On linear internal waves on the sea, strongly vertically trapped

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Summary. — We study some explicit cases of marine thermocline. We focus our attention on the strongly vertically trapped internal waves, which in our cases allow an explicit dispersion relation and a simple behaviour in terms of elementary functions. The explicit form of the Väisälä-Brunt frequency $N^2(z)$ is proportional to $1/|z-z_0|$ in one case and to $A^2-B^2(z-z_0)^2$ in the other. A comparison with some experimental data concerning the Ligurian Sea is actually in course.

Riassunto. — In relazione a determinate condizioni di superficie, la struttura verticale del mare si caratterizza mediante una brusca variazione nella densità. Nel presente lavoro vengono studiate le onde interne che vi risultano fortemente intrappolate, ottenendo relazione di dispersione, velocità di gruppo e correlazione in termini di funzioni elementari per due situazioni sperimentali individuabili analiticamente attraverso la frequenza di Väisälä-Brunt $N^2(z)$ proporzionale a $1/|z-z_0|$ in un caso ed uguale a $A^2-B^2(z-z_0)^2$ nell'altro. È in corso un confronto con i dati provenienti da campagne di misura effettuate nel Mar Ligure.

1. – Introduction

An interesting problem in the energy balance of a sea, concerns the internal waves and their energies. These are waves which propagate horizontally and their largest amplitude is related to the vertical variations of the density $\varrho(z)$. This is due to the vertical variations of temperature and salinity, which are originated by the intense ex-

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change of moment and heat with the mowing atmosphere. More in detail, one could first separate a "mixed layer" which has a vertical extension of some tens of meters under the air-sea surface. This layer is mixed by the turbulence propagating downward from the moving atmosphere (this is one of the effects of winds, storms, atmospheric turbulence etc.). Its energy propagates downward freely and makes temperature, salinity, density and motion-homogeneous in this "mixed layer". The surface under this mixed layer is the natural domain of propagation for the most intense internal waves. It has however to be remarked that it is not very easy to distinguish what percentage of atmospheric energy generates internal waves and what percentage of energy is used by the system to erode the underlying stratified region. Practically, moreover, one can remark that many times the mixed layer has not a very sharp division with the underlying stratified region, but has a vertical extension of $50 \div 500$ meters. The resulting periodic phenomena, the internal waves, are in this case related to a smoother variation of the density than in the case of the sharp division between the mixed layer and the deeper stratified region.

Its stratification is a rather curious phenomenon: one can experimentally remark many sheets of an horizontal extension of kilometers and this is a surprising contrast with the vertical extension of few centimeters. Practically, forgetting this "fine-structure", one could see it as a stable region of slowly varying density $\varrho_0(z)$.

In the lowest part, $\varrho_0(z)$ decreases with the depth as a slow exponential $\exp(-\alpha z)$, where α is a constant.

In this contest, we have remarked that the surface between the mixed layer and the stratified thermocline is the domain of many interesting and important phenomena, related to the internal waves distribution of energy inside the fluid ([6]). It can be shown, more precisely, that the knowledge of exact shape of $\varrho_0(z)$, the static density profile, could give essential informations concerning the internal waves structure, their correlations and their energetics.

More explicitly, it has to be added that the experimental evidence stresses that this is not really a sharp surface, but it appears rather as a vertical region of transition between the homogeneous mixed layer and the stratified thermocline. This appears interesting because the internal waves can be described by a simple equation (if one assumes the linear waves and if the Boussinesq approximation is assumed valid: see, for example, Phillips ([6]) and Thorpe ([7])) where the explicit shape of $\varrho_0(z)$ plays an explicit role.

Now, this equation has simple dispersion relation and solution in some cases rather well known in the literature. These are supplied when the Vaisala-Brunt frequency

$$N^2(z) = -\frac{\mathrm{d}\varrho_0}{\mathrm{d}z} \cdot \frac{g}{\varrho_0} > 0$$

has a $\delta(z)$ behaviour or $N^2(z) = \text{const.}$ behaviour (Phillips [6]) and when

$$N^2(z) = \begin{cases} 0 & \text{for } 0<z<d \\ e^{-az} & \text{for } z>d \end{cases} \quad \begin{array}{l} (d \text{ is the depth of the discontinuity} \\ \text{in the density}) \end{array}$$

as in the classical analysis of Garret and Munk [3] and in few other cases studied by Thorpe [8].

The realistic cases are rather different; one could easily solve them numerically, but this would imply some loss of informations. For this region we have studied two rather realistic profiles

$$N^2(z) = a^2/|z-z_0| \quad \text{and} \quad N^2(z) - A^2 - B^2(z-z_0)^2$$

These profiles can, in some cases, simulate correctly the experimental situation and they also allow an explicit calculation of the internal waves, their dispersion relation, their group and wave velocities, their correlations.

An experimental verification is actually in course [1].

2. Linear theory of marine internal waves

The theory of internal waves is rather well known (Phillips [6], Thorpe [8]). In the following we will repeat the essential results on the time evolution of these waves.

In fact, in the case that the Boussinesq approximation can be assumed and the earth's rotation can be disregarded, the velocity components satisfy the equations [6,7].

$$\partial_t u + \frac{1}{\varrho_0}\frac{\partial p}{\partial x} = 0$$

$$\partial_t v + \frac{1}{\varrho_0}\frac{\partial p}{\partial y} = 0$$

$$\partial_t w + \frac{1}{\varrho_0}\frac{\partial p}{\partial z} + g\frac{\varrho'}{\varrho_0} = 0$$

where p is the departure from the hydrostatic pressure and one has assumed

$$\varrho = \varrho_0(z) + \varrho'(x, y, z, t)$$

with the density variation $\varrho' \ll \varrho_0$.

In the same frame of approximation, one also assumes that the water is incomprensible:

$$\partial_x u + \partial_y v + \partial_z w = 0$$

$$\frac{d\varrho}{dt} = \frac{\partial\varrho'}{\partial t} + u\frac{\partial\varrho'}{\partial x} + v\frac{\partial\varrho'}{\partial y} + w\frac{\partial\varrho_0}{\partial z} = 0$$

Calculating first the time derivative of the vorticity, one has

$$\partial^2_{tt}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial t}\frac{g}{\varrho_0}\frac{\partial\varrho'}{\partial x} = 0$$

$$\partial^2_{tt}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial t}\frac{g}{\varrho_0}\frac{\partial\varrho'}{\partial y} = 0$$

and, taking into account the $\frac{d\varrho}{dt} = 0$ relation, one has

$$\frac{\partial}{\partial t}\frac{g}{\varrho_0}\frac{\partial\varrho'}{\partial x} = \frac{g}{\varrho_0}\frac{\partial}{\partial x}\frac{\partial}{\partial t}\varrho' = -\frac{g}{\varrho_0}\frac{d\varrho_0}{dz}\frac{\partial w}{\partial z}$$

$$\frac{\partial}{\partial t}\frac{g}{\varrho_0}\frac{\partial\varrho'}{\partial y} = \frac{g}{\varrho_0}\frac{\partial}{\partial y}\frac{\partial}{\partial t}\varrho' = -\frac{g}{\varrho_0}\frac{d\varrho_0}{dz}\frac{\partial w}{\partial y}$$

Taking then the horizontal divergence, one has the basic relation

$$\partial^2_{tt}(\partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz})\, w(\underline{x}, z, t) + N^2(z)(\partial^2_{xx} + \partial^2_{yy})\, w(\underline{x}, z, t) = 0 \quad [2.1]$$

where $N^2(z) = -\frac{g}{\varrho_0}\frac{d\varrho_0}{dz} > 0$ is the Väisälä-Brunt frequency.

Assuming a plane progressive wave solution of the form

$$w(\underline{x}, z, t) = W(z)\exp i(K_x x + K_y y + K_z z - \omega t)$$

one easily arrives to the equation

$$\frac{d^2W(z)}{dz^2} + \left(\frac{N^2(z)}{\omega^2}K^2 - K^2\right)W(z) = 0 \quad [2.2]$$

with

$$W(0) = 0 \qquad \text{at the rigid free surface } (^9)$$
$$W(-d) = 0 \qquad \text{at the bottom}$$

The informations concerning the stratification are included in the particular shape of $N^2(z)$. This is rather constant in the mixed layer. In the lower region it has many sharp variations (the fine-structure).

These variations give the classical behaviour of $N^2(z)$ in the thermocline when, in some sense, they are averaged in z. At last $N^2 \longrightarrow$ const. value in the deepest regions can be found.

As the various preceding cases have been studied, we have focused our attention to two explicitly solvable cases:

$$a) \qquad N^2(z) = \alpha^2/|z-z_0|$$
$$b) \qquad N^2(z) = A^2 - B^2(z-z_0)^2$$

where α, A, B, are constants to be determined on experimental ground. The depth z_0 is that corresponding to the zone of highest variation of the Vaisala-Brunt frequency $N^2(z)$. The cases seem to be general enough to approximate realistic cases, particularly in the parabolic case.

3. The explicit case $N^2(z) = \alpha^2/|z-z_0|$

We are now going to study the case above mentioned $N^2(z) = \alpha^2/|z-z_0|$. The equation [2.2] now results

$$\frac{d^2 W(z)}{dz^2} = \left(K^2 - \frac{K^2\alpha^2}{\omega^2}\,\frac{1}{|z-z_0|}\right) W(z) \qquad [3.1]$$

We assume that the distance among z_0, the bottom $z = -d$ and the surface $z = 0$ could be considered large. In practice z_0 is 20-50 meters for localized seas (^{1}) and hundred meters for the Ocean. The depth d of the bottom is usually fixed to be larger, in order to avoid bottom effects. For strongly trapped internal waves, the vertical region of interest is determined in order to fix the vertical scale of motion. It usually is of the order of magnitude of few meters. Outside this region $z = z_0$, the Väisalä-Brunt frequency decays rather rapidly, as a power of z. Much more quick, however, is the decay of the solution W, which results in general a negative exponential. So one could also assume for these waves an idealized boundary condition

$$W(\pm\infty) = 0$$

which simplifies the calculations. Then one can say that our eigenvalue equation can assume an infinity of eigensolutions, labelled by $\mu = 1, 2, 3\ldots$

The general solution is (see Appendix):

$$W_\mu = A_\mu \varrho^{-K(z-z_0)} L^1_{\mu-1} \{2K|z-z_0|\} \qquad [3.2]$$

where A_μ is a constant and the function $L^1_{\mu-1}(\varrho)$ is called Laguerre polynomial. At $z = z_0$, the function is complicated: the equation shows that its second derivative diverges. Then W_μ results a continuous function symmetric around the peak z_0 One can moreover say (see Appendix) that the solution exist if and only if

$$\frac{a^2 K}{2\omega^2} = \mu = 1, 2, \ldots$$

It's interesting to note that the preceding dispersion relation for these internal waves is similar to that of the two-fluid system.

Then one can calculate the group velocity c_g

$$c_g = \frac{d\omega}{dK} = \frac{1}{2} \frac{a^2}{\mu(2K)^{1/2}}$$

One has also to remark that W_μ is proportional to exp-$K|z-z_0|$ so that the behaviour of the wave decays exponentially outside the zone of sharp variation of $N^2(z)$.

This implies that we must consider only waves with a large K value fixed by the dimension of the physically interesting region through the relation $K \sim 1/L$. One could enlarge the preceding treatment of the individual structure of internal waves by considering the correlation function. This quantity is

$$B_{\mu\mu'}(K) = \overline{W_\mu W_{\mu'}} = \int_{-\infty}^{+\infty} W_\mu W^*_{\mu'} \, dz'$$

It appears interesting because it is a powerful tool in the comparison between theoretical models and experimental data [4]. In our case it results:

$$B_{\mu\mu'}(K) = \frac{1}{2KL} \int_0^{L/2} A_\mu A^*_{\mu'} \, e^{-2K(z-z_0)} L^1_{\mu-1} L^{1*}_{\mu'-1} \, d\{2K|z'-z_0|\}$$

The Laguerre polynomial are on orthogonal set, therefore one has

$$B_{\mu\mu'}(K) = \begin{cases} 0 & \text{for } \mu \neq \mu' \\ \dfrac{1}{2L} \{|A_\mu|(\mu-1)!\}^2 \, K^{-1} & \text{for } \mu = \mu' \end{cases}$$

this K^{-1} behaviour of the correlation function is rather interesting and is in agreement with the results of Phillips (6) for a range of the spectrum of the lowest internal mode. If one wants to calculate also the cross-correlation $\overline{WV}$, the continuity equation

$$i\, K_y V + \frac{\partial}{\partial z} W = 0$$

must be used. This can give the matrix element

$$\int \left(\frac{dW}{dz}\right) W \, d z' = 0$$

One could add that the matrix element

$$\int \left(\frac{dW_n}{dz}\right) W_m \, dz'$$

is not zero if (and only if) $n = m \pm 1$. In more detail

$$\int \left(\frac{dW_n}{dz}\right) W_{n-1} \, dz' = \sqrt{\frac{n}{2}}$$

and

$$\int \left(\frac{dWn}{dz}\right) W_{n+1} \, dz' = -\sqrt{\frac{n+1}{2}}$$

To finish, also the case

$$N^2(z) = \frac{\alpha^2}{z} - \frac{\beta^2}{z^2} \qquad \alpha, \beta \text{ const.}$$

could be exactly treated. It results, however, of different physical interest because it describes an instable case, whereas in this note one studies stable phenomena only.

4. The case of a parabola $N^2(z) = A^2 - B^2(z-z_0)^2$

By repeating the preceding considerations, one can arrive to the equation:

$$\frac{d^2W(z)}{dz^2} = \frac{B^2K^2}{\omega^2}(z-z_0)^2\, W(z) - \left\{\frac{A^2K^2}{\omega^2} - K^2\right\} W(z); \quad A^2 > \omega^2$$

with the boundary conditions (6),

$$W(0) = 0$$
$$W(-d) = 0$$

In this case also there is an infinity of solutions (see Appendix):

$$W_q = A_q\, e^{-\frac{BK}{2\omega}(z-z_0)^2}\, H_q\left\{\left(\frac{KB}{\omega}\right)^{1/2}(z-z_0)\right\} \qquad q = 1, 2, \ldots \qquad [4.1]$$

Where A_q is a constant and $H_q(\varrho)$ is an Hermite polynomial. From this equation one can see that the wave amplitude decreases more rapidly than in the preceding case (eq. [3.2]) as $|z-z_0|$ increases.

There is also in this case a dispersion relation

$$2q+1 = \frac{A^2K}{\omega B} - \frac{K\omega}{B}$$

It has to be remarked that ω decreases as q increases, that is, for higher modes, the dimension L of the system fixes a K value $K \sim 1/L$. From the above equation, one can derive:

$$\omega = \frac{-CB + (C^2B^2 + 4K^2A^2)^{1/2}}{2K} \qquad C = q+1 \qquad [4.2]$$

and then the group velocity can be deduced

$$c_g = \left\{\frac{2A^2}{(C^2B^2 + 4A^2K^2)^{1/2}} + \frac{1}{2K^2}\left[CB - (C^2B^2 + 4\,A^2K^2)^{1/2}\right]\right\}$$

In this case also it is easy to calculate the correlation function

$$B_{qq'}(K) = \frac{1}{L}\int_0^{L/2} A_q A_{q'}^*\, e^{-\frac{KB}{\omega}(z'-z_0)^2}\, H_q H_{q'}^*\, dz' =$$

$$= \frac{1}{L} \left(\frac{\omega}{KB}\right)^{1/2} \int_0^{L/2} A_q \, A^*_{q'} \, H_q H^*_{q'} \, e^{-\frac{KB}{\omega}(z'-z_0)^2} \, \mathrm{d}\left[\left(\frac{KB}{\omega}\right)^{1/2} (z' - z_0)\right]$$

which, remembering the orthogonality of the Hermite polynomials, becomes

$$B_{qq'}(K) = \begin{cases} 0 & \text{for } q \neq q' \\ |A_q|^2 \left(\dfrac{\pi\omega}{KB}\right)^{1/2} \dfrac{2^q}{L} \, q! & \text{for } q = q' \end{cases} \qquad [4.3]$$

This result can be written in a more clear form by using the dispersion relation.

In fact, from eq. [4.2] one has that

$$\omega K^{-1} \sim K^{-2}$$

for which eq. [4.3] becomes

$$B_{qq'}(K) \propto K^{-1/2}$$

In this case, the behaviour of the correlation function is different from the other one.

As in the preceding section, if one wants to calculate the correlation $\overline{WV}$, one must use the continuity equation, giving V as a function of $\dfrac{\mathrm{d}W(z)}{\mathrm{d}z}$. Now, the resulting matrix element

$$\int \left(\frac{\mathrm{d}W(z')}{\mathrm{d}z'}\right) W \mathrm{d}z'$$

can be found in the literature (5).

The profile

$$N^2(z) = A^2 - B^2 (z - z_0)^2 + C (z - z_0) + \mathrm{D} (z - z_0)^3$$

can also be calculated. The result is exact for $C \neq 0$ and with some approximation for $D \neq 0$.

To finish, we want to note that results obtained in this note for the internal waves in the presence of a strong stratification are used for a comparison with experimental data concerning the Ligurian sea (1).

APPENDIX

a) *The* $\frac{a^2}{|z - z_0|}$ *case.*

This equation is rather well known. We follow here a description of the solution which is widely used in quantum mechanics (2). In adimensional form, it results

$$\frac{d^2u}{d\varrho^2} + \left(\frac{\lambda}{\varrho} - \frac{1}{4}\right) u = 0$$

Asimptotically the solution results

$$u \sim e^{\pm \varrho/2}$$

and if one wants u to be finite,

$$u \sim e^{-\varrho/2}$$

Multiplication by a polynomial doesn't change the asimptotic value

$$u = F(\varrho)\, e^{-\varrho/2}$$

This gives

$$\frac{d^2F}{d\varrho^2} - \frac{dF}{d\varrho} + \frac{\lambda}{\varrho} F = 0$$

Putting now

$$F = \sum_{K=1} A_K \varrho^K$$

inside the preceding equation, one has

$$(\lambda - 1) A_1 + 2A_2 = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$K(K+1)A_{K+1} + (\lambda - K) A_K = 0$$

Now, if one assumes that the series is infinite, then

$$\frac{A_{K+1}}{A_K} = \frac{K - \lambda}{K(K+1)} \xrightarrow[K \to \infty]{} \frac{1}{K}$$

which correspond to an exponential behaviour of $F(\varrho)$, is in contradiction with our hypothesis $u \sim e^{-\varrho/2}$.

Then the series cannot be infinite.

The other only possibility is that λ be an integer n.

Then one has the "eigenvalue condition"

$$\lambda = n$$
$$A_K = 0 \qquad K > n$$

The polynomials obtained in this way are related to the associated Laguerre polynomials:

$$L_n^1(\varrho) = e^{\varrho} \frac{d^n}{d\varrho^n} (\varrho^n e^{-\varrho})$$

The first one are:

$$L^1_0 = 1; \qquad L^1_1 = 4-2\varrho; \qquad L^1_2 = 18 - 18\varrho - 3\varrho^2$$
$$L^1_3 = 96 - 144\varrho + 48\varrho^2 - 4\varrho^3$$

b) *The $A^2-B^2(z-z_0)^2$ case*

We repeat here the preceding treatment of the equation. In adimensional form

$$\frac{d^2 u}{d\varrho^2} + (\lambda - \varrho^2)\, u = 0$$

The asimptotic value being

$$u \sim e^{-\varrho^2/2}$$

If one assumes

$$u = H(\varrho) e^{-\varrho^2/2}$$

then obtains, for the basic equation

$$\frac{d^2 H}{d\varrho^2} - 2\varrho \frac{dH}{d\varrho} + (\lambda - 1)\, H = 0. \qquad [A.1]$$

Now, writing for H

$$H = \sum_{n=1}^{N} a_n \varrho^n$$

the differential equation [A.1] gives

$$\frac{a_{s+2}}{a_s} = \frac{(2s+1) - \lambda}{(s+2)(s+1)} \qquad s \geq 0$$

Separating the even or add solutions, on then has

$$\lambda = 2n + 1$$

The solutions are even or add polynomials, the Hermite polynomials

$$H_q(\varrho) = (-1)^q e^{\varrho/2} \frac{d^n e^{-\varrho^2}}{d\varrho^n}$$

The first one are

$$H_0 = 1; \quad H_1 = 2\varrho; \quad H_2 = 4\varrho^2 - 2; \quad H_3 = 8\varrho^3 - 12\varrho$$

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