + 2\,\frac{w}{R+z}\,. \tag{1}$$

1.2. – If we use the expression [1] of χ, and indicate as $Gc_vQ/\mathbf{R}$ the amount of heat received per unit time per unit mass by an element of gas of fixed mass, the fundamental equations, are the equation of state

$$p = \mathbf{R}\,T\,\varrho \tag{2}$$

the continuity equation

$$\frac{D\bar{\varrho}}{Dt} = -\bar{\varrho}\,\chi \qquad [3]$$

and the equation expressing the thermal balance

$$\bar{\varrho}\,Q\,G = \gamma\,\bar{p}\,\chi + \frac{D\bar{p}}{Dt} \qquad [4]$$

If: (i) we remember that $\bar{p} = p_0 + p$, $\bar{\varrho} = \varrho_0 + \varrho$, $\bar{T} = T_0 + T$ and neglect the second order terms of the quantities p, ϱ, T, χ; (ii) Q is of the same order of χ; (iii) we assume a time factor $e^{i\sigma t}$; (iv) the ellipticity of the Earth is negligible; (v) the vertical acceleration is negligible; then the above equations [3] and [4] may be written as follows:

$$i\,\sigma\,\varrho + w\,\frac{d\varrho_0}{d\,z} + \varrho_0\,\chi = 0 \qquad [5]$$

$$G\,\varrho_0\,Q = i\,\sigma\,p - w\,g\,\varrho_0 + \gamma\,p_0\,\chi\;; \qquad [6]$$

this last equation may assume the different form

$$G\,Q/R = i\,\sigma\,T + w\,\frac{d\,T_0}{d\,z} + (\gamma - 1)\,T_0\,\chi\,. \qquad [7]$$

In the same hypotheses, the equations of motion assume the form

$$i\,\sigma\,u - 2\,\omega v\,\cos\vartheta = -\frac{1}{(R+z)}\,\frac{\partial}{\partial\,\vartheta}\left(\frac{p}{\varrho_0} + \Omega\right) \qquad [8]$$

$$i\,\sigma\,v + 2\,\omega u\,\cos\vartheta = -\frac{1}{(R+z)\,\mathrm{sen}\,\vartheta}\,\frac{\partial}{\partial\,\Phi}\left(\frac{p}{\varrho_0} + \Omega\right) \qquad [9]$$

$$\frac{\partial\,p}{\partial\,z} = -g\,\varrho - \varrho_0\,\frac{\partial\,\Omega}{\partial\,z}\,. \qquad [10]$$

Concerning the tide-producing potential Ω, its dependence on height z may be written (5) in the form $\Omega \propto (R+z)^2/R^2$.

If equation [6] is differentiated and use made of equations [5] and [10], we can write

$$\frac{\partial\,w}{\partial\,z} = (1-\gamma)\,\chi + \gamma\,H\,\frac{\partial\,\chi}{\partial\,z} - \frac{G}{g\,\varrho_0}\,\frac{\partial}{\partial\,z}\,(\varrho_0\,Q) + 2\,\frac{w}{R+z} + \qquad [11]$$

$$-\frac{i\,\sigma}{g}\,\frac{\partial\,\Omega}{\partial\,z}\,.$$

If the expressions of u and v deduced from [8] and [9] are now substituted in [1], we obtain the fundamental equation

$$\chi \frac{\partial w}{\partial z} - 2 \frac{w}{R+z} = \frac{i\sigma}{4\omega^2 (R+z)^2} F\left(\frac{p}{\varrho_0} + \Omega\right) \qquad [12]$$

where F is the differential operator

$$F = \frac{1}{\operatorname{sen}\vartheta} \frac{\partial}{\partial\vartheta} \left[\frac{\operatorname{sen}\vartheta}{f^2 - \cos^2\vartheta} \left(\frac{\partial}{\partial\vartheta} - i \frac{\operatorname{cotg}\vartheta}{f} \frac{\partial}{\partial\Phi}\right)\right] +$$

$$+ \frac{1}{f^2 - \cos^2\vartheta} \left[\frac{i \operatorname{cotg}\vartheta}{f} \frac{\partial^2}{\partial\vartheta\partial\Phi} + \frac{1}{\operatorname{sen}^2\vartheta} \frac{\partial^2}{\partial\Phi^2}\right].$$

We now differentiate equations [11] and [12] with respect to z, then add the equations that we obtain and rearrange their terms, neglecting those containing $\frac{w}{(R+z)^2}$, $\frac{\partial^2 g}{\partial z^2}$, $\frac{\partial g}{\partial z}$, $\frac{\partial\Omega}{\partial z}$, $\frac{\partial^2\Omega}{\partial z^2}$, which are of the second order with respect to $1/(R+z)$ or $1/R$.

If we make appropriate substitutions, the resultant equation may be written as follows

$$\gamma H \frac{\partial^2\chi}{\partial z^2} + \gamma\left(\frac{dH}{dz} - 1\right)\frac{\partial\chi}{\partial z} - \frac{2}{R+z}\left[-2\frac{\partial w}{\partial z} + \right.$$

$$\left. + \frac{i\sigma}{4\omega^2 (R+z)^2} F\left(\frac{p}{\varrho_0} + \Omega\right)\right] + \frac{\partial}{\partial z}\left\{\frac{(R+z)^2}{R^2}\left[-\frac{\partial Q}{\partial z} + \right.\right. \qquad [13]$$

$$\left.\left. + \frac{Q}{H}\left(1 + \frac{dH}{dz}\right)\right]\right\} + \frac{i\sigma}{4\omega^2 (R+z)^2} F\left[\frac{\partial}{\partial z}\left(\frac{p}{\varrho_0} + \Omega\right)\right] = 0 .$$

The main difference in the above equation with respect to that obtained by Wilkes, is the presence of a term which containes a factor $2/(R+z)$.

By making use of equations [6] and [12] in the equation [13] we may thus write the equation

$$\gamma H \frac{\partial^2\chi}{\partial z^2} + \gamma\left(\frac{dH}{dz} + 1\right)\frac{\partial\chi}{\partial z} + \frac{1}{4\omega^2 (R+z)^2} F\left[g\left(1 - \gamma - \gamma\frac{dH}{dz} + \right.\right.$$

$$\left.\left. + 2\frac{H}{R+z}\right)\chi\right] - \frac{2}{R+z}\left[\chi - 3\frac{\partial w}{\partial z} - 2\frac{w}{R+z}\right] +$$

$$+ \frac{\partial}{\partial z}\left\{\frac{(R+z)^2}{R^2}\left[-\frac{\partial Q}{\partial z} + \frac{Q}{H}\left(1 + \frac{dH}{dz}\right)\right]\right\} + \qquad [14]$$

$$+ \frac{1}{4\omega^2 (R+z)^2} F\left[G Q \left(\frac{1}{H} + \frac{1}{H}\frac{dH}{dz} - \frac{2}{R+z}\right)\right] = 0 .$$

We at once see that if $Q = 0$ and with the same approximations, our equations [13] and [14] are equivalent to equation [11] of *I*. Concerning the supplementary terms with respect to the corresponding equation of Wilkes, we see that in our approximation we can neglect the term in w which is of the second order in $\frac{1}{R+z}$; on the order hand, the term in $\frac{\partial w}{\partial z}$ may be eliminated if appropriate use of equation [11] is made.

1.3. – We now put

$$\left.\begin{aligned} \chi\,(z, \vartheta, \Phi) &= \chi\,(z)\,\psi\,(\vartheta, \Phi) \\ Q\,(z, \vartheta, \Phi) &= q\,(z)\,\psi\,(\vartheta, \Phi) \\ \Omega\,(z, \vartheta, \Phi) &= \Omega\,(z)\,\psi\,(\vartheta, \Phi) \\ \tau &= \frac{1}{R} \end{aligned}\right\} \qquad [15]$$

Thus in the equation [14] we can separate the variables z and (ϑ, Φ); if we retain only the terms of order zero and one with respect to the parамet τ, we obtain the final equations in χ and ψ

$$F\,(\psi) + 4\,\frac{R^2\,\omega^2}{G\,h}\,\psi = 0 \qquad [16]$$

and

$$\gamma H \frac{d^2 \chi}{d z^2} + \gamma \left(\frac{d H}{d z} - 1 + 6\,\tau H\right) \frac{d \chi}{d z} + \left\{\gamma \frac{d H}{d z} + \right.$$
$$\left. + \gamma - 1 + 2\,\tau \left[2\,h - \gamma H - 3\,\gamma h - 2\,z \left(\gamma \frac{d H}{dz} + \gamma - 1\right)\right]\right\} \frac{\chi}{h} +$$
$$- \frac{q}{H h}\left(1 + \frac{d H}{d z}\right) + 2\,\tau \frac{q}{H h}\left[H + (z + 4\,h)\left(1 + \frac{d H}{d z}\right)\right] +$$
$$- 8\,\tau \frac{d q}{d z} + \left\{\frac{d}{d z}\left[\frac{q}{H}\left(1 + \frac{d H}{d z}\right)\right] - \frac{d^2 q}{d z^2}\right\}(1 + 2\,\tau z) = 0 \qquad [17]$$

where h is a constant of separation of the variables.

The equation [16] is the same considered by Wilkes and the other authors who have elaborated the tidal theory.

Its solution can be written (³) in the form

$$\psi\,(\vartheta, \Phi) = \Theta\,(\vartheta)\,e^{is\Phi} \qquad [18]$$

where s is an integer, provided that the separation constant h has certain discrete values h_r $(r = 1, 2, \ldots)$ depending on s and σ. The equation [17] instead obviously coincides with the classical equation if we put $\tau = 0$.

Now the new expressions of the velocity components u, v, w, and of p/ϱ_0, which reduce to the classical expressions as we put $\tau = 0$, are the following

$$\frac{p}{\varrho_0} = -\Omega(1+6h\tau) - i\,\frac{Gh}{\sigma}\left[\gamma\left(-\chi + H\frac{d\chi}{dz}\right) + \frac{q}{H}\left(1 + \frac{dH}{dz}\right) + \right.$$
$$\left. -\frac{dq}{dz} - 2i\,\frac{Gh}{\sigma}\,\tau\right\}\gamma\left[(2H - 2h - z)\chi + H(z+2h)\frac{d\chi}{dz}\right] +$$
$$+\frac{q}{H}\left[2h - 3H + 2z + 2(h+z)\frac{dH}{dz}\right] - 2(z+h)\frac{dq}{dz}\Big\}\,\Theta(\vartheta)\,e^{is\Phi}\,e^{i\sigma t}. \qquad [19]$$

$$u = \frac{i\sigma}{4\omega^2(R+z)(f^2-\cos^2\vartheta)}\left(\frac{p}{\varrho_0} + \Omega\right)\left(\frac{d}{d\vartheta} + \frac{s\cot g\,\vartheta}{f}\right)\Theta(\vartheta)\,e^{is\Phi}\,e^{i\sigma t} \qquad [20]$$

$$v = -\frac{\sigma}{4\omega^2(R+z)(f^2-\cos^2\vartheta)}\left(\frac{p}{\varrho_0}+\Omega\right)\left(\frac{\cos\vartheta}{f}\frac{d}{d\vartheta} + \frac{s}{\operatorname{sen}\vartheta}\right)\Theta(\vartheta)\;e^{is\Phi}\;e^{i\sigma t} \qquad [21]$$

$$w = -\frac{i\sigma\Omega}{G}(1+2\tau z+6\tau h) + h\left[\gamma\left(\frac{H}{h}-1\right)\chi + \gamma H\frac{d\chi}{dz} + \right.$$
$$+q\left(\frac{1}{H} + \frac{1}{H}\frac{dH}{dz} - \frac{1}{h}\right) - \left.\frac{dq}{dz}\right] + 2h\tau\Big\{2\gamma\left[(H-h-z)\chi + \right.$$
$$\left.+H(z+h)\frac{d\chi}{dz}\right] + \frac{q}{H}\left[2h - 3H + z\left(3-\frac{H}{h}\right) + (2h+3z)\frac{dH}{dz}\right] +$$
$$-(2h+3z)\frac{dq}{dz}\Big\}\;\Theta(\vartheta)\;e^{is\Phi}\;e^{i\sigma t}. \qquad [22]$$

If we write Θ^s_r and χ^s_r the expressions of $\Theta(\vartheta)$ and $\chi(z)$ corresponding to the value h_r of h, the general solution χ of the equation [14], including the time factor, is, in form of a double series

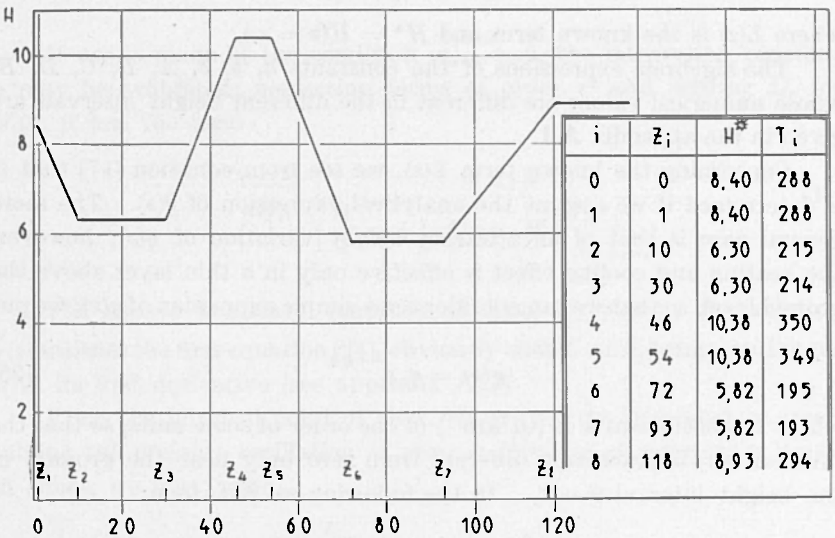
$$\chi(z,\vartheta,\Phi) = \Sigma_{rs}\,A_{rs}\,\chi^s_r(z)\,\Theta^s_r(\vartheta)\,e^{\,i(s\Phi+\sigma t)} \qquad [23]$$

where A_{rs} are constant coefficients.

1.4. – The tidal equation [17] is of the second order, so that the necessary boundary conditions are two: one obvious condition is that the vertical velocity component $w(z)$ vanishes on the ground; the second condition is that either that the total energy in the tidal wave has a finite value (6) or that at high level, just below the region where the energy is finally absorbed, the direction of propagation must be upwards [3]; the physical meaning of this last condition (radiation condition) is that, in effect, the upper atmosphere absorbs the tidal energy flux from the ground, without reflecting it.

2. – *The solution of the tidal equation*

2.1. – We shall study separately both the cases of gravitational and thermal oscillations, assuming a height variation of the scale height H, indicated in fig. 1. We do not consider other different height variations of H, because many results on this aspect of the problem have



i	z_i	H^*	T_i
0	0	8.40	288
1	1	8,40	288
2	10	6.30	215
3	30	6.30	214
4	46	10,38	350
5	54	10,38	349
6	72	5,82	195
7	93	5,82	193
8	118	8,93	294

Fig. 1. – Height variation of the scale height H. The quantities H and z are in km; the temperatures T_i are in °K.

already been given by Jacchia and Kopal [7]. The model of the atmosphere of fig. 1 is the linear approximation of the most satisfactory profile obtained by these authors. An interesting point is that our assumptions H = a constant or H = a linear function of z are not the same as T = a constant and T = a linear function of z, because $H = RT_0/g$, so that, for example the hypotheses H = a constant is equivalent to the other $T_0 \propto g = G(1 - 2\tau z)$, *i. e.* to a very slight linear height variation of the temperature.

We study the resonance spectrum of the atmosphere at ground level and the height variation of the pressure oscillation for the cases $\tau = 0$ and $\tau \neq 0$.

In the above particular model of the atmosphere, the fundamental equation [17] may be written in the form

$$\left.\begin{aligned} &\frac{d^2\chi}{dz^2} + a\frac{d\chi}{dz} + (b + cz)\,\chi = L(z) \quad \text{for } H = \text{a constant} \\ &\frac{d^2\chi}{dz^2} + \frac{A+B\,(z-z_i)}{1+C\,(z-z_i)}\,\frac{d\chi}{dz} + \\ &\quad + \frac{D+Ez}{1+C\,(z-z_i)}\,\chi = L(z) \quad \text{for } H = H^* - \beta\,(z-z_i) \end{aligned}\right\} \qquad [24]$$

where $L(z)$ is the known term and $H^* = H(z = z_i)$.

The algebraic expressions of the constants a, b, c, A, B, C, D, E, whose numerical values are different in the different height intervals are given in the appendix **A.1**.

Concerning the known term $L(z)$, we see from equation [17] that it is determined if we assume the analytical expression of $q(z)$. The more general case is that of an arbitrary height variation of $q(z)$; however, the heating and cooling effect is effective only in a thin layer above the ground level, so that we can consider some simple expression of $q(z)$; we put

$$q\,(z) = q_0\, e^{-kz} \qquad [25]$$

where the coefficient k is (in km^{-1}) of the order of some units, so that the function $q(z)$ is practically different from zero only near the ground, in the height interval $0 - z_1$. In the following we put, then:

$$\left.\begin{aligned} &q\,(z) = q_0\, e^{-kz} \quad \text{for } z < z_1 \ (H = \text{a constant}) \\ &q\,(z) = 0 \quad \text{for } z \geqslant z_1 \end{aligned}\right\} \qquad [26]$$

If use of [26] is made, we obtain for $z < z_1$ the following espression of the known term $L(z)$

$$L\,(z) = q_0\,(\mu + \nu z)\, e^{-kz} \qquad [27]$$

where we have put

$$\left.\begin{aligned} &\mu = \frac{1}{\gamma H}\left[\frac{k}{H} + k^2 + \frac{1}{hH} - 2\tau\left(4\,k + \frac{4}{H} + \frac{1}{h}\right)\right] \\ &\nu = 2\tau\frac{1}{\gamma H}\left(\frac{k}{H} + k^2 - \frac{1}{hH}\right). \end{aligned}\right\} \qquad [28]$$

For atmospheric levels $z \geqslant z_1$, on the other hand, we put $L(z) = 0$. A solution of equation [24] has to be found for which, corresponding to each height z_i, both $\chi(z)$ and $\frac{d\chi}{dz}$ must be continuous.

2.2. – *The case of purely gravitational oscillations.* The solution of equation [24] for the particular case $L(z) = 0$ can be written in the form

$$\chi(z) = a\,\chi^*(z) \qquad [29]$$

where $\chi^*(z)$ is the particular solution for which

$$\chi^*(z_s) = 1\,. \qquad [30]$$

If use is made of the condition $w_{z=0} = 0$, the integration constant a may be evaluated: neglecting terms of order τ^2 and writing Ω_0 for $\Omega(0)$, it has the form

$$a = \frac{i\sigma\Omega_0}{\gamma G h}\,\frac{1+2\tau h}{\left(\frac{H}{h}-1\right)\chi^*_{z=0} + H\left(\frac{d\chi^*}{dz}\right)_{z=0}}\,. \qquad [31]$$

The second boundary condition is that, for $z \geqslant z_s$, the solution $\chi(z)$ satisfies the first equation [24], obviously with $L = 0$, being continuous with its first derivative (see appendix **A.2**).

From the physical point of view, we are chiefly interested in ascertaining the pressure oscillation at ground level (or at some other level); in effect, we may write

$$p = -p_0\,\frac{\Omega_0}{Gh}\,\frac{1+2\tau h}{\left[\left(\frac{H}{h}-1\right)\chi^*_0 + H\left(\frac{d\chi^*}{dz}\right)_0\right]}\,\chi^*_0 = P e^{i\varphi} = -\Omega_0\varrho_0\lambda \qquad [32]$$

where the quantity λ is the "resonance magnification". We have calculated the resonance spectrum as a function of the parameter h, for $\tau = 0$ and $\tau \neq 0$; precisely, in fig. 2, we show the diagrams of the quantity $10\log_{10}|p|$ and of the phase-angle φ of p; for the tide-producing potential at the ground, we have assumed $\Omega_0 = 10^4$ C.G.S. As an effect of considering the height variation of g and of the radius vector, the resonance spectrum exhibits a slight shift to the right; the resonance occurs for $h = h_1 = 7.955$ when $\tau = 0$ and for

$h = h_2 = 7.843$, instead, when $\tau \neq 0$. The corresponding variation of the resonance period is (7) from 11.96 to 12.01 hours, *i. e.* about three minutes.

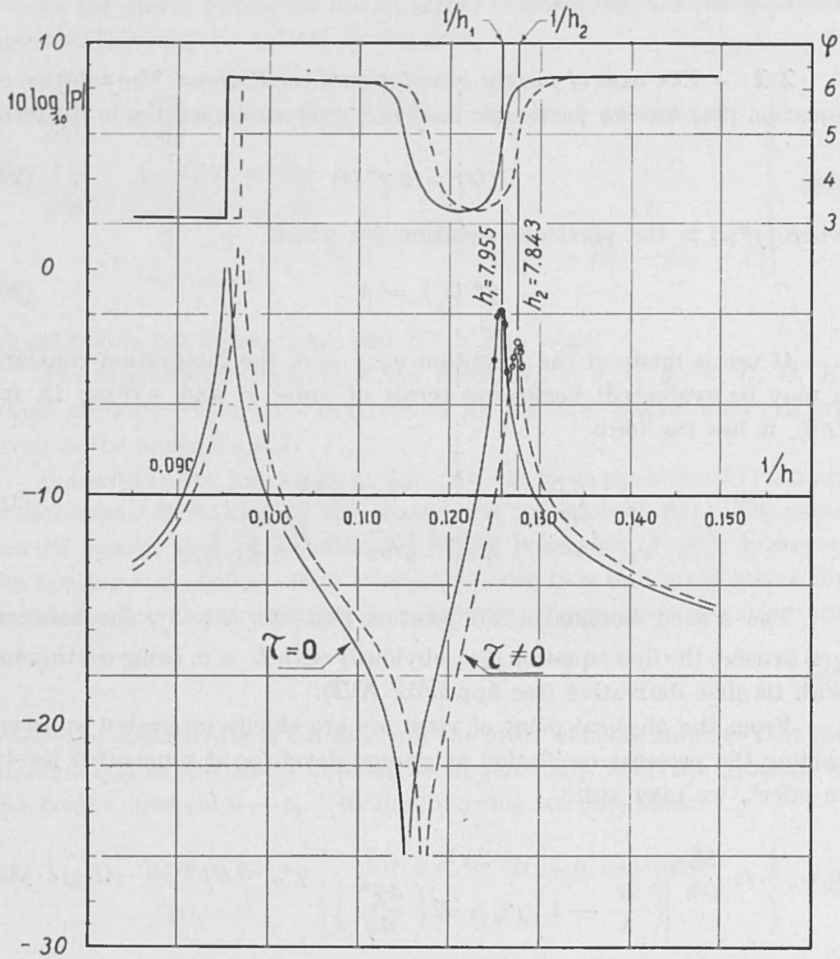


Fig. 2. – The resonance spectra at ground level. The pressure oscillation p is in mm$_{Hg}$; the phase-angle φ in radiants; the parameter h in km.

The second resonance period also slightly increases from 10.88 to 10.94 hours, *i. e.* about three minutes.

The resonance amplitude of p at ground level for $\tau \neq 0$ is approximately 1.5 times smaller than for $\tau = 0$. Such a remarkable difference is not surprising, for the very reason that one is concerned with a resonance phenomenon; on the other hand, also the numerical results of Jacchia and Kopal clearly indicate that even small differences in the model atmosphere give remarkable differences in the resonance magnification. A more important feature, of which we must take account, is the following: for values of h, greater than h_1 and smaller than h_2, the phase angles φ in the two cases $\tau = 0$ or $\tau \neq 0$ are very different in consequence, for a given model atmosphere, we may obtain opposite signs of p/p_0 in either case. This fact may have important effects in the practical application of tidal theory, *i. e.* in the interpretation of experimental data.

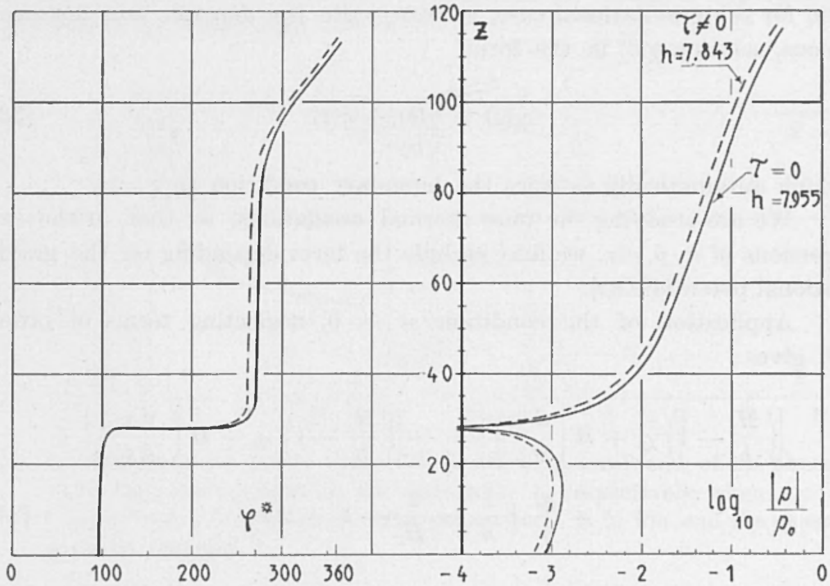


Fig. 3. – The height variation of $\log_{10} \frac{|p|}{p_0}$ and φ^*. The phase-angle φ^* is in degrees; the height z in km.

Concerning the height variation of the resonance magnification, we see from fig. 3 that the ratio of the amplitudes in the two cases is approximately constant. With regard to the phase φ^*, the greatest difference between the two cases is about 10°; however, notwithstanting such a slight difference, there is the important feature that the phase-angle φ^* has the value 270° at considerably different heights: at about 35 km if $\tau = 0$

and at about 80 km if $\tau \neq 0$. Such a feature may be more clearly seen if one draws the diagram of the real part of $\frac{p}{p_0}$ (which is the effective value of $\frac{p}{p_0}$ at time $t = 0$). In other words the nodal point of the pressure oscillation occurs at remarkably different heights, in the two cases.

2.3. – *The case of purely thermal oscillations.* In $\eta(z)$ is a solution, in the interval $0 \leqslant z \leqslant z_1$, of the equation [24] satisfying the conditions $\eta(z) = \left(\frac{d\eta}{dz}\right) = 0$ for $z \geqslant z_1$ (appendix **A**.3) and $\bar{\chi}(z)$ is the solution [29] for the gravitational case, we can write the general, non homogeneous, solution $\chi(z)$ in the form

$$\chi(z) = \bar{\chi}(z) + \eta(z) \tag{33}$$

which automatically satisfies the boundary condition at $z = z_s$.

We are studying the pure thermal oscillations, so that in the expressions of w, p, etc. we may esclude the term depending on the gravitational potential $\Omega(z)$.

Application of the condition $w = 0$, neglecting terms of order τ^2, gives

$$\left[\left(\frac{H}{h} - 1\right)\bar{\chi}_0 + H\left(\frac{d\bar{\chi}}{dz}\right)_0\right] = -\left[\left(\frac{H}{h} - 1\right)\eta_0 + H\left(\frac{d\eta}{dz}\right)_0\right] + \\ + \frac{q_0}{\gamma}\left(\frac{1}{h} - \frac{1}{H} - k + 2\tau\right). \tag{34}$$

If use is now made of [31] and [29], we may replace the left hand term of the above expression by the term $\frac{i\sigma\Omega_0}{\gamma G h}(1+2\tau h)$, so that we finally obtain the following expression connecting q_0 and Ω_0

$$q_0 = i\sigma\Omega_0 \frac{1 + 2\tau h}{Gh\left[\left(\frac{1}{h} - \frac{1}{H} - k + 2\tau\right) - \gamma\left(\frac{H}{h} - 1\right)M_0 - \gamma H\left(\frac{dM}{dz}\right)_0\right]} \tag{35}$$

From the above expression we may draw the graphs of q_0 as a function of the parameter h; the function $M(z)$ is defined in Appendix **A.3.**

Concerning the pressure oscillation p' at ground level, if use is made of [32] and [35] we may write

$$p' = p\left\{1 - \frac{(1-\gamma H M_0)(1+2\tau h)}{\lambda h\left[\frac{1}{h} - \frac{1}{H} - k + 2\tau - \gamma\left(\frac{H}{h} - 1\right)M_0 - \gamma H\left(\frac{dM}{dz}\right)_0\right]}\right\} = -\Omega_0 \varrho_0 \lambda' \quad [36]$$

where p and λ are those given in [32] and the quantity λ' is the new " resonance magnification " factor, which has to be compared with the corresponding λ we have obtained in **2**.**2**. By easy calculation, we may see that the above results are the same as those of Wilkes, if appropriate sustitutions are made with $\tau = 0$.

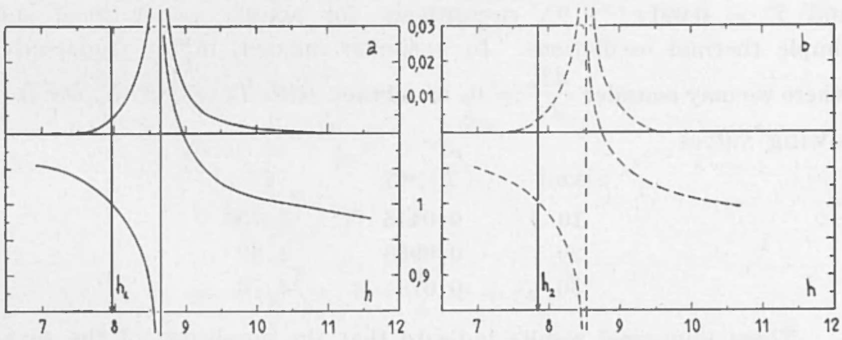


Fig. 4. – Dependence of the ratio λ'/λ (lower diagrams) and of its phase-angle (upper-diagrams) on the parameter h, respectively when $\tau = 0$ (case a) and $\tau \neq 0$ (case b). The parameter h is in km and the phase-angle in radiants.

The physical meaning of the equation [36] is that if λ is small, then the purely thermal pressure oscillation p' may be considerably different from the corresponding value p of the gravitational case; however, when λ is great, *i. e.* in the effective case corresponding to the resonance, the term $\{\ldots\}$ in [36] is very close to unity, so that in effect the pure thermal oscillations are substantially identical with the pure gravitational oscillations. Such a conclusion is confirmed by figs. 4 a and b, which show the values and the phase angles of the ratio λ'/λ. The relative shift of the diagrams in the cases $\tau = 0$ and $\tau \neq 0$ is as might have been expected from the resonance spectra of fig. 2.

Equation [35] gives us the amplitude q_0, corresponding to any value of Ω_0, of the heat transfer which produces approximately the same pressure oscillation determined by a gravitational potential whose value at ground level is just Ω_0.

2.4. – *The temperature oscillation* – The theoretical and numerical results of the foregoing sections allow us to write the height dependence of the temperature oscillation T; in the following, we are interested in the evaluation of the orders of magnitude so that, for the sake of simplicity, we at once assume $\tau = 0$. If use is made of equation [7], we obtain

$$\frac{T}{T_0} = \frac{1}{i\sigma}\left[\frac{Gq}{RT_0} - \frac{w}{T_0}\frac{dT_0}{dz} - (\gamma - 1)\chi\right] = \left|\frac{T}{T_0}\right| e^{i\xi} . \qquad [37]$$

If $h = 7.955$ and $z = 0$ (ground level) we have $T = 0.071\, e^{i\,1.698}$ °C and $T = 0.091\, e^{i\,2.17}$ °C, respectively for simple gravitational and simple thermal oscillations. In a similar manner, in the tropopause, where we may consider $\frac{dT_0}{dz} = 0$, we obtain, with $T_0 = 288$ °K, the following values

z (km)	$\lvert T \rvert$ (°C)	ξ
10	0.0446	1.756
20	0.0968	4.69
30	0.678	4.76

These numerical results indicate that the amplitude of the semidiurnal temperature oscillation is of the order of tenths of a Celsius (or a Kelvin) degree, both at ground level and in the tropopause; however it is quickly increasing with the height z.

3. – *Concluding remarks.*

The results we have obtained allow us to conclude that the assumption of the effective height variation of the gravity acceleration and of the Earth's radius vector implies some important refinement of the tidal theory. As concerns its application to practical cases, such refinements may lead to considerable differences in the interpretation of experimental data (for example metereological or cosmic ray data), if we assume $\tau \neq 0$ or $\tau = 0$. On the other hand, the difference in the calculated values of the parameter h corresponding to the resonance implies that the construction of a more refined model atmosphere requires the use of the tidal theory in the actual form.

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Appendix **A.1**. – The algebraic expression of the constants in the equation [24] are the following

$$a=-\frac{1}{H^*}+6\tau \quad ; \quad b=\frac{\gamma-1}{\gamma h H^*}+2\tau\left(\frac{2}{\gamma H^*}-\frac{3}{H^*}-\frac{1}{h}\right) \quad ; \quad c=4\tau\frac{1-\gamma}{\gamma h H^*} \; ;$$

$$A=\frac{\beta-1}{H^*}+6\tau \quad ; \quad B=6\tau\frac{\beta}{H^*} \quad ; \quad C=\frac{\beta}{H^*} \; ;$$

$$D=\left(\frac{\gamma-1}{\gamma}+\beta\right)\frac{1}{hH^*}+2\tau\left(\frac{2}{\gamma H^*}-\frac{3}{H^*}-\frac{1}{h}\right) \quad ; \quad E=-\frac{2\tau}{hH^*}\left(2\frac{\gamma-1}{\gamma}+3\beta\right) .$$

Appendix **A.2**. – We put

$$\varkappa = \left| b-\frac{1}{4}a^2+cz_s \right|^{1/2}$$

$$\delta = \frac{2}{3\,|c|}\,\varkappa^3$$

$$y = \frac{2}{3\,|c|}\left| b-\frac{1}{4}a^2+cz \right|^{3/2}$$

where a, b, c are the constants of equation [24] relative to the heights $z \geqslant z_8$

Then a solution of the first equation [24], may easily be expressed by modified Bessel functions of the second kind as follows

$$\chi^* = \beta_1\, e^{-\frac{a}{2}z}\, y^{1/3}\, K_{1/3}(y) \qquad \text{if } b-\frac{1}{4}a^2+cz_8<0$$

or, by Bessel functions of the third kind,

$$\chi^* = \beta_2\, e^{-\frac{a}{2}z}\, y^{1/3}\, i^{-5/6}\, H^{(\,)}_{1/3}(y) \qquad \text{if } b-\frac{1}{4}a^2+cz_8>0$$

The constants β_1 and β_2 have values which satisfy the condition [30], *i. e.* $\chi^*(z_8)=1$. The second boundary condition is instead expressed in the form

$$\left(\frac{d\chi^*}{dz}\right)_{z_8} = -\frac{a}{2}-\varkappa\,\frac{K_{2/3}(\delta)}{K_{1/3}(\delta)}$$

or, respectively,

$$\left(\frac{d\chi^*}{dz}\right)_{z_0} = -\frac{a}{2} + i^{2/3}\,\varkappa\,\frac{H^{(2)}_{2/3}(\delta)}{H^{(2)}_{1/3}(\delta)}$$

Appendix **A.3**. – The function $\eta(z)$ satisfying the conditions

$$\eta(z) = \left(\frac{d\eta}{dz}\right) \equiv 0 \qquad \text{for } z \geqslant z_1$$

may be found as follows.

We put

$$x = \left(k - \frac{a}{2}\right)(z_1 - z)$$

$$\eta(z) = q_0\,M(z) = q_0\,m\,e^{-k z_1 - \frac{a}{2}(z - z_1)}\,\Psi(x) \qquad \text{for } z < z_1$$

with $m = \dfrac{\mu + \nu z_1}{k^2 - a\,k + b + cz_1}$ and μ, ν given by equations [28].

If moreover we put

$$s = \frac{\left|b - \frac{1}{4}a^2 + cz_1\right|^{1/2}}{k - \frac{a}{2}}\,; \qquad l = \frac{\nu}{(\mu + \nu z_1)\left(k - \frac{a}{2}\right)}\,;$$

$$r = \frac{c}{(k^2 - a\,k + b + cz_1)\left(k - \frac{a}{2}\right)}\,; \qquad \varepsilon = \text{the sign of}\left(b - \frac{1}{4}a^2 + cz_1\right)$$

we have to find the function $\Psi(x)$ for which

$$\frac{d^2\Psi}{d\,x^2} + \varepsilon\,s^2\,\Psi - r\,(1 + \varepsilon\,s^2)\,x\,\Psi = (1 + \varepsilon\,s^2)\,(1 - l\,x)\,e^x\,.$$

The parameter r is very small, so that the equation may be solved by the perturbation method, putting

$$\Psi(x) = \bar{\Psi}(x) + r\,\bar{\bar{\Psi}}(x)$$

The initial conditions of $\eta(z)$ may be written in the form

$$\bar{\Psi}(0) = \bar{\Psi}'(0) = \bar{\bar{\Psi}}(0) = \bar{\bar{\Psi}}'(0) = 0 \quad \text{for } z \geqslant z_1\,.$$

One can easily express the function $\bar{\Psi}$ in the form

$$\bar{\Psi}(x) = \left(1 + \frac{2\,l}{1 - s^2} - l\,x\right)e^x - \left(1 + \frac{2\,l}{1 - s^2}\right)\cosh s\,x\, +$$

$$-\frac{1}{s}\left(1 + l\,\frac{1 + s^2}{1 - s^2}\right)\text{senh}\;s\,x$$

if $\varepsilon = -1$; the corresponding expression of $\Psi(x)$ for the case $\varepsilon = +1$ is obtained by substituting *is* in the place of *s* in the above expression.

The function $\overline{\overline{\Psi}}$ has instead the form

$$\overline{\overline{\Psi}}\,(x) = \left\{ x - \frac{2}{1 - s^2} - l \left[x^2 - \frac{6}{1 - s^2}\, x + 2\, \frac{5 + s^2}{(1 - s^2)^2} \right] \right\} e^{x} +$$

$$+ \frac{2}{1 - s^2} \left(1 + l\, \frac{5 + s^2}{1 - s^2} \right) \cosh s x + \frac{1}{s}\, \frac{1 + s^2}{1 - s^2} \left(1 + 4\, \frac{l}{1 - s^2} \right) \operatorname{senh} s x +$$

$$- \frac{1}{4\, s^3} \left(1 + 2\, \frac{l}{1 - s^2} \right) (s^2 x^2 \operatorname{senh} s x - s x \cosh s x + \operatorname{senh} s x) +$$

$$- \frac{\varpi}{4\, s^3} \left(1 + l\, \frac{1 + s^2}{1 - s^2} \right) (s x \cosh s x - \operatorname{senh} s x)$$

if $\varepsilon = -1$; the quantity s has again to be substituted by *is* if $\varepsilon = +1$.

ABSTRACT

In this paper, we reconsider the general theory of atmospheric tidal oscillations, assuming the correct height variation of the Earth's radius vector R and of the acceleration of gravity g. The fundamental tidal equation is solved separately for the case of purely gravitational and purely thermal oscillations. In the condition of resonance the pressure oscillation p is substantially identical in the two cases; with respect to the classical case of constant R and g the resonance period in our generalized case is some minutes greater than the period calculated by Wilkes and the resonance spectrum, as a whole, exhibits a slight shift.

Concerning the amplitude of p it is approximately 1.5 times smaller than in the case of Wilkes; an important feature is that the height of the nodal point of the pressure oscillation p, i. e. the height at which the sign of p is changed, is considerably greater in our case (~ 80 km) than in the Wilkes case (~ 35 km).

RIASSUNTO

In questo lavoro si considera in forma generale la teoria delle oscillazioni di marea della atmosfera, di origine sia gravitazionale sia termica, assumendo la corretta variazione con la quota della accelerazione di gra-

vità g e del raggio vettore R. La equazione fondamentale che descrive il fenomeno di marea viene risolta separatamente nei due casi di oscillazioni puramente gravitazionali e di oscillazioni puramente termiche, per un modello di atmosfera (fig. 1) ottenuto approssimando con tratti lineari la effettiva variazione con la quota della scala delle altezze H.

Rispetto al caso classico che g ed R non variino con la quota, si constata nel caso puramente gravitazionale un aumento del periodo di risonanza della atmosfera di qualche minuto; la ampiezza di risonanza al suolo è invece circa 1.5 volte inferiore; il più notevole effetto è tuttavia un notevole innalzamento, da circa 35 a circa 80 km, della quota a cui la oscillazione di pressione cambia di segno. Risultati sostanzialmente analoghi valgono per la oscillazione di origine puramente termica, in quanto in condizioni di risonanza le oscillazioni della pressione nei due casi tendono a identificarsi.

Si calcola infine la ampiezza di oscillazione della temperatura prodotta dalla oscillazione di pressione e si trova che essa è dell'ordine dei decimi di grado centigrado, in accordo con le indicazioni sperimentali.

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