On the linear theory of viscoelastic porous media (*)

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Summary. — This paper is concerned with a reciprocity theorem and a variational theorem in the linear theory of viscoelastic porous media in the quasi-static case.

Riassunto. — In questo lavoro viene trattato un teorema di reciprocità ed un teorema variazionale nell'ambito della teoria lineare dei mezzi porosi viscoelastici nel caso quasi-statico.

Introduction.

The theory of deformation of viscoelastic porous solids containing a viscous fluid has been developed by Biot (1), who has also obtained a variational theorem which leads to practical methods for the treatments of dynamical problems and stress analysis in viscoelastic porous materials.

A viscoelastic porous solid is represented as a viscoelastic skeleton, with a statistical distribution of interconnected pores containing a viscous fluid.

Predelanu (15) has established a reciprocity theorem by making use of the Laplace transforms. Predelanu and Nan (16) have made some applications of such a theorem.

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Ieşan [8,9,10] has established a method by means of which he was able to give reciprocity theorems in the dynamic theory of continua with arbitrary initial conditions without using Laplace transform, in the case of non-homogeneous boundary conditions.

Ieşan [11] has also given a method to obtain variational theorems of Gurtin type [6] using these new reciprocity theorems.

For viscoelastic non-porous media, the reciprocity theorems and the variational theorems have been given by Leitman [13] and Ieşan [12].

Using the Ieşan's method, we have already found a reciprocity theorem and a variational theorem for isotropic and anisotropic porous materials [2,3,4,5].

In this paper we want to give the reciprocity theorem and the variational theorem in the linear theory of viscoelastic porous media.

Basic equations.

Throughout this paper we employ a rectangular coordinate system, Ox_i $(i = 1, 2, 3)$, and the usual indicial notations.

Let $\bar{B}$ be a regular (in the sense of Kellog) region of space occupied by a viscoelastic porous medium, whose boundary is Σ. Moreover B is the interior of $\bar{B}$, n_i are the components of the unit outward normal to Σ.

For convenience and clarity in presentation, all regularity hypotheses on considered functions will be omitted.

On these grounds, the field equations in the linear theory of viscoelastic porous media, for the quasi-stationary case, are:

1) – the constitutive equations:

$$\sigma_{ij} = \int_{-\infty}^{t} [2N(t-\tau)\, \dot{e}_{ij}(\tau) + A(t-\tau)\, \dot{e}(\tau)\, \delta_{ij} + Q(t-\tau)\, \dot{\varepsilon}(\tau)\, \delta_{ij}]\, d\tau, \qquad [1.a]$$

$$\sigma = \int_{-\infty}^{t} [Q(t-\tau)\, \dot{e}(\tau) + R(t-\tau)\, \dot{\varepsilon}(\tau)]\, d\tau, \qquad [1.b]$$

2) – the strain-displacement relations:

$$2\, e_{ij} = u_{i,j} + u_{j,i} \qquad [2]$$

3) – the equations of equilibrium:

$$(\sigma_{ij} + \sigma\delta_{ij})_{,j} + \varrho F_i = 0, \qquad [3]$$

4) – the law governing the fluid flow:

$$k\,\sigma_{,ii} = \dot{\varepsilon} - \dot{e} + W. \qquad [4]$$

In the above equations we have used the following notations: u_i – the components of the displacement vector for the solid phase; σ_{ij}, e_{ij} – the components of the stress and strain tensors of the solid; σ – the hydrostatic state of the fluid; e, ε – the dilatations of the solid and of the fluid, respectively; ϱ – the density of the solid; F_i – the components of the body forces; k – the permeability of the medium; N, A, Q and R – characteristic functions of the viscoelastic medium; W – the output of the fluid source. A comma denotes partial differentiation with respect to space variables x_k, and a superimposed dot denotes partial differentiation with respect to the time t.

Let us also introduce the notion of admissible state $S = \{u_i, e_{ij}, \varepsilon, \sigma_{ij}, \sigma\}$ by which we mean a set of functions u_i, e_{ij}, ε, σ_{ij}, σ defined on $\bar{B} \times (-\infty, \infty)$ and such that the following symmetry relations:

$$e_{ij} = e_{ji}, \qquad \sigma_{ij} = \sigma_{ji},$$

are satisfied.

Since a viscoelastic porous material 'remembers' its past history, we must prescribe the above functions in the body up to some initial time t_0. The initial data consist of the functions $\{\overset{\circ}{u}_i, \overset{\circ}{e}_{ij}, \overset{\circ}{\varepsilon}, \overset{\circ}{\sigma}_{ij}, \overset{\circ}{\sigma}\} = S_0$ defined on $\bar{B} \times (-\infty, t_0)$. Without loss of generality we take $t_0 = 0$. Therefore the initial history condition is:

$$S = S_0, \text{ on } \quad \bar{B} \times (-\infty, 0). \qquad [5]$$

If S is an admissible state which satisfies the initial history condition [5], then automatically e and ε satisfy the following initial conditions:

$$e(x,\ 0) = \lim_{t \to 0^-} \overset{\circ}{e}(x, t) = a(x) \qquad [6.a]$$

$$\varepsilon(x,\ 0) = \lim_{t \to 0^-} \overset{\circ}{\varepsilon}(x, t) = q(x) \qquad [6.b]$$

where a and q are prescribed functions.

Moreover, we must adjoin the boundary conditions (4)

$$\sigma\, n_i = f p_i, \qquad \text{on } \Sigma, \tag{7.a}$$

$$\sigma_{ij}\, n_j = (1 - f)\, p_i, \qquad \text{on } \Sigma, \tag{7.b}$$

where f is a constant, called porosity, defined by Biot (1), and p_i are prescribed functions.

By a solution of the considered problem we mean an admissible state S which satisfies to the field equations [1]-[4], the initial history condition [5] and the boundary conditions [7].

Preliminaries.

Let f, g and h be functions defined on $\overline{B} \times [0, \infty)$, continuous on $[0, \infty)$ with respect to the time t for each $x \epsilon \overline{B}$, we denote by $f{*}g$ the convolution of f and g:

$$[f{*}g]\,(x, t) = \int_0^t f\,(x, t - \tau)\; g\,(x, \tau)\; d\tau$$

We will have occasion to use the following well-known properties of the convolution (7)

$$f{*}g = g{*}f$$

$$f{*}(g{*}h) = (f{*}g){*}h = f{*}g{*}h$$

$$f{*}(g + h) = f{*}g + f{*}h$$

Henceforth we will denote by l the function defined on $[0, \infty)$ by:

$$l(t) = 1 \tag{8}$$

Let Ω be the function defined on $\overline{B} \times [0, \infty)$ by:

$$\Omega = l{*}W - (q - a) \tag{9}$$

It is easy to prove:

Theorem 1. If the functions u_i, σ satisfy the equations [4] and the initial conditions [6], then:

$$kl{*}\sigma_{,ii} = \varepsilon - e + \Omega. \tag{10}$$

We will have also occasion to use the "convolution scalar product" defined as ([14])

$$(f \circledcirc g) = \int_B f*g \ \mathrm{d}B. \qquad [11]$$

If we introduce the notations:

$$\bar{\sigma}_{ij} = 2 \int_{-\infty}^{0} \overset{\circ}{N}(t-\tau)\, \overset{\circ}{e}_{ij}(\tau)\, \mathrm{d}\tau + \delta_{ij} \int_{-\infty}^{0} [A\,(t-\tau)\, \overset{\circ}{e}\,(\tau) + Q\,(t-\tau)\, \overset{\circ}{\varepsilon}\,(\tau)]\mathrm{d}\tau,$$

$$\bar{\sigma} = \int_{-\infty}^{0} [Q\,(t-\tau)\, \overset{\circ}{e}\,(\tau) + R\,(t-\tau)\, \overset{\circ}{\varepsilon}\,(\tau)]\, \mathrm{d}\tau \qquad [12]$$

the equations [1] can be written as:

$$\sigma_{ij} = \bar{\sigma}_{ij} + \frac{\mathrm{d}}{\mathrm{d}t}\,(2\,N*e_{ij} + A*e\delta_{ij} + Q*\varepsilon\delta_{ij}), \qquad [13.a]$$

$$\sigma = \bar{\sigma} + \frac{\mathrm{d}}{\mathrm{d}t}\,(Q*e + R*\varepsilon). \qquad [13.b]$$

THE RECIPROCITY THEOREM.

We consider the body subjected to two different systems of loadings:

$$L^{(\alpha)} = \left\{ F_i^{(\alpha)},\ p_i^{(\alpha)},\ W^{(\alpha)},\ S_0^{(\alpha)} \right\},\ \alpha = 1,\ 2, \qquad [14]$$

and the two corresponding configurations:

$$C^{(\alpha)} = \left\{ u_i^{(\alpha)},\ e^{(\alpha)},\ \varepsilon^{(\alpha)},\ \sigma_{ij}^{(\alpha)},\ \sigma^{(\alpha)} \right\},\ \alpha = 1,\ 2. \qquad [15]$$

Theorem 2. If a viscoelastic porous material is subjected to two different systems of loadings [14], then between the two corresponding configurations [15], there is the following reciprocity relation:

$$\int_\Sigma (p_i^{(1)} + \bar{p}_i^{(1)})*u_i^{(2)}\, \mathrm{d}\Sigma + kf \int_\Sigma l*(p_i^{(1)} + \bar{q}_i^{(1)})*{,}\sigma_i^{(2)}\, \mathrm{d}\Sigma +$$

$$+ \int_B \varrho\,(F_i^{(1)} + \bar{F}_i^{(1)})*u_i^{(2)}\, \mathrm{d}B + k \int_B l*\bar{\sigma}_{,i}^{(1)}*\sigma_{,i}^{(2)}\, \mathrm{d}B -$$

$$-\int_B (\sigma^{(1)} - \bar{\sigma}^{(1)}) * \Omega^{(2)} \mathrm{d}B = \int_\Sigma (p_i^{(2)} + \bar{p}_i^{(2)}) * u_i^{(1)} \, \mathrm{d}\Sigma +$$

$$+ kf \int_\Sigma l * (p_i^{(2)} + \bar{q}_i^{(2)}) * \sigma,_i^{(1)} \, \mathrm{d}\Sigma + \int_B \varrho \, (F_i^{(2)} + \bar{F}_i^{(2)}) * u_i^{(1)} \, \mathrm{d}B +$$

$$+ k \int_B l * \sigma,_i^{(2)} * \sigma,_i^{(1)} \, \mathrm{d}B - \int_B (\sigma_i^{(2)} - \sigma^{(2)}) * \Omega^{(1)} \mathrm{d}B \qquad [16]$$

where:

$$\bar{p}_i^{(\alpha)} = -(\bar{\sigma}_{ij}^{(\alpha)} + \bar{\sigma}^{(\alpha)} \delta_{ij}) \, n_j, \qquad [17]$$

$$\bar{q}_i^{(\alpha)} = -(1/f) \, \bar{\sigma}^{(\alpha)} \, n_i, \qquad [18]$$

$$\varrho \, F_i^{(\alpha)} = (\sigma_{ij}^{(\alpha)} + \sigma^{(\alpha)} \delta_{ij}),_j \qquad [19]$$

Proof. From equations [13], we can write:

$$\sigma_{ij}^{(1)} - \bar{\sigma}_{ij}^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t} (Q * \varepsilon^{(1)}) \, \delta_{ij} = \frac{\mathrm{d}}{\mathrm{d}t} (2 \, N * e_{ij}^{(1)} + A * e^{(1)} \, \delta_{ij}) \quad [20.a]$$

$$\sigma_{ij}^{(2)} - \bar{\sigma}_{ij}^{(2)} - \frac{\mathrm{d}}{\mathrm{d}t} (Q * \varepsilon^{(2)}) \, \delta_{ij} = \frac{\mathrm{d}}{\mathrm{d}t} (2 \, N * e_{ij}^{(2)} + A * e^{(2)} \, \delta_{ij}) \quad [20.b]$$

$$\sigma^{(1)} - \bar{\sigma}^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t} (Q * e^{(1)}) = \frac{\mathrm{d}}{\mathrm{d}t} (R * \varepsilon^{(1)}) \qquad [21.a]$$

$$\sigma^{(2)} - \bar{\sigma}^{(2)} - \frac{\mathrm{d}}{\mathrm{d}t} (Q * e^{(2)}) = \frac{\mathrm{d}}{\mathrm{d}t} (R * \varepsilon^{(2)}) \qquad [21.b]$$

By making the convolution product of both members of every equation [20] and [21] with $e_{ij}^{(2)}$, $e_{ij}^{(1)}$, $\varepsilon^{(2)}$ and $\varepsilon^{(1)}$, respectively, we get:

$$[\sigma_{ij}^{(1)} - \bar{\sigma}_{ij}^{(1)} - \delta_{ij} \frac{\mathrm{d}}{\mathrm{d}t} (Q * \varepsilon^{(1)})] * e_{ij}^{(2)} = e_{ij}^{(2)} * \frac{\mathrm{d}}{\mathrm{d}t} (2 \, N * e_{ij}^{(1)}) +$$

$$+ e^{(2)} * \frac{\mathrm{d}}{\mathrm{d}t} (A * e^{(1)}) \qquad [22.a]$$

$$[\sigma_{ij}^{(2)} - \bar{\sigma}_{ij}^{(2)} - \frac{\mathrm{d}}{\mathrm{d}t} (Q * \varepsilon^{(2)}) \, \delta_{ij}] * e_{ij}^{(1)} = e_{ij}^{(1)} * \frac{\mathrm{d}}{\mathrm{d}t} (2 \, N * e_{ij}^{(2)}) +$$

$$e^{(1)} * \frac{\mathrm{d}}{\mathrm{d}t} (A * e^{(2)}) \qquad [22.b]$$

$$[\sigma^{(1)} - \bar{\sigma}^{(1)} - \frac{\mathrm{d}}{\mathrm{d}t} (Q * e^{(1)})] * \varepsilon^{(2)} = \varepsilon^{(2)} * \frac{\mathrm{d}}{\mathrm{d}t} (R * \varepsilon^{(1)}) \qquad [23.a]$$

$$[\sigma^{(2)} - \bar{\sigma}^{(2)} - \frac{d}{dt}(Q*e^{(2)})]*\varepsilon^{(1)} = \varepsilon^{(1)}*\frac{d}{dt}(R*\varepsilon^{(2)}) \qquad [23.b]$$

It is easy to show:

$$e_{ij}^{(1)}*\frac{d}{dt}(N*e_{ij}^{(2)}) = e_{ij}^{(2)}*\frac{d}{dt}(N*e_{ij}^{(1)}), \qquad [24.a]$$

$$e^{(1)}*\frac{d}{dt}(A*e^{(2)}) = e^{(2)}*\frac{d}{dt}(A*e^{(1)}), \qquad [24.b]$$

$$\varepsilon^{(1)}*\frac{d}{dt}(R*\varepsilon^{(2)}) = \varepsilon^{(2)}*\frac{d}{dt}(R*\varepsilon^{(1)}). \qquad [23.c]$$

Taking into account the relation [24], from equations [22] and [23] we get:

$$(\sigma_{ij}^{(1)} - \bar{\sigma}_{ij}^{(1)})*e_{ij}^{(2)} - \frac{d}{dt}(Q*\varepsilon^{(1)})*e^{(2)} = (\sigma_{ij}^{(2)} - \bar{\sigma}_{ij}^{(2)})*e_{ij}^{(1)} -$$
$$- \frac{d}{dt}(Q*\varepsilon^{(2)})*e^{(1)} \qquad [25]$$

$$(\sigma^{(1)} - \bar{\sigma}^{(1)})*\varepsilon^{(2)} - \frac{d}{dt}(Q*e^{(1)})*\varepsilon^{(2)} = (\sigma^{(2)} - \bar{\sigma}^{(2)})*\varepsilon^{(1)} -$$
$$- \frac{d}{dt}(Q*e^{(2)})\varepsilon^{(1)} \qquad [26]$$

By adding these two relations, we have:

$$(\sigma_{ij}^{(1)} - \bar{\sigma}_{ij}^{(1)})*e_{ij}^{(2)} + (\sigma^{(1)} - \bar{\sigma}^{(1)})*\varepsilon^{(2)} =$$
$$= (\sigma_{ij}^{(2)} - \bar{\sigma}_{ij}^{(2)})*e_{ij}^{(1)} + (\sigma^{(2)} - \bar{\sigma}^{(2)})*\varepsilon^{(1)}. \qquad [27]$$

If we introduce the notation:

$$I_{\alpha\beta} = \int_B [(\sigma^{(\alpha)}{}_{ij} - \bar{\sigma}^{(\alpha)}{}_{ij})*e_{ij} + (\sigma^{(\alpha)} - \bar{\sigma}^{(\alpha)})*\varepsilon(\beta)]\, dB;\ \alpha, \beta = 1, 2, \qquad [28]$$

from the relation [27] we obtain:

$$I_{12} = I_{21}. \qquad [29]$$

Using the equations [2], [3], [10], [17]-[19], we find:

$$I_{\alpha\beta} = \int_\Sigma (p_i^{(\alpha)} + \bar{p}_i^{(\alpha)})*u_i(\beta)\, d\Sigma + kf \int_\Sigma l*(p_i^{(\alpha)} + \bar{q}_i^{(\alpha)})*\sigma,_i(\beta) +$$

$$+ \int_B (\varrho F_i^{(\alpha)} + \varrho F_i^{(\alpha)}) * u_i^{(\beta)} \, dB - k \int_B l * \sigma,_i^{(\alpha)} * \sigma,_i^{(\beta)} \, dB + \qquad [30]$$

$$+ \int_B l * \bar{\sigma},_i^{(\alpha)} * \sigma,_i^{(\beta)} \, dB ;$$

therefore the reciprocity relation [16] is obtained from [29] and [30], and the theorem is then proved.

If we assume $S_0^{(\alpha)} = 0$, the relation [16] becomes.

$$\int_\Sigma p_i^{(1)} * u_i^{(2)} \, d\Sigma + kf \int_\Sigma l * p_i^{(1)} * \sigma,_i^{(2)} \, d\Sigma + \int_B \varrho F_i^{(1)} * u_i^{(2)} \, dB -$$

$$- \int_B \sigma^{(1)} * \Omega^{(2)} \, dB = \int_\Sigma p_i^{(2)} * u_i^{(1)} \, d\Sigma + kf \int_\Sigma l * p_i^{(2)} * \sigma,_i^{(1)} \, d\Sigma +$$

$$+ \int_B \varrho F_i^{(2)} * u_i^{(1)} \, dB - \int_B \sigma^{(2)} * \Omega^{(1)} dB . \qquad [31]$$

In the case of homogeneous boundary conditions, equation [31] leads to:

$$\int_B (\varrho F_i^{(1)} * u_i^{(2)} - \sigma^{(1)} * \Omega^{(2)}) \, dB = \int_B (\varrho F_i^{(2)} * u_i^{(1)} - \sigma^{(2)} * \Omega^{(1)}) \, dB . \qquad [32]$$

Furthermore we have:

$$\Omega^{(\alpha)} = l * W^{(\alpha)} ; \qquad [33]$$

and equation [32] becomes:

$$\int_B (\varrho F_i^{(1)} * u_i^{(2)} - l * \sigma^{(1)} * W^{(2)}) \, dB = \int_B (\varrho F_i^{(2)} * u_i^{(1)} - l * \sigma^{(2)} * W^{(1)}) \, dB \qquad [34]$$

Some applications of the reciprocity theorem.

Let us firstly consider applications of equation [31] which corresponds to the case with $S_0 = 0$.

Let us assume that the following loading system:

$$F_j^{(1)} = \delta \, (x - \xi) \; \delta(t) \; \delta_{ij} ,$$

$$p_i^{(1)} = W^{(1)} = a^{(1)} = q^{(1)} = 0, \qquad [35]$$

where δ is the Dirac measure, will generate the following functions:

$$u_j^{(i)}(x, \xi, t); \ \sigma^{(i)}(x, \xi, t) \qquad [36]$$

The Krönecker symbol which appears in [35] expresses the fact that only the i — th componet of the force is $\neq 0$. The indices between parenteses in [36] mean that u_j and σ are quantities corresponding to such a force. Then [31] gives

$$\varrho u_j(\xi, t) = \int_\Sigma p_i * u_i^{(j)} \, d\Sigma + \int_B \varrho F_i * u_i^{(j)} \, dB + + \int_B \Omega * \sigma^{(j)} \, dB + kf \int_\Sigma l * p_i * \sigma,_i^{(j)} \, d\Sigma \qquad [37]$$

Let us now assume that the following loading system:

$$W^{(1)} = \delta(x - \xi) \, \delta(t)$$
$$F_j^{(1)} = p_j^{(1)} = a^{(1)} = q^{(1)} = 0 \qquad [38]$$

will generate the following functions:

$$v_i(x, \xi, t); \quad \gamma(x, \xi, t) \qquad [39]$$

Then the relation [31] gives:

$$\sigma(\xi, t) = \frac{d}{dt} \left\{ \int_\Sigma p_i * v_i \, d\Sigma + kf \int_\Sigma l * p_i * \gamma,_i \, d\Sigma + + \int_B \varrho F_i * v_i \, dB + \int_B \Omega * \gamma \, dB \right\}.$$

Let us, finally, assume that the following loading system:

$$q^{(1)} = \delta(x - \xi) \qquad [40]$$
$$F_i^{(1)} = p_i^{(1)} = W^{(1)} = a^{(1)} = 0$$

will generate the functions:

$$w_i(x, \xi, t); \quad \pi(x, \xi, t). \qquad [41]$$

Then from the relation [9], it follows that:

$$\Omega^{(1)} = - q^{(1)} = - \delta(x - \xi)$$

and, from equation [31], we obtain:

$$\sigma(\xi, t) = -\left\{ \int_{\Sigma} p_i * w_i \, d\Sigma + kf \int_{\Sigma} l * p_i * \pi_{,i} \, d\Sigma + \right.$$

$$\left. + \int_{B} \varrho \, F_i * w_i \, dB + \int_{B} \Omega * \pi \, dB \right\} . \qquad [42]$$

Let us now consider an application of equation [16] corresponding to the case $S_o \neq 0$. Let us suppose that the following loading system:

$$F_i^{(1)} = p_i^{(1)} = \Omega^{(1)} = 0, \quad S_o \neq 0, \qquad [43]$$

with, in particular

$$\overline{F}_j^{(1)} = \delta(x - \xi)\, \delta(t), \qquad [44]$$

will generate the functions:

$$s_j^{(i)}(x, \xi, t), \; \vartheta^{(i)}(x, \xi, t). \qquad [45]$$

Then equation [16] gives:

$$\varrho u_j(\xi, t) = -\int_{\Sigma} \bar{p}_i^{(1)} * u_i^{(i)} \, d\Sigma - kf \int_{\Sigma} l * \bar{\gamma}_i^{(1)} * \sigma_i^{(j)} \, d\Sigma -$$

$$- k \int_{B} l * \vartheta_{,i} * \sigma_{,i}^{(j)} \, dB + \int_{B} (\vartheta^{(j)} - \bar{\vartheta}) * \Omega \, dV + \int_{\Sigma} (p_i + p_i) * s_i^{(j)} \, d\Sigma +$$

$$+ kf \int_{\Sigma} l * (p_i - \bar{\gamma}_i) * \vartheta_{,i}^{(j)} \, d\Sigma + \int_{B} \varrho \, (F_i - \overline{F}_i) * s_i^{(j)} \, dB +$$

$$+ k \int_{B} l * \bar{\sigma}_{,i} * \vartheta_{,i}^{(j)} \, dB . \qquad [46]$$

THE VARIATIONAL THEOREM.

We want firstly to write the field equations in terms only of the components of the displacement vector, u_i, of the solid and of the hydrostatic state of stress, σ, of the liquid.

From equations [1.b] and [8], we get:

$$l * (\sigma - \bar{\sigma}) = Q * e + R * \varepsilon. \qquad [47]$$

It is easy to prove:

Theorem 3. Let M be the set of all functions $\{u_i, \sigma\}$, in which σ is such that the initial conditions [6b] are satisfied. Then the functions of M satisfy the equations [4] and the initial conditions [6], if and only if the relation [10] is satisfied.

From the relation [10], we get:

$$kR*l*\sigma_{,ii} = R*\varepsilon - R*e + R*\Omega \qquad [48]$$

From equations [47] and [48], we obtain:

$$l* \left\{kR*\sigma_{,ii} - (\sigma - \bar{\sigma})\right\} + (Q + R)*e = R*\Omega. \qquad [49]$$

From equations [13.a] and [47], we obtain:

$$l*R*(\sigma_{ij} - \bar{\sigma}_{ij}) = 2\, N*R*e_{ij} + (A*R - Q*Q)*e\delta_{ij} + Q*l*(\sigma - \bar{\sigma})\,\delta_{ij}. \qquad [50]$$

From equation [3], we have:

$$l*R*\sigma_{ij,j} + l*R*\sigma_{,i} + \varrho l*R*F_i = 0. \qquad [51]$$

Then, from [50] and [51], we can write:

$$2\, N*R*e_{ij,j} + (A*R - Q*Q)*e_{,i} + Q*l*(\sigma_{,i} - \bar{\sigma}_{,i}) + l*R*\bar{\sigma}_{ij,j} +$$
$$+ l*R*\sigma_{,i} + \varrho l*R*F_i = 0.$$

This relation may be written as:

$$N*R*u_{i,jj} + (N*R + A*R - Q*Q)*u_{r,ri} + l*(Q + R)*\sigma_{,i} +$$
$$+ \varrho l*R*F_i + l*R*\bar{\sigma}_{ij,j} - l*Q*\sigma_{,i} = 0. \qquad [52]$$

In what follows without loss of generality we consider the case in which $S_0 = 0$. In this case the field equations become:

$$l* \left\{kR*\sigma_{,ii} - \sigma\right\} + (Q + R)*u_{s,s} = R*\Omega \qquad [53]$$

$$N*R*u_{i,jj} + (N*R + A*R - Q*Q)*u_{r,ri} +$$
$$+ l*(Q + R)*\sigma_{,i} + \varrho l*R*F_i = 0. \qquad [54]$$

The reciprocity theorem, given by equation [32], with the equations [53] and [54], leads to:

$$\int_B l*R*(\varrho F_i^{(1)}*u_i^{(2)} + \Omega^{(1)}*\sigma^{(2)})\, \mathrm{d}B =$$
$$= \int_B l*R*(\varrho F_i^{(2)}*u_i^{(1)} + \Omega^{(2)}*\sigma^{(1)})\, \mathrm{d}B. \qquad [55]$$

The equation [53] can be written as:

$$g* \left\{kR*\sigma,_{ll} - \sigma\right\} + l*(Q + R)*u_{s,s} = l*R*\Omega, \qquad [56]$$

where the function g is defined by:

$$g = l*l = t.$$

Let us, now, introduce the notations:

$$A_i U = - N*R*u_{i,jj} - (N*R + A*R - Q*Q)*u_{r,ri} - l*(Q + R)*\sigma,_i \qquad [57]$$

$$A_4 U = g* \left\{kR*\sigma,_{ll} - \sigma\right\} + l*(Q + R)*u_{s,s}, \qquad [58]$$

where

$$U = \left\{u_i,\sigma\right\}. \qquad [59]$$

With these notations the equations [54] and [56] can be written in the form:

$$\mathcal{A}U = F \qquad [60]$$

where

$$F = \left\{\varrho l*R*F_i,\ l*R*\Omega\right\} \qquad [61]$$

$$\mathcal{A}U = \left\{A_i U,\ A_4 U\right\} \qquad [62]$$

Also equation [55] by the notations [57] and [58] can be expressed in another form. Let us write:

$$U = \left\{u_i^{(1)},\ \sigma^{(1)}\right\},\ V = \left\{u_i^{(2)},\ \sigma^{(2)}\right\}$$

then, from equations [60], [61] and [62], we get:

$$A_i U = \varrho\, l*R*F_i^{(1)};\ A_4 U = e*R*\Omega^{(1)}$$

$$A_i V = \varrho\, l*R*F_i^{(2)};\ A_4 V = l*R*\Omega^{(2)}.$$

Then the reciprocity theorem [55] becomes:

$$\int_B \mathcal{A}U*V\mathrm{d}B = \int_B U*\mathcal{A}V\mathrm{d}B \qquad [63]$$

or, taking the relation [11] into account, we can write:

$$(\mathcal{A}U \circledcirc V) = (U \circledcirc \mathcal{A}V) \qquad [64]$$

This relation shows that the operator $\mathcal{A}$ is symmetrical in convolution.

Let us consider the functional:

$$\mathcal{F}(u) = (\mathcal{A}U \circledcirc U) - 2\,(U \circledcirc F). \qquad [65]$$

If the operator $\mathcal{A}$ is symmetric in convolution, then:

$$\delta\mathcal{F}(U) = 0, \text{ on } D_a \qquad [66]$$

if and only if $U \in D_a$ satisfies to the equation [60]. D_a is the domain of definition of the operator $\mathcal{A}$.

Let us now consider the vectors $U \equiv \{u_i, \sigma\}$ and $V \equiv \{v_i, \eta\}$. In the case of homogeneous boundary conditions, we have:

$$(\mathcal{A}U \circledcirc V) = \int_B l*R*(\sigma_{ij} + \sigma\delta_{ij})*v_{i,j}\,dB - \int_B g*kR*\sigma_{,i}*\eta_{,i}\,dB +$$
$$+ \int_B g*\sigma*\eta\,dB + \int_B l*(Q + R)*l*\eta\,dB \qquad [67]$$

Moreover, from equation [67], we have:

$$(\mathcal{A}U \circledcirc V) = \int_B l*R*(\sigma_{ij} + \sigma\delta_{ij})*u_{i,j}\,dB + \int_B l*(Q + R)*u_{r,r}*\sigma\,dB +$$
$$+ \int_B g*\sigma*\sigma\,dB - \int_B g*kR*\sigma_{,i}*\sigma_{,i}\,dB \qquad [68]$$

In this case, we can write:

$$l*R*\sigma_{ij} = 2\,N*R*e_{ij} + (A*R - Q*Q)*e\,\delta_{ij} + Q*l*\sigma\delta_{ij} \qquad [69]$$

Therefore equation [68] becomes:

$$(\mathcal{A}U \circledcirc U) = \int_B \{[N*R*(u_{i,j} + u_{j,i}) + (A*R - Q*Q)*u_{s,s}\delta_{ij} +$$
$$+ Q*l*\sigma\delta_{ij}]*u_{i,j} + l*R*\sigma*u_{r,r} + l*(Q + R)*u_{r,r}*\sigma +$$
$$+ g*\sigma*\sigma - g*kR*\sigma_{,i}*\sigma_{,i}\}\,dB \qquad [70]$$

In our case, the functional [65] has the expression:

$$\Phi(U) = (\mathcal{A}U \circledcirc U) - 2\int_B \{u_i*\varrho l*R*F_i + \sigma*l*R*\Omega\}\,dB. \qquad [71]$$

Thus we get:

Theorem 4. Let $m \subset M$ be the set of all vectors $U = \{u_i, \sigma\}$, which satisfy the homogeneous boundary conditions. Then

$$\delta\Phi\,(U) = 0, \text{ on } m$$

if and only if $U \in m$ is a solution of the considered problem [60].

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