

Evaluation of the aerothermodynamic field produced by a pseudospheric body of mercury type at $M = 22.6$ flying in air in thermodynamic equilibrium (*)

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SUMMARY. — The differential equations valid after the shock are first given in curvilinear coordinates; the chosen unknowns are the two velocity components and the entropy and enthalpy. A function of entropy and enthalpy is then determined, by means of which all the thermodynamic variables of the fluid are "coherently" approximated. Later on, the density and all the other kinematic and thermodynamic variables are calculated immediately after the shock, taking the angle σ as a parameter. The shape of the body is now taken into account and a convenient shape of the shock wave is given.

The differential equations are then integrated with a step-by-step procedure, until the stagnation entropy is reached on the body.

Finally the pressure and the temperature on the body are given. A sonic-to-stagnation pressure of 0.6 is the result, instead of 0.523 for a perfect gas.

RIASSUNTO. — Sono prima ricavate dalla teoria dello strato d'urto sottile, le equazioni valide dietro l'onda d'urto prodotta da un corpo non appuntito viaggiante a velocità ipersonica in atmosfera parzialmente dissociata che si suppone in equilibrio termodinamico. Come esempio si è considerata la capsula Mercury nella fase di rientro alla quota di 60 Km ed alla velocità di 7500 m/sec. ($M_\infty = 22,6$) con la parte frontale in prossimità dell'asse, completamente sferica.

Prima di iniziare lo studio del campo aerotermodinamico dietro l'onda d'urto con l'« Inverse Problem », sono determinati i coefficienti di una funzione dell'entropia ed entalpia da cui dipendono tutte le grandezze interessanti il fluido, funzione che tiene conto della dissociazione del gas.

Di poi viene calcolata la densità nei punti immediatamente dietro l'onda per vari valori dell'angolo che la tangente alla stessa forma con l'as-

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se del corpo, adoperando un metodo di approssimazioni successive, senza assegnare ancora la forma dell'onda d'urto.

Dal calcolo delle densità sono note velocità, entropia ed entalpia dietro l'urto. Assegnando ora un'onda probabile si inizia il calcolo del campo dietro l'urto con le equazioni in forma adimensionale mediante un procedimento al passo, fino a giungere alla forma del corpo corrispondente all'onda d'urto data.

L'ultimo passo del procedimento di calcolo fornisce temperature e pressioni sul corpo. Calcolato così il campo subsonico dietro l'onda d'urto fino ed oltre la linea sonica, il campo supersonico potrà, da questo punto, essere studiato col metodo delle caratteristiche.

Symbols

p	= Pressure
p_o	= Reference pressure (atmospheric pressure at sea level)
ρ	= Density
T	= Temperature in °K
T_o	= Reference temperature = 273.16°K
ρ_o	= Reference density = p_o/R_oT_o
R_o	= Gas constant referred to one Kg. of air = $6.886 \times 10^{-2} \frac{\text{Kcal}}{\text{Kg} \cdot \text{°K}}$
R	= Gas constant referred to one Kgr.mole
h	= Enthalpy
S	= Entropy
F	= p/p_o
G	= ρ/ρ_o
θ	= T/T_o
η	= $\frac{h}{R_oT_o} - 300$
ξ	= $\frac{S}{R_o} - 48$
σ	= Shock wave angle
σ'	= $\frac{\pi}{2} - \sigma$
x	= Curvilinear coordinate parallel to the shock wave
y	= Coordinate orthogonal to the shock wave
$r; z$	= Cylindrical coordinates
$K(x)$ or $K(\sigma')$	= Local curvature of the shock wave ($y = 0$)
H	= $1 - Ky$ Correction factor for the curvature of the other coordinate lines ($y = \text{const.}$)

- a = Sound velocity
 a_0 = 332 m/sec. = Reference velocity = Sound velocity at T_0
 γ = $C_p/C_v = 1.4$
 V = Flow velocity
 $u; v$ = Components of V along x and y
 u_∞ = $V_\infty \cos \sigma = V_\infty \sin \sigma'$
 v_∞ = $V_\infty \sin \sigma = V_\infty \cos \sigma'$
 M = Mach number
 $n; b_i; c_i$ = Coefficients of the function $f(\xi; \eta)$
 Z = Dissociation coefficient

Indexes

- o = Reference conditions
 ∞ = Free stream conditions
 s = Stagnation conditions
 $(oru)_x = \frac{\partial (oru)}{\partial x}$; $q_x = \frac{\partial q}{\partial x}$; $F_\xi = \frac{\partial F}{\partial \xi}$
 $\left(\frac{\partial F}{\partial \eta}\right)_\xi$ = Derivative of F with respect to η at constant ξ .

I - INTRODUCTION.

The hypersonic flow introduces new problems in aerodynamics, which complicate the solutions in this field of applied physics.

We are interested in very high stagnation temperatures, and then chemical phenomena are present; the bow shock connected with a blunt body, as necessary in re-entry problems, gives such high entropy gradients that the classical potential approach is obviously completely wrong.

The relaxation times in the physical and chemical transformations of the molecular species can modify the flow properties between two extreme cases, namely the equilibrium flow and the completely frozen flow.

These are some of the difficulties encountered in hypersonic problems; other difficulties arise from consideration of the boundary layer and the local mean free path of the molecules.

We are dealing with the particular case of the continuous flow of a real gas in equilibrium, in a thin layer between shock and body. Viscous effects are not considered; the inverse method is used. The thin shock layer theory, developed in order to calculate with a good degree

of approximation the flow field produced by a blunt body at a very high Mach number, warrants the supposition that the shock shape is very similar to the body shape because of the small stand-off distance between the shock and the body. The intrinsic weakness of the inverse method — small changes in shock shape correspond to bigger changes in body shape — is much less important at high than at low Mach numbers.

With a given shock shape, the corresponding body found does not coincide exactly with the desired body but it does correspond to a uniform stream before the shock.

Many authors have considered the problem of the blunt body in supersonic and hypersonic flow during the last few years. Van Dyke (7) and Van Dyke and Gordon (8) make the calculations for a family of shock waves of conical shape; this gives, as Van Dyke shows, similar body shapes, at least up to the sonic point. These authors are dealing with a perfect gas.

Some modifications to Van Dyke's method are suggested by Vinokur and Sanders (21) in order to account for different shock shapes; for enthalpy they propose the formula:

$$h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} + A$$

where γ and A are constants, but different before and after the shock.

Vaglio-Laurin and Ferri (4), also for a perfect gas, make the numerical calculation of the subsonic region before an arbitrary axisymmetrical body. They introduce two new independent variables, one of which is related to the stream function. The stream lines, the shock line and the body are known in the plane of the new variables. At the end, the results are reduced to the physical plane.

Vaglio-Laurin (2), on the same basis, takes into account the P.L.K. perturbation method, consisting in stretching the coordinates in order that an already known solution overlaps the field of a body having a slightly different profile.

Vaglio-Laurin and Trella (3) carried out the calculations for many bodies, in various re-entry conditions, for a perfect gas or equilibrium gas; equilibrium equations are given in a polynomial form of the entropy in function of enthalpy and pressure. In the shock jump they express enthalpy in terms of pressure and density

$$h = \frac{\Gamma}{\Gamma - 1} \frac{p}{\rho}$$

with Γ slightly varying along the shock.

Our aim is to approximate the Mollier diagram in terms of a function $f(\xi; \eta)$ of entropy and enthalpy, by which function all the thermodynamic quantities are expressed.

A pseudospheric body with a 20° conic afterbody, as the "Mercury Project", is taken as an example.

II - EQUATIONS OF MOTION DOWNSTREAM OF THE SHOCK.

Let x and y be the curvilinear coordinates on the shock (Fig. 1).

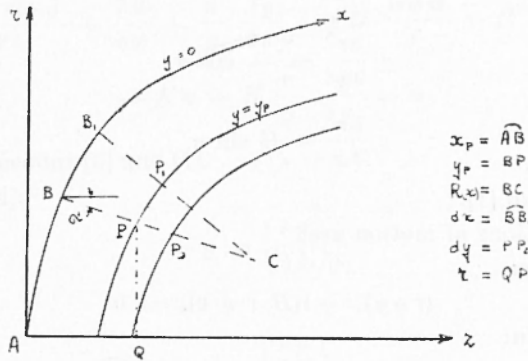


Fig. 1

Let $y = 0$ be the equation of the shock wave and $K(x)$ its local curvature:

$$K(x) = \frac{1}{R(x)} = - \left(\frac{\partial \sigma}{\partial x} \right)_{y=0} = \left(\frac{\partial \sigma'}{\partial x} \right)_{y=0} \quad [1]$$

x being the curvilinear abscissa along the shock wave. The definition of the coordinates is completed stating that the line

$$x = \text{const.}$$

is a straight line perpendicular to the local shock wave direction, and that the y coordinate of a point P is the distance of P from the shock wave along the line $x = \text{const.}$

The curvature of the line $y = \text{const}$ is:

$$\frac{1}{R(x) - y} = \frac{K(x)}{1 - y K(x)} = K(x) / H(x; y) \quad [2]$$

with:

$$H(x; y) = 1 - y K(x) \quad [3]$$

An elementary arc is given by:

$$ds^2 = \overline{PP_1}^2 + \overline{PP_2}^2$$

or:

$$ds^2 = H^2 dx^2 + dy^2 \quad [4]$$

We now need the distance $PQ = r$ from the symmetry axis. It is easy to see that:

$$r = \int_0^x \sin \sigma dx - y \cos \sigma \quad [5]$$

and:

$$\frac{\partial r}{\partial y} = -\cos \sigma \quad [6]$$

$$\frac{\partial r}{\partial x} = H \sin \sigma \quad [7]$$

with the help of [1].

The equations of motion are:

Continuity:

$$(r \varrho u)_x + (H r \varrho v)_y = 0 \quad [8]$$

Momentum:

$$u u_x + H v u_y - K u v = -\frac{1}{\varrho} p_x \quad [9]$$

$$u v_x + H v v_y + K u^2 = -\frac{H}{\varrho} p_y \quad [10]$$

Entropy:

$$H \frac{\partial S}{\partial y} = -\frac{u}{v} \frac{\partial S}{\partial x} \quad [11]$$

We choose S and h as thermodynamic variables; so we wish to eliminate p and ϱ from [8], [9] and [10].

From the known formula:

$$T dS = dh - \frac{dp}{\varrho} \quad [12]$$

we have:

$$-\frac{1}{\varrho} \frac{\partial p}{\partial x} = -\frac{\partial h}{\partial x} + T \frac{\partial S}{\partial x}$$

but:

$$h + \frac{u^2}{2} + \frac{v^2}{2} = \text{const} \quad [13]$$

Then:

$$-\frac{1}{\rho} \frac{\partial \rho}{\partial x} = u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} + T \frac{\partial S}{\partial x}$$

and [9] becomes:

$$H \frac{\partial u}{\partial y} = Ku + \frac{\partial v}{\partial x} + \frac{T}{v} \frac{\partial S}{\partial x} \quad [14]$$

Instead of equation [10], the energy equation [13] in closed form is preferred. We now use [8] to get $\partial v/\partial y$. Equation [8] can be easily transformed into:

$$\begin{aligned} \frac{u H \sin \sigma}{r} + \frac{\partial u}{\partial x} + \frac{u}{\rho} \frac{\partial \rho}{\partial x} - v H \frac{\cos \sigma}{r} - H \frac{\partial v}{\partial y} - \\ - K v + H \frac{v}{\rho} \frac{\partial \rho}{\partial y} = 0 \end{aligned} \quad [15]$$

taking into account [6] and [7].

From [12]:

$$\rho = \left(\frac{\partial p}{\partial h} \right)_S \quad [16]$$

Further:

$$a^2 = \left(\frac{\partial p}{\partial \rho} \right)_S = \frac{\left(\frac{\partial p}{\partial h} \right)_S}{\left(\frac{\partial \rho}{\partial h} \right)_S} = \frac{\rho}{\left(\frac{\partial \rho}{\partial h} \right)_S}$$

or:

$$\frac{1}{\rho} \left(\frac{\partial \rho}{\partial h} \right)_S = \frac{1}{a^2} \quad [17]$$

Now:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial x} = \frac{1}{\rho} \left\{ \frac{\partial \rho}{\partial S} \frac{\partial S}{\partial x} + \frac{\partial \rho}{\partial h} \frac{\partial h}{\partial x} \right\}$$

and:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial x} = \frac{1}{\rho} \frac{\partial \rho}{\partial S} \frac{\partial S}{\partial x} - \frac{1}{a^2} \left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} \right) \quad [18]$$

using [17] and [13].

In the same way we can obtain:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial y} = \frac{1}{\rho} \frac{\partial \rho}{\partial S} \frac{\partial S}{\partial y} - \frac{1}{a^2} \left(u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} \right) \quad [19]$$

If we introduce [18] and [19] into [15], the terms containing the entropy vanish by virtue of [11] and we get:

$$\left(1 - \frac{u^2}{a^2}\right) \frac{\partial u}{\partial x} + H \left(1 - \frac{v^2}{a^2}\right) \frac{\partial v}{\partial y} - \frac{uv}{a^2} \left(H \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \\ - K v + H \frac{u \sin \sigma - v \cos \sigma}{r} = 0$$

We express the term $H \frac{\partial u}{\partial y}$, given by [14]:

$$\left(1 - \frac{u^2}{a^2}\right) \frac{\partial u}{\partial x} + H \left(1 - \frac{v^2}{a^2}\right) \frac{\partial v}{\partial y} - \frac{2uv}{a^2} \frac{\partial v}{\partial x} - \frac{u}{a^2} T \frac{\partial S}{\partial x} - \\ - \left(1 + \frac{u^2}{a^2}\right) K v + H \frac{u \sin \sigma - v \cos \sigma}{r} = 0 \quad [20]$$

On the symmetry axis:

$$\lim_{x \rightarrow 0} \left\{ H \frac{u \sin \sigma - v \cos \sigma}{r} \right\} = \left[\frac{\partial u}{\partial x} - K v \right]_{x=0} \quad [21]$$

Note that equations [11] and [14] could be obtained directly from Crocco's theorem, written in curvilinear coordinates. Equations [14], [21], [11] and [13] contain u ; v ; S and h as unknowns; in an adimensional form:

$$H \frac{v}{a_0} \frac{\partial \frac{u}{a_0}}{\partial y} = K \frac{u}{a_0} \frac{v}{a_0} + \frac{v}{a_0} \frac{\partial \frac{v}{a_0}}{\partial x} + \frac{\theta}{\gamma} \frac{\partial \xi}{\partial x} \quad [22]$$

$$\left(1 - \frac{u^2}{a^2}\right) \frac{\partial \frac{u}{a_0}}{\partial x} + H \left(1 - \frac{v^2}{a^2}\right) \frac{\partial \frac{v}{a_0}}{\partial y} - \frac{2uv}{a^2} \frac{\partial \frac{v}{a_0}}{\partial x} - \frac{\theta}{\gamma} \frac{u}{a_0} \frac{a_0^2}{a^2} \frac{\partial \xi}{\partial x} - \\ - \left(1 + \frac{u^2}{a^2}\right) K \frac{v}{a_0} + H \left(\frac{u}{a_0} \sin \sigma - \frac{v}{a_0} \cos \sigma\right) / r = 0 \quad [23]$$

$$\frac{2}{\gamma} \eta + \frac{u^2}{a_0^2} + \frac{v^2}{a_0^2} = \frac{2}{\gamma} \eta_{\max} \quad [24]$$

$$H \frac{v}{a_0} \frac{\partial \xi}{\partial y} = - \frac{u}{a_0} \frac{\partial \xi}{\partial x} \quad [25]$$

III - DETERMINATION OF THE FUNCTION $f(\xi; \eta)$

For a perfect gas we have, from [12]:

$$dp = \rho dh - p \frac{ds}{R_0}$$

$$\left(\frac{\partial p}{\partial s} \right)_h = - \frac{p}{R_0}$$

and, after integration:

$$\frac{p}{p_0} = e^{-\frac{s}{R_0}} f(h) \quad [26]$$

But:

$$\left(\frac{\partial p}{\partial h} \right)_s = \rho = p_0 e^{-\frac{s}{R_0}} f'(h)$$

and, dividing:

$$\frac{f(h)}{f'(h)} = \frac{p}{\rho} = R_0 T = \frac{\gamma - 1}{\gamma} h$$

Then:

$$f(h) = h^{3.5}$$

for $\gamma = 1.4$

The result is that, for a perfect gas:

$$\frac{p}{p_0} = \left(\frac{h}{h_0} \right)^{3.5} e^{-\frac{s - s_0}{R_0}} \quad [27]$$

If we introduce adimensional quantities:

$$\frac{p}{p_0} = F = (b_0 + b_1 \eta)^{3.5} e^{-\xi} \quad [28]$$

where η and ξ are the adimensional enthalpy and entropy differences evaluated from a given state.

For the real case of air in thermodynamic equilibrium we put, in analogy with [26]:

$$F = C_0 10^{-C_1 \xi - C_2 \xi^2} [f(\xi; \eta)]^n \quad [29]$$

where C_0 ; C_1 ; C_2 and n are constants.

For function $f(\xi; \eta)$ we assume, in analogy with [28]:

$$f(\xi; \eta) = g_0(\xi) + \eta g_1(\xi) + \eta^2 g_2(\xi) \quad [30]$$

Functions $g_k(\xi)$ are given in order to approximate the Mollier diagram for equilibrium air [22] in the field:

$$280 < \frac{h}{R_o T_o} < 360$$

$$48 < \frac{S}{R_o} < 52$$
[31]

In this field $f(\xi; \eta)$ has been evaluated from [29], [33] and

$$\frac{F}{G} = \frac{p/p_o}{\rho/\rho_o} = Z \frac{p}{\rho} - \frac{Z}{T_o} = \frac{f(\xi; \eta)}{n f_x(\xi; \eta)}$$

Dropping very few terms, this gives:

$$f(\xi; \eta) = (1 + b_1 \xi) + (b_2 + b_4 \xi)\eta + b_7 \xi \eta^2$$

and:

$$\begin{aligned} n &= 8.2192 \\ b_1 &= -0.00416 \\ b_2 &= 0.0032706 \\ b_4 &= 0.0000723 \\ b_7 &= 0.98 \cdot 10^{-7} \end{aligned}$$
[32]

The value of n is determined so as to give $f(o; \eta)$ a linear expression in η

$$f(o; \eta) = 1 + 0.0032706 \eta$$

as is the case in a perfect gas, for any ξ , from [28].

Writing [16] in adimensional form, we now obtain from [29]:

$$G = \frac{p}{\rho_o} = \frac{\partial F}{\partial \eta} = C_o n \cdot 10^{-C_1 \xi - C_2 \xi^2} [f(\xi; \eta)]^{n-1} \frac{\partial f}{\partial \eta}$$
[33]

The speed of sound is given by [17], or

$$\frac{R_o T_o}{a^2} = \frac{a_o^2}{\gamma a^2} = \frac{1}{G} \left(\frac{\partial G}{\partial \eta} \right)_{\xi} = \left(\frac{\partial \lg G}{\partial \eta} \right)_{\xi}$$

and:

$$\frac{a_o^2}{a^2} = (n-1) \gamma \frac{f_{\eta}(\xi; \eta)}{f(\xi; \eta)} + \gamma \frac{f_{nn}(\xi; \eta)}{f_{\eta}(\xi; \eta)}$$
[34]

We must now express the temperature.

From [12], we get:

$$T = \left(\frac{\partial h}{\partial S} \right)_p = - \frac{\left(\frac{\partial p}{\partial S} \right)_h}{\left(\frac{\partial p}{\partial h} \right)_S}$$

or:

$$\theta = \frac{-\frac{\partial F}{\partial \xi}}{\frac{\partial F}{\partial \eta}} = \frac{-\frac{\partial F}{\partial \xi}}{G} \quad [35]$$

and from [33] and [29]:

$$\theta = \frac{1}{f_\eta(\xi, \eta)} \left\{ \frac{3.02585}{n} (C_1 + 2 C_2 \xi) f(\xi; \eta) - f_\xi(\xi; \eta) \right\} \quad [36]$$

Note that from [35], we can get:

$$\frac{1}{Z} = \frac{\rho R_o T}{p} = \frac{G \theta}{F} = - \frac{\partial \lg F}{\partial \xi} \quad [37]$$

or, from [29]:

$$\frac{1}{Z} = 2.302585 (C_1 + 2 C_2 \xi) - \frac{n f_\xi(\xi; \eta)}{f(\xi; \eta)} \quad [38]$$

The validity of [38] for $\eta = 0$ and:

$$48 < \xi < 52$$

gives:

$$\begin{cases} C_1 = 0.2675 \\ C_2 = -0.000922 \end{cases} \quad [39]$$

The constant C_0 is given by [29]:

$$C_0 = F(O; O) = 0.5217 \quad [40]$$

Equation [29] is then completely specified. Equations [33], [36] and [34] give the density, the temperature and the velocity of sound. Equations [36] and [34] will be used in the field; equations [29] and [33] will be used on the shock, to get initial conditions. Formulae [29], [33], [36] and [34] have been controlled for values of ξ and η other than those used for calculating coefficients: the agreement is within 2-3 percent. It could be increased by taking into account a third order term η^3 , in [30].

IV - INITIAL CONDITIONS ON THE SHOCK.

On the shock (v. Fig. 2 below), we have:

$$\frac{v}{v_\infty} = \varepsilon = \frac{\rho_\infty}{\rho} \quad [41]$$

$$p = p_\infty + \rho_\infty v_\infty^2 (1 - \varepsilon) \quad [42]$$

$$h = h_\infty + \frac{1}{2} v_\infty^2 (1 - \varepsilon^2) \quad [43]$$

$$u = u_\infty \quad [44]$$

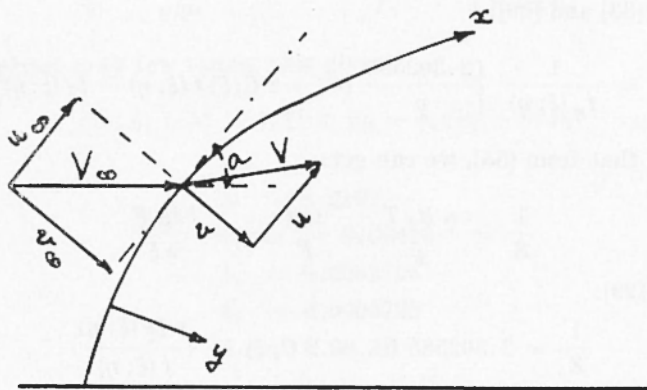


Fig. 2

Putting $v_\infty = V_\infty \sin \sigma$ and noting [22] that at an altitude of 60 km $T_\infty = T_0$ or $\theta_\infty = 1$:

$$G = \frac{G_\infty}{\varepsilon} \quad [45]$$

$$F = F_\infty + 1.4 G_\infty M_\infty^2 (1 - \varepsilon) \sin^2 \sigma \quad [46]$$

$$\eta = 0.7 M_\infty^2 (1 - \varepsilon^2) \sin^2 \sigma - 296.5 \quad [47]$$

$$\frac{f(\xi; \eta)}{n f_\eta(\xi; \eta)} = \frac{F}{G} \quad [48]$$

$$G = G(\xi; \eta) \quad [49]$$

$$\varepsilon' = \frac{G_\infty}{G(\xi; \eta)} \quad [50]$$

with:

$$\left\{ \begin{array}{l} G_\infty = F_\infty = 3.5 \cdot 10^{-4} \\ M_\infty = 22.6 \\ V_\infty = 7,500 \text{ m/sec} \end{array} \right. \quad 90^\circ \geq \sigma \geq 60^\circ \quad [51]$$

Equations [45] to [50] are iterated until we get $\epsilon = \epsilon'$ (v. Fig. 3 below).

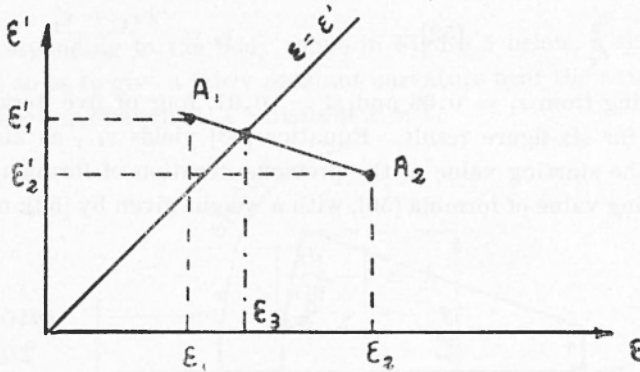


Fig. 3

Table no. 1

σ'	ϵ	η	ξ	u/a_0	v/a_0
0°	0.0604011	59.727650	3.7321361	0	1.364483
1°	0.0604070	59.618930	3.7282209	0.394202	1.364277
2°	0.0604246	59.292750	3.7164604	0.788403	1.364181
3°	0.0604539	58.749670	3.6968749	1.182379	1.363804
4°	0.0604951	57.990230	3.6694715	1.575900	1.363272
5°	0.0605481	57.015410	3.6342657	1.968885	1.362599
6°	0.0606131	55.826350	3.5912786	2.361370	1.361766
7°	0.0606901	54.424600	3.5405367	2.753090	1.360795
8°	0.0607792	52.811740	3.4820692	3.143900	1.359660
9°	0.0608805	50.989850	3.4159125	3.533810	1.358371
10°	0.0609942	48.961110	3.3421087	3.922820	1.356946
11°	0.0611205	46.727950	3.2067018	4.310470	1.355359
12°	0.0612595	44.293180	3.1717462	4.696760	1.353427
13°	0.0614112	41.659690	3.0752968	5.081701	1.351747
14°	0.0615762	38.830740	2.9714196	5.465060	1.349715
15°	0.0617544	35.809770	2.8601824	5.846837	1.347525
16°	0.0619463	32.600450	2.7416635	6.226807	1.345177
17°	0.0621523	29.206690	2.6159485	6.604743	1.342686
18°	0.0623723	25.632640	2.4831201	6.980873	1.340056
19°	0.0626071	21.882690	2.3432865	7.354743	1.337265
20°	0.0628568	17.961380	2.1965430	7.726355	1.334320
21°	0.0631219	13.873460	2.0433010	8.095707	1.331235
22°	0.0634030	9.623940	1.8828080	8.462575	1.327997
23°	0.0637003	5.218020	1.7160652	8.826731	1.324611
24°	0.0640146	0.661080	1.5429261	9.188403	1.321096
25°	0.0643466	— 4.041360	1.3635380	9.547138	1.317424
26°	0.0646968	— 8.883560	1.1780605	9.902936	1.313607
27°	0.0650660	— 13.859630	0.9866639	10.255798	1.309664
28°	0.0654549	— 18.963480	0.7895265	10.605490	1.305570
29°	0.0658645	— 24.188910	0.5868409	10.962033	1.301349
30°	0.0662957	— 29.529540	0.3788113	11.295180	1.297000

Two simple formulae are used instead of the graphical method illustrated in Fig. 3:

$$\lambda = \frac{\varepsilon'_1 - \varepsilon'_2}{\varepsilon_2 - \varepsilon_1} \quad [52] \quad \varepsilon_3 = \frac{\lambda \varepsilon_2 + \varepsilon'_2}{\lambda + 1} \quad [53]$$

Starting from $\varepsilon_1 = 0.06$ and $\varepsilon_2 = 0.07$, four or five iterations are sufficient for six-figure result. Equation [53] yields ε_{t+1} as an average between the starting value of the previous iteration of formula [45] and the resulting value of formula [50], with a weight given by [52]: note that:

$$\lambda \simeq \left[- \frac{d\varepsilon'}{d\varepsilon} \right]_{\varepsilon' = \varepsilon}$$

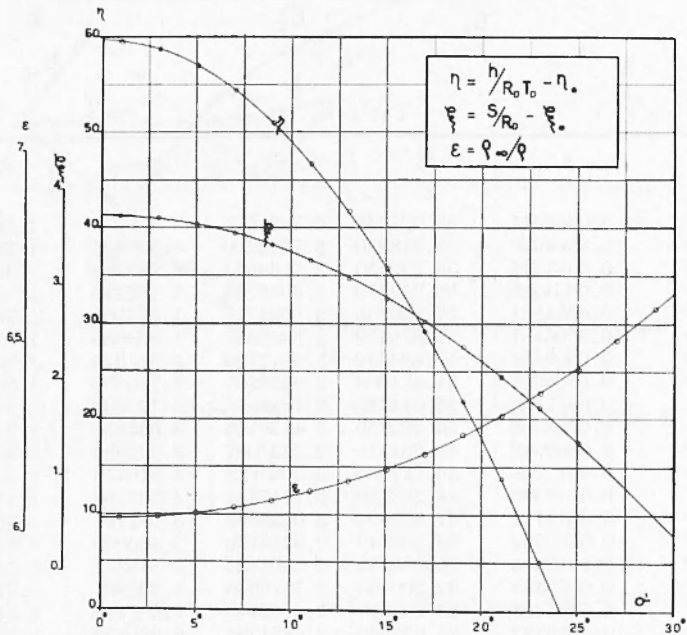


Fig. 4

When the true value of ε is obtained, we get:

$$\left\{ \begin{array}{l} v = \varepsilon v_\infty = \varepsilon \bar{V}_\infty \sin \sigma \\ u = u_\infty = V_\infty \cos \sigma \end{array} \right. \quad [54]$$

Table I contains the results in function of $\sigma' = \frac{\pi}{2} - \sigma$. Fig. 4 shows ε, ξ, η along the shock.

V - THE SHAPE OF THE SHOCK.

Corresponding to the body shape in Figure 5 below, a shock shape is chosen so as to give a fairly constant curvature near the axis, followed by an inflection of about 1.4 radians at $x \approx 1$.

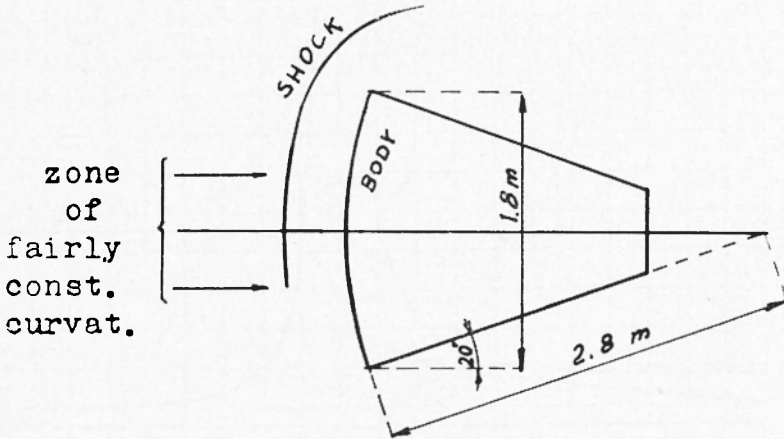


Fig. 5

The type of function representing the curvature is then:

$$K(x) = \frac{1}{3 - 3bx^4 + bx^8} \tag{55}$$

which has a maximum just at $x = \sqrt[4]{1.5} \approx 1.1$. The condition that the total inflection be about 1.4 radians implies $b = 1.25$ (Fig. 6). From [55] we then get:

$$\int_0^x K(x) dx = \sigma' = \frac{\pi}{2} - \sigma$$

By reversal of this formula x is determined as a function of σ or σ' . Table II and Fig. 7 show x and K .

The shape of the shock is then given by (v. Fig. 1):

$$r = \int_0^x \sin \sigma dx = \int_0^{\sigma'} \cos \sigma' \frac{d\sigma'}{K(\sigma')}$$

$$z = \int_0^x \cos \sigma dx = \int_0^{\sigma'} \sin \sigma' \frac{d\sigma'}{K(\sigma')}$$

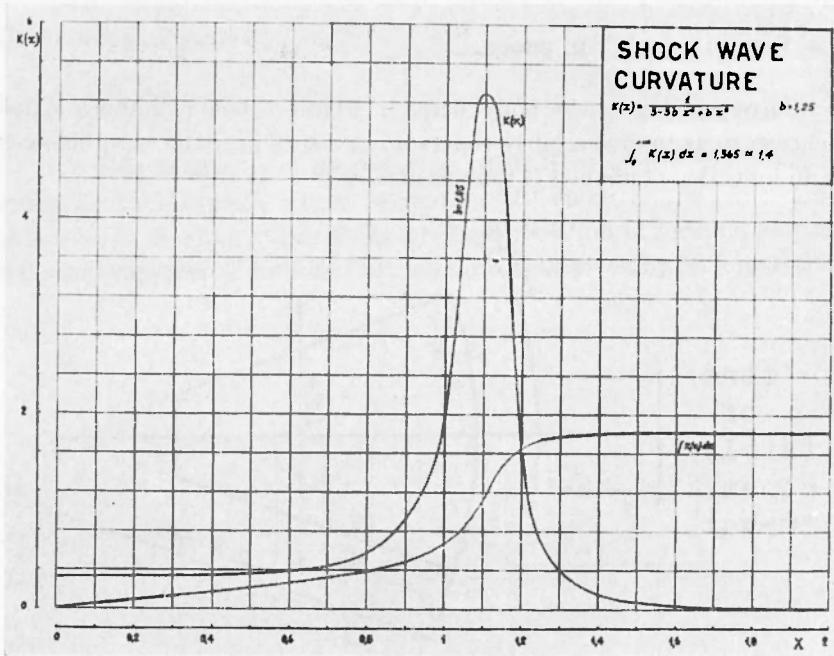


Fig. 6

Table no. 2

σ'	X	$K(\sigma')$	σ'	X	$K(\sigma')$
0	0.00	0.333333	16	0.760713	0.530544
1	0.05236	0.333336	17	0.792204	0.582585
2	0.104716	0.333383	18	0.820678	0.642483
3	0.156967	0.333383	19	0.846574	0.712626
4	0.209419	0.333413	20	0.869741	0.791277
5	0.261565	0.335295	21	0.890746	0.881625
6	0.313515	0.337411	22	0.909424	0.980279
7	0.364956	0.340852	23	0.926411	1.092132
8	0.415808	0.346165	24	0.941471	1.211032
9	0.465671	0.354281	25	0.95528	1.344798
10	0.514263	0.364453	26	0.967502	1.483055
11	0.561300	0.378859	27	0.978849	1.638133
12	0.606286	0.397503	28	0.988874	1.793741
13	0.649008	0.421324	29	0.998330	1.968372
14	0.689069	0.450494	30	1.006661	2.138292
15	0.726414	0.487209			

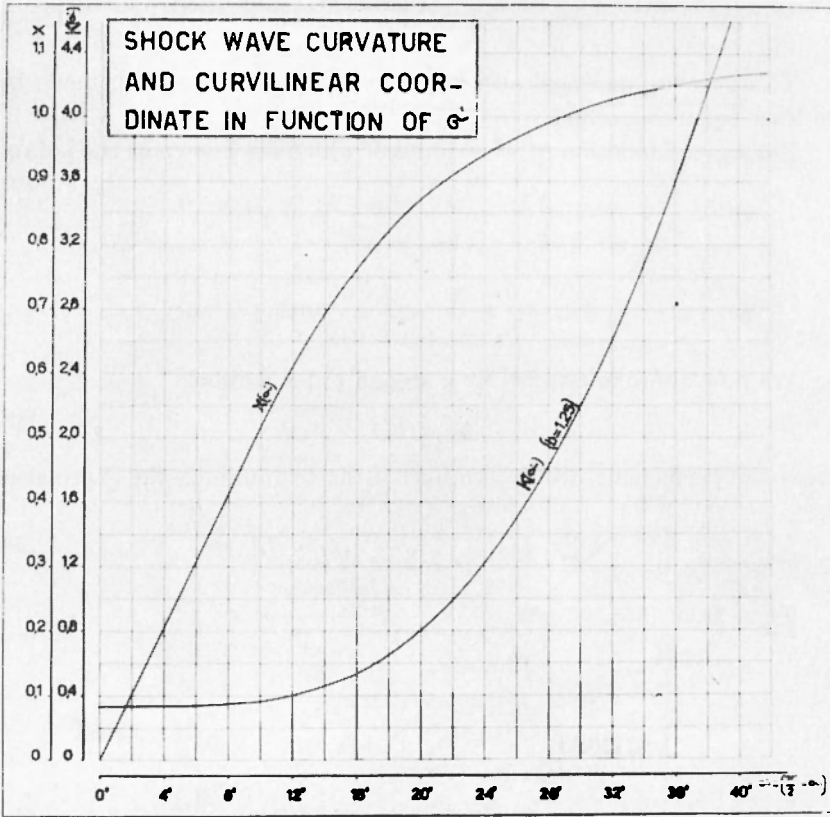


Fig. 7

VI - EVALUATION OF THE FIELD BEHIND THE SHOCK.

The evaluation of the aerothermodynamic field is done by a step-by-step procedure.

Consider equations [22] to [25].

If we suppose all variables along the line to be known, it is possible to calculate the three x -derivatives as:

$$\left(\frac{\partial \xi}{\partial x}\right)_y \equiv K \left(\frac{\partial \xi}{\partial \sigma'}\right)_y \tag{56-1}$$

$$\left(\frac{\partial u/a_0}{\partial x}\right)_y \equiv K \left(\frac{\partial u/a_0}{\partial \sigma'}\right)_y \tag{56-2}$$

$$\left(\frac{\partial v/a_0}{\partial x} \right)_y = K \frac{\partial v/a_0}{\partial \sigma'}_y \quad [56-3]$$

To improve numerical accuracy this calculation has been made by the least squares method.

For a given function $f(t)$ the values of which are known at the points:

$$\begin{aligned} t_{-2} &= -2h; & t_{-1} &= -h; & t_0 &= 0 \\ t_1 &= h; & t_2 &= 2h \end{aligned}$$

let: $f_i = f(t_i)$

and: $f'_i = f'(t_i)$

We now approximate $f(t)$ by a second order parabola

$$f(t) = a_0 + a_1 t + a_2 t^2 \quad [57]$$

where the coefficients are determined so as to minimize the expression:

$$\sum_{i=-2}^2 \left[a_0 + a_1 t_i + a_2 t_i^2 - f_i \right]^2 \quad [58]$$

Then from [56] we get:

$$\begin{aligned} f'_0 &= a_1 \\ f'_1 &= a_1 + 2a_2 h \\ f'_2 &= a_1 + 4a_2 h \end{aligned}$$

or:

$$f'_0 = \frac{2f_1 + f_2 - f_{-1} - 2f_{-2}}{10h} \quad [59]$$

$$f'_1 = \frac{1}{70h} \{34f_2 - 3f_1 - 20f_0 - 17f_{-1} + 6f_{-2}\} \quad [60]$$

$$f'_2 = \frac{1}{70h} \{54f_2 - 13f_1 - 40f_0 - 27f_{-1} + 26f_{-2}\} \quad [61]$$

We took 31 points on the shock at $\sigma' = 0^\circ; 1^\circ; 2^\circ \dots 30^\circ$. The x derivatives are evaluated, as in [56], by means of σ' -derivatives, which are given by [59], for ordinary points, and by [60] [61] for the last two points. This has the effect of slightly smoothing the process as the finite difference scheme is based on minimizing the error of the locally approximating parabola. The step in y was $\Delta y = 0.005$ for $0 < y < 0.08$ and $\Delta y = 0.002$ for $0.08 < y < 0.12$ near the body.

A second smoothing procedure was necessary along $y = 0.04$ and later on $y = 0.08$ and $y = 0.10$.

This smoothing procedure was carried out by means of [57], that is:

$$[f_0]_{\text{smooth}} \simeq a_0$$

$$[f_1]_{\text{smooth}} \simeq a_0 + a_1 h + a_2 h^2$$

$$[f_2]_{\text{smooth}} \simeq a_0 + 2 a_1 h + 4 a_2 h^2$$

OR:

$$[f_0]_{\text{smooth}} = \frac{1}{35} \{-3 f_2 + 12 f_1 + 17 f_0 + 12 f_{-2}\} \quad [62]$$

$$[f_1]_{\text{smooth}} = \frac{1}{35} \{9 f_2 + 13 f_1 + 12 f_0 + 6 f_{-2} - 5 f_{-3} - 3 f_{-2}\} \quad [63]$$

$$[f_2]_{\text{smooth}} = \frac{1}{35} \{31 f_2 + 9 f_1 - 3 f_0 - 5 f_{-1} + 3 f_{-2}\} \quad [64]$$

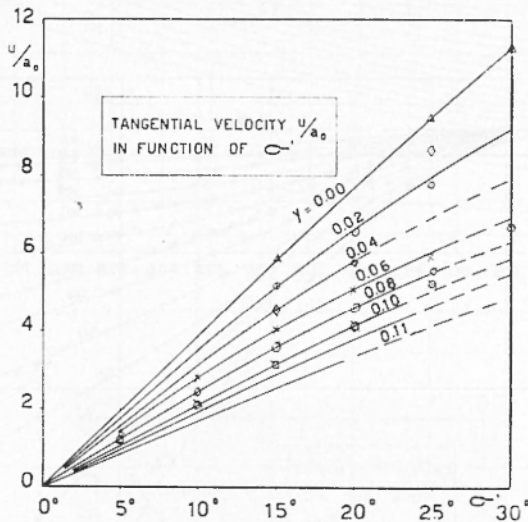


Fig. 8

Fig. 8, Fig. 9 and Fig. 10 show the two velocity components and the entropy in the field. Fig. 11 shows the Mach number: the sonic line is given by an intersection $M = 1$. Fig. 11 is made up with the help of [34]. Fig. 12 and 13 give the shape of the body: $r = 0$ on the axis and $\xi = \xi_{\text{max}}$ along $\sigma' = \text{const} > 0$. In the same Fig. 13 we find the stream lines as constant- ξ -lines. Fig. 14 gives the body shape. In curvilinear coordinates the value of γ varies, but very little, from .125 on the axis to .128 at the sonic point.

Figures 15 and 16 give pressure and temperature on the body.

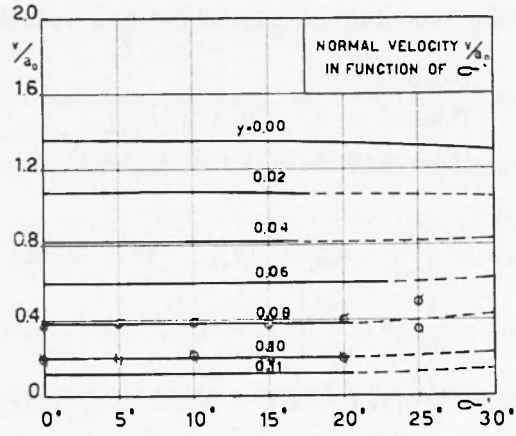


Fig. 9

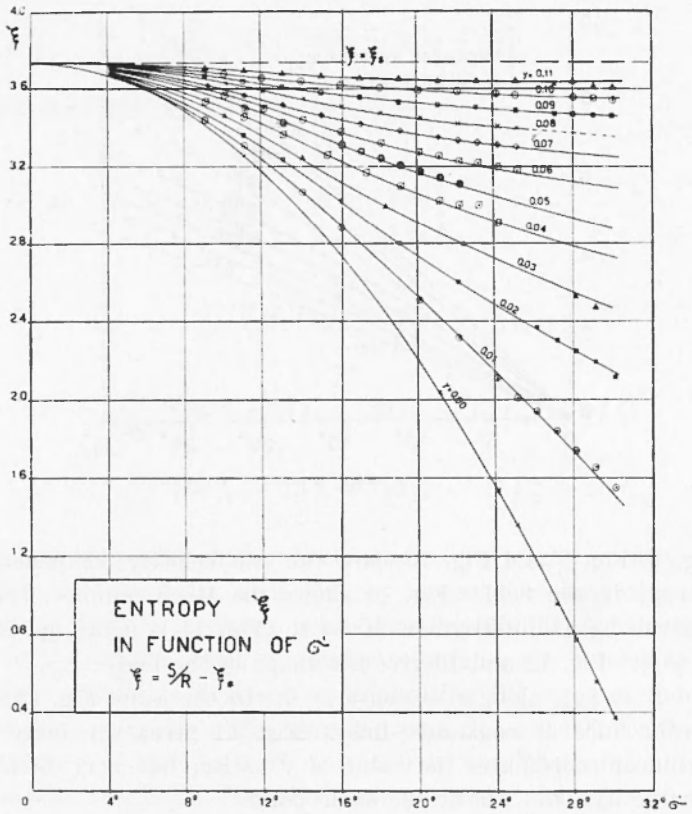


Fig. 10

Figure 17 and 18 represent them in function of r . The integration of the pressure gives a drag coefficient of:

$$C_D = 1.71$$

[65]

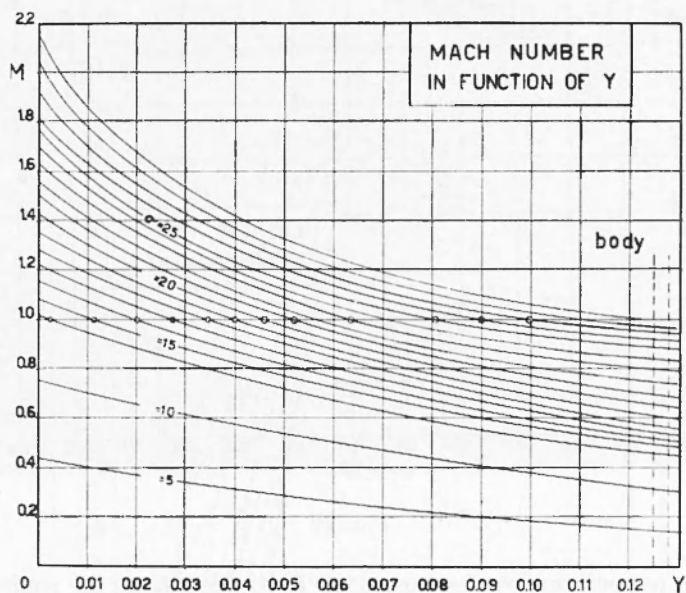


Fig. 11

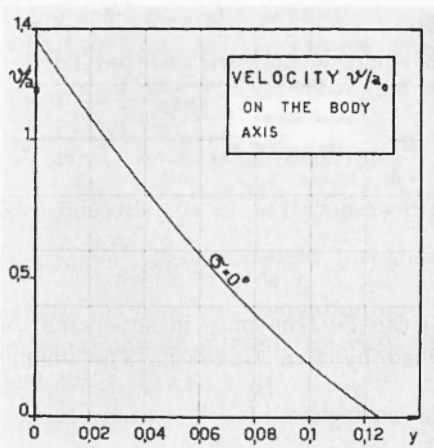


Fig. 12

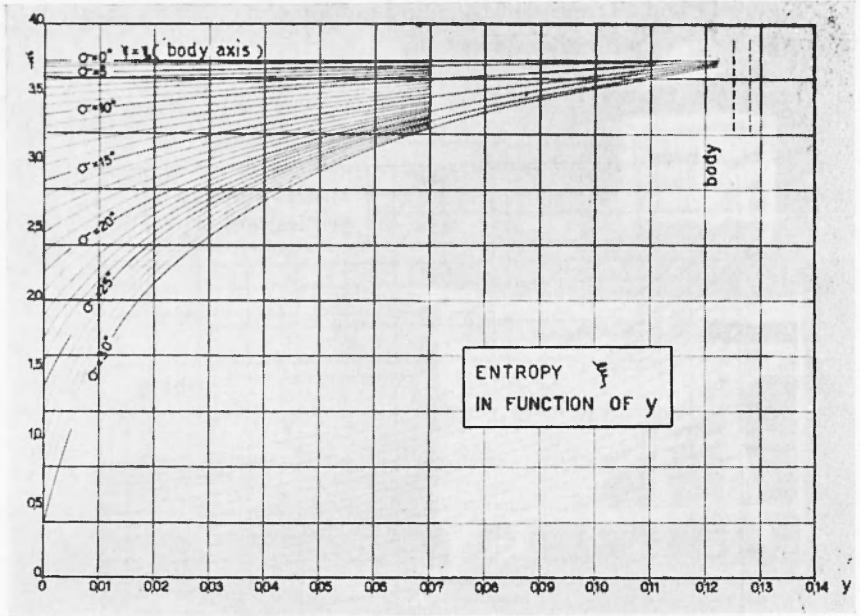


Fig. 13

The temperature varies from $6520^{\circ}K$ to $6390^{\circ}K$ at the sonic point.

VII - DIVISION BY ZERO IN COMPUTING $\frac{\partial v}{\partial y}$ AND $\frac{\partial \xi}{\partial y}$.

If we consider equation [23], we see that if:

$$v = \pm a \tag{66}$$

the coefficient of $\frac{\partial v}{\partial y}$ is zero.

This question does not arise in the subsonic region where, by definition:

$$\sqrt{u^2 + v^2} \leq a$$

Condition [66] can be true only in supersonic regions. The same conclusion is reached by Van Dyke⁽⁸⁾. The analogous term in Van Dyke's work is the coefficient of $\frac{Q_n}{Q}$, namely term D .

Figure 19 below, reproduced from Van Dyke's study, shows line $D = 0$.

Van Dyke in his study seeks an explanation as to whether line $D = 0$ has any simple physical significance.

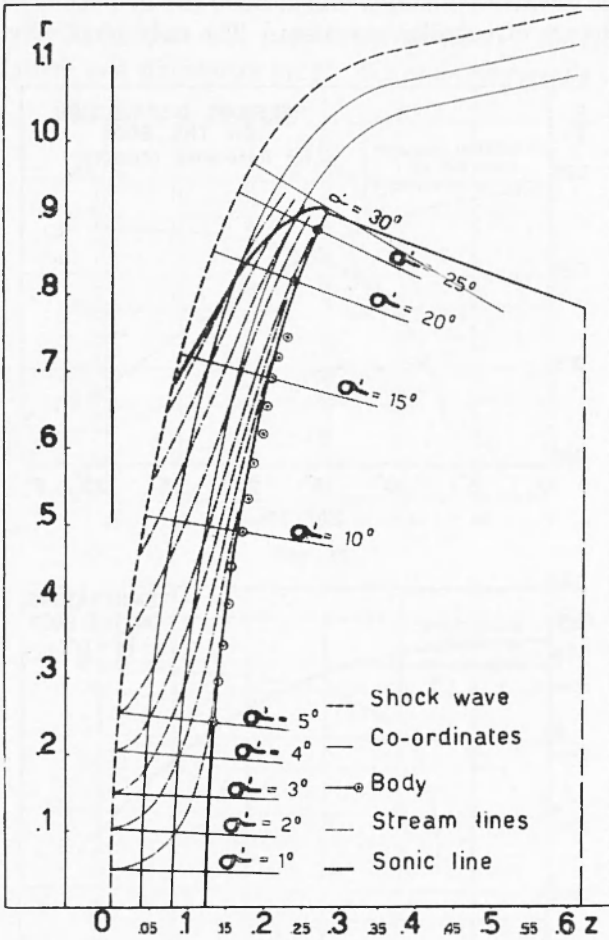


Fig. 14

As the equation for $\frac{\partial v}{\partial y}$ is obtained by introducing a stream function in order to satisfy the continuity equation and by eliminating the pressure from the equations of motion, we may infer that condition $D = 0$ is equivalent to:

$$\left(1 - \frac{v^2}{a^2}\right) = 0 \tag{67}$$

which is the coefficient of $\frac{\partial v}{\partial y}$ in the continuity equation [23]. If this is

so, we may say that the $D = 0$ line has, but not completely, a physical significance: it would be a line along which *one* of the velocity components equals the velocity of sound [67]. Equation [67] is not an invariable one in a change of coordinate system. The only physically significant

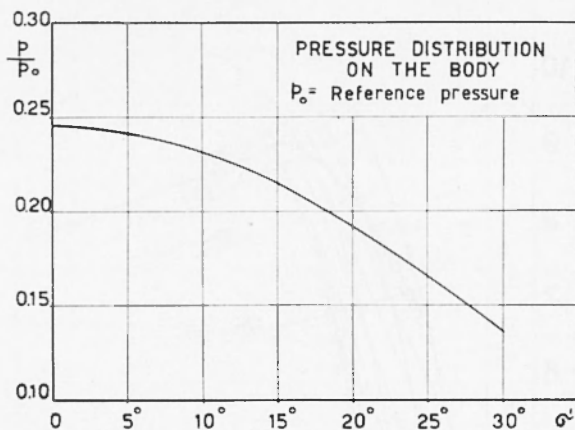


Fig. 15

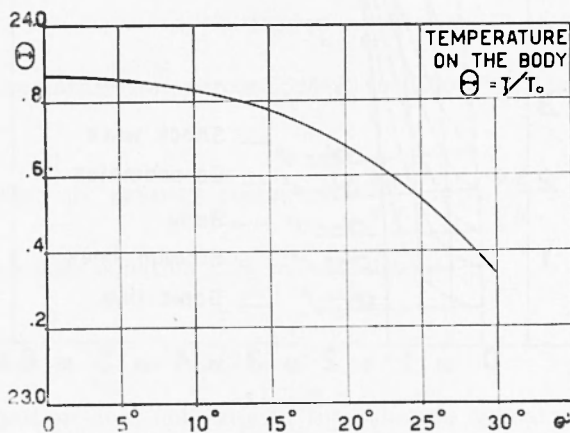


Fig. 16

thing we can infer is that the line in question lies entirely within the supersonic region.

We can conclude that this difficulty arises only when we are well in the supersonic region, where the method of the characteristics must be applied. Moreover, in the case of a very blunt shock shape chosen in order to get a sharp-ended blunt body, as in Fig. 5, the body profile

runs about parallel to the shock profile and component v is about zero along the body.

This condition, on the contrary, is negative if solving equation [25] for entropy. Component v is indeed the coefficient of $\partial\xi/\partial y$: when it vanishes, there is a singularity for ξ . We must necessarily infer that at

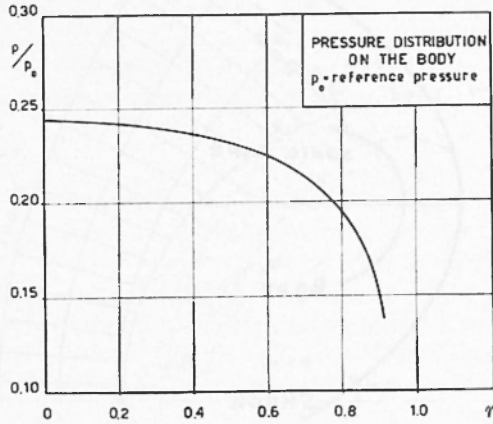


Fig. 17

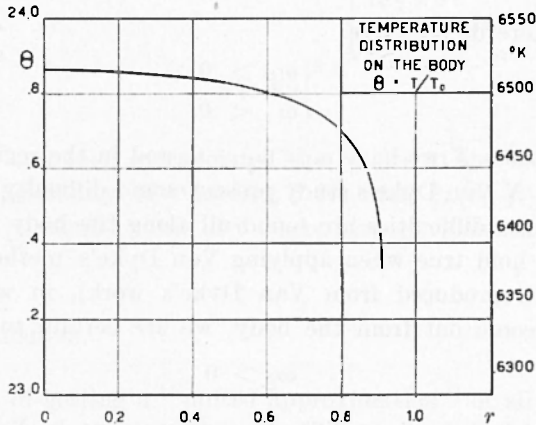


Fig. 18

the same station the term $\partial\xi/\partial x$ vanishes too and the singularity is purely mathematical. If we consider the quoted work of Van Dyke, we see something analogous, the "bad" coefficient being:

$$\left(\omega + \frac{\xi \omega_{\xi}}{2} \right) \tag{68}$$

where ω is related to the stream function ψ : $\omega = \frac{2\psi}{\xi^2}$ and ξ is a curvilinear coordinate.

Van Dyke states that the quantity [68] cannot vanish, except in points such as A (Fig. 20 below) internal to the body profile.

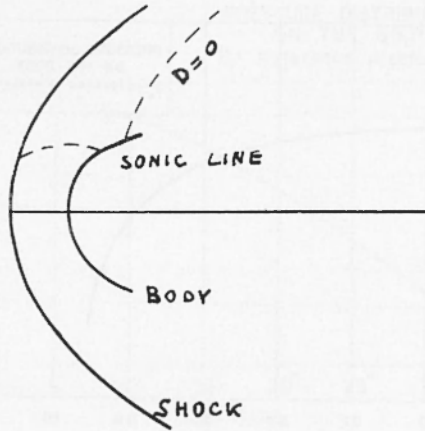


Fig. 19

Only there do we have:

$$\begin{cases} \omega_{\xi} > 0 \\ \omega < 0 \end{cases}$$

On the nose N we have $\omega = \omega_{\xi} = 0$, and in the region of the stagnation point N Van Dyke's study presents some difficulty. In our case, however, these difficulties are found all along the body profile and the same would hold true when applying Van Dyke's method: in the case of Fig. 20 (reproduced from Van Dyke's work), in which the lines $\eta = \text{const}$ come out from the body, we are certain to have:

$$\omega_{\xi} > 0$$

all along the body and no difficulty arises except in N . In a case like that of the present study, in which the body profile is very near to a coordinate line, we should get:

$$\omega \simeq 0$$

along the body, and necessarily, if the $\omega = 0$ line is also an $\eta = \text{const}$ line:

$$\omega_{\xi} \simeq 0$$

so that [68] would vanish very near the body profile (Note that [68] is nothing but ψ_{ξ}/ξ). Nothing has been done in our study to avoid this difficulty; we have smoothed the solution by means of [62], [63] and [64]

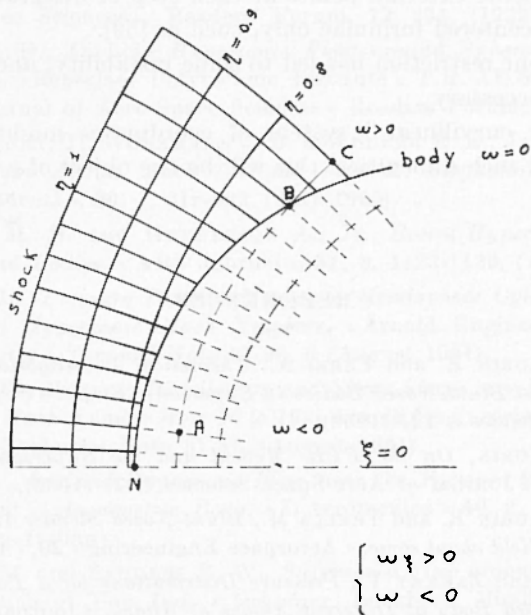


Fig. 20

till $y = 0.122$ and then extrapolated the results to the body. We are studying the possibility of "leading" function $\xi_x(x_1; y)$ (x_1 fixed) to vanish at that particular value of y where $v(x_1; y)$ vanishes.

VIII - CONCLUSIONS.

A method of getting a "united approximation" of all the thermodynamic functions is given. The velocity components and the entropy are taken as unknowns, thus preparing the study of an axisymmetrical body at an angle of attack where neither the potential flow nor the stream function can be considered. Two "divisions by zero" are illustrated and it is shown that one of them certainly cannot occur in the subsonic field: it is then of no importance in connection with this study or similar ones. A way of avoiding the second division by zero is also indicated.

A least squares method is used for differentiating the functions; a similar method is used three times for a smoothing process.

A wider field of Mollier diagram should be approximated in order to dispose of some external points at each step of integration to enable us to employ centered formulae only, such as [59].

The present restriction has led to some instability, and a smoothing process was necessary.

A "polar curvilinear" system of coordinates might be useful in order to avoid such difficulties: this will be the object of a further study.

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