Ray theory for pre-stressed media

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Summary. — Because of the similarity between the equations of motion governing infinitesimal vibration due to a small perturbing force superimposed on an already existing state of finite stress and the equations of linear anisotropic elasticity, methods of analysis used in one may be extended to the second. In particular, in this paper, the technique of ray expansions is considered. Methods for calculation of rays and amplitude coefficients of the ray series are given. A seismic ray is described by a system of ordinary differential equations of first order which can be solved by standard numerical techniques. Another system of ordinary differential equations is introduced to compute amplitude coefficients.

Riassunto. — Per la similarità che sussiste tra le equazioni del moto che governano le vibrazioni infinitesime dovute ad una piccola perturbazione sovrapposta ad uno stato pre-esistente di sforzo finito e le equazioni della elasticità lineare anisotropa, metodi di analisi usati in un caso possono essere estesi all'altro. In particolare, in questo lavoro, si considera la teoria dei raggi. Vengono dati metodi per il calcolo dei raggi e dei coefficienti di ampiezza. Un raggio sismico è descritto da un sistema di equazioni differenziali ordinarie del prim'ordine che può essere risolto con tecniche numeriche standard. Un altro sistema di equazioni differenziali è introdotto per il calcolo delle ampiezze.

1. Introduction

In connection with a more detailed study of the seismic source mechanism and the structure of the Earth's crust and upper mantle,

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great attention has been devoted to the consequences of the fact that the Earth is in a state of pre-stress. This state is attributed to a number of causes, such as self-gravitation, rotation and tectonics ([8,12]).

We have recently derived a theory of small deformations superimposed upon large ones, suitable for the study of the seismological effects of the existence of such a pre-stress. The field equations are obtained by postulating energy balances and imposing invariance under rigid body motions ([5]). The equations of motion turn out to be formally equivalent to those of linear infinitesimal anisotropic elasticity, although the elasticity tensor, in our case, possesses only the major symmetry. Because of this similarity, some methods of analysis used in infinitesimal anisotropic elasticity may be extended to our theory of pre-stressed media. In particular, in this paper, the technique of ray expansion will be considered.

The theory of ray series has been well developed for isotropic media and has brought about a number of very valuable results ([1]). It was first applied to anisotropic media by Babich ([2]), who derived differential equations for the wave fronts and the amplitude coefficients of the ray series. Babich's approach has been reformulated by Cerveny ([6]) who has obtained a system of equations which allows numerical solutions by standard procedures.

From the viewpoint of applications in seismology the kinematic description of elastic waves and the calculation of the zeroth amplitudes of a ray series is of great importance. Some results in the description of wave processes in special cases of anisotropic media have been given by a number of authors ([10]).

2. Equations of rays in pre-stressed media

In our analysis the motion will be referred to a reference configuration and to a fixed set of rectangular cartesian axes. The body in the reference configuration is assumed to be homogeneous and the coordinates of a material particle in the reference configuration are X_A, $A = 1,2,3$ with respect to these axes. In the subsequent motion of the body this particle has coordinates x_i,

$$x_i = x_i\,(X_A, t)$$

In order to distinguish between the umperturbed motions and the perturbed ones, the terms "*primary*" and "*secondary*" state will be employed. In our intentions "*primary*" means "*prior to the occurrence of an earthquake*". When referred to the *primary* configuration the linearized equations of motion for a pre-stressed hyperelastic medium are ([5]):

$$(d_{ijkl}\, u_{kl})_{,j} + \varrho\, (F^*_i - F_i) = \varrho\, \ddot{u}_i \qquad [1]$$

where u_i are the components of the displacement vector field; F^*_i and F_i are the components of the body forces in the *secondary* and *primary* state, respectively; ϱ is the mass density. A superposed dot denotes the material derivative, while partial derivatives will be denoted by a comma preceding a subscript. The elasticity tensor d_{ijkl} is given by:

$$d_{ijkl} = t_{jl}\, \delta_{ik} + c_{ijkl} \qquad [2]$$

where t_{jl} is the pre-stress tensor. d_{ijkl} possesses only the major symmetry:

$$d_{ijkl} = d_{klij} \qquad [3]$$

which is intimately connected with the assumption that the considered medium is hyperelastic ([11]). Moreover, c_{ijkl} possesses the following symmetry properties:

$$c_{ijkl} = c_{klij} = c_{jikl} = c_{ijlk} \qquad [4]$$

We shall consider the case when the difference $(F^*_i - F_i)$ is equivalent to a point impulsive force acting at the origin. The equation [1] is replaced by:

$$(d_{ijkl}\, u_{kl})_{,j} - \varrho\, \ddot{u}_i = 0 \qquad [5]$$

for $t > 0$ and $\vec{x} \neq 0$, together with suitable initial conditions. The solutions of the equations of motion [5] are sought. These solutions are non-analytic along certain moving surfaces which are called wave-fronts. A wave-front will be described by the following equation:

$$t = S\,(\vec{x}) \quad ; \qquad [6]$$

then [5] can be solved by assuming a ray series solution of the form:

$$u_i\,(\vec{x}, t) = \sum_{n=0}^{\infty} A_i^{(n)}\,(\vec{x})\, E_n\,(t - S\,(\vec{x})) \qquad [7]$$

where the functions $E_n(\mu)$ satisfy the relation:

$$E'_{n+1}\,(\mu) = E_n\,(\mu) \qquad [8]$$

The ray series includes, as we have already pointed out, solutions which are discontinuous at the wave-fronts. It follows from [8] that the order of discontinuity of E_{n+1} is one less than that of E_n. By substituting [7] into [5] and writing h_i for $S_{,i}$ we get:

$$\sum_{n=0}^{\infty} \Big\{ \Big[d_{ijkl}\, A_{k,l}^{(n)}(x_i) \Big]_{,j} E_n (t - \vec{S}(\mathrm{x})) + \Big[d_{ijkl}\, h_j\, A_{k,l}^{(n)}(x_i) + (A_k^{(n)}(x_i)\, d_{ijkl}\, h_l)_{,j} \Big] E_{n-1} (t - \vec{S}(\mathrm{x})) + (A_k^{(n)}(x_i)\, d_{ijkl}\, h_j\, h_l)\, E_{n-2} (t - \vec{S}(\mathrm{x})) \Big\} = \varrho \sum_{n=0}^{\infty} A_i^{(n)}(x_i)\, E_{n-2} (t - \vec{S}(\mathrm{x}))$$

The summation can be eliminated and the latter equation can be cast into the form:

$$\vec{\mathrm{N}}(\vec{\mathrm{A}}^{(n)}) - \vec{\mathrm{M}}(\vec{\mathrm{A}}^{(n-1)}) + \vec{\mathrm{L}}(\vec{\mathrm{A}}^{(n-2)}) = 0 \qquad [9]$$

for $n = 0,1,2\ldots$, and $\vec{\mathrm{A}}^{(-1)} = \vec{\mathrm{A}}^{(-2)} = 0$, by the definition. The vector operators $\vec{\mathrm{L}}, \vec{\mathrm{M}}$ and $\vec{\mathrm{N}}$ are given by:

$$\mathrm{N}_i(\vec{\mathrm{A}}^{(n)}) = \Gamma_{ik}\, A_k^{(n)} - A_i^{(n)} \qquad [10a]$$

$$\mathrm{M}_i(\vec{\mathrm{A}}^{(n)}) = \varrho^{-1} h_j\, d_{ijkl}\, A_{k,l} + \varrho^{-1} (d_{ijkl}\, h_l\, A_k^{(n)})_{,j} \qquad [10b]$$

$$\mathrm{L}_i(\vec{\mathrm{A}}^{(n)}) = \varrho^{-1} (d_{ijkl}\, A_{k,l}^{(n)})_{,j} \qquad [10c]$$

where:

$$\Gamma_{jk} = \varrho^{-1} (h_k\, h_l\, t_{jl} + h_i\, h_l\, c_{ijkl}) \qquad [11a]$$

$$h_i = S_{,i} \qquad [11b]$$

The system [9] is the basic system of equations of ray theory for a pre-stressed medium. It can be used, when certain initial conditions are given, to determine $\vec{S}(\mathrm{x})$ and $\vec{\mathrm{A}}^{(n)}(\vec{\mathrm{x}})$. The system is recurrent. For $n = 0$, [9] reduces to:

$$(\Gamma_{jk} - \delta_{jk})\, A_k^{(0)} = 0 \qquad [12]$$

which represents a system of three algebraic equations for $A_1^{(0)}$, $A_2^{(0)}$, $A_3^{(0)}$. The form of [12] leads us to consider the eigenvalue problem for the matrix Γ_{jk}. This matrix is symmetric and positive definite and its eigenvalues are then real and positive. They can be determined finding the roots of the characteristic equation:

$$\mathrm{Det}\,(\Gamma_{jk} - H\,\delta_{jk}) = 0 \qquad [13]$$

and will be denoted by H_m, $m = 1,2,3$. $H_m\left(\vec{h}, \vec{x}\right)$ are homogeneous functions of the second order in $\vec{h}$.

If the three H_m's are distinct, the corresponding eigenvectors $\vec{g}^{(m)}$ can be determined from the equations:

$$(\Gamma_{jk} - H_m\,\delta_{jk})\,g_k^{(m)} = 0 \qquad [14]$$

where no summation is intended over m.

We can say the system [12] has a non-zero solution only in the case when any of the eigenvalues of Γ_{jk} is equal to one, i.e., if the H_m's are distinct, [12] has a non-trivial solution only in the following three cases: $H_1 = 1$ and $H_2 \neq 1$, $H_3 \neq 1$; $H_2 - 1$ and $H_1 \neq 1$, $H_3 \neq 1$; $H_3 = 1$ and $H_1 \neq 1$, $H_2 \neq 1$. The equations:

$$H_m\left(\vec{h}, \vec{x}\right) = 1 \qquad m = 1, 2, 3, \qquad [15]$$

are non-linear partial differential equations for $S\left(\vec{x}\right)$, which describes the propagation of a wave front. Thus, in a pre-stressed medium, with d_{ijkl} and its derivatives continuous, three independent wave-fronts can propagate. One of them corresponds to the so-called quasi-compressional waves, the others to two quasi-shear waves. These wave fronts are generally independent. In the degenerate case of two identical eigenvalues, there will be only two independent wave fronts. This result has been already obtained in an independent way by Boschi ([4]). The three equations [15] can be solved by means of the characteristics ([7]).

We have already pointed out that H_m's are homogeneous functions of h_i: thus Euler's theorem on homogeneous functions apply to find:

$$2\,H_m = h_i\,\frac{\partial H_m}{\partial h_i} \qquad [16]$$

Equation [16] allows us to obtain the equations of the characteristics in a easy way. We get:

$$\frac{\mathrm{d}x_i}{\mathrm{d}S} = \frac{1}{2}\frac{\partial H_m}{\partial h_i} \qquad [17a]$$

$$\frac{\partial h_i}{\partial S} = -\frac{1}{2}\frac{\partial H_m}{\partial x_i} \qquad [17b]$$

In the system [17] the expression for H_m is complicated because H_m is solution of a cubic equation. Fortunately we do not need the analytical expression for H_m, we need only the analytical expression for the partial derivatives of H_m, which can be found from [13] by means of the theorem on the implicit functions. Thus we obtain:

$$\frac{\partial H_m}{\partial x_i} = \frac{\partial \Gamma_{jk}}{\partial x_i}\frac{D_{jk}}{D} \qquad [18a]$$

$$\frac{\partial H_m}{\partial h_i} = \frac{\partial \Gamma_{jk}}{\partial h_i}\frac{D_{jk}}{D}\,, \quad m = 1,2,3 \qquad [18b]$$

where

$$\begin{aligned}
D_{11} &= (\Gamma_{22}-1)(\Gamma_{33}-1)-\Gamma^2_{23}\\
D_{22} &= (\Gamma_{11}-1)(\Gamma_{33}-1)-\Gamma^2_{13}\\
D_{33} &= (\Gamma_{11}-1)(\Gamma_{22}-1)-\Gamma^2_{12}\\
D_{12} &= D_{21} = \Gamma_{13}\Gamma_{23}-(\Gamma_{33}-1)\Gamma_{12}\\
D_{13} &= D_{31} = \Gamma_{12}\Gamma_{23}-(\Gamma_{22}-1)\Gamma_{13}\\
D_{23} &= D_{32} = \Gamma_{12}\Gamma_{13}-(\Gamma_{11}-1)\Gamma_{23}\\
D &= \mathrm{tr}\,(D_{jk})
\end{aligned} \qquad [19]$$

We will give in the Appendix the explicit expression of each term D_{jk} as a function of the elasticity tensor and the pre-stress. From [11] we deduce that:

$$\frac{\partial \Gamma_{jk}}{\partial h_i} = \varrho^{-1}(d_{ijkl}+d_{ikjl})\,h_l \qquad [20a]$$

$$\frac{\partial \Gamma_{jk}}{\partial x_i} = \varrho^{-1}\,d_{ljks,i}\,h_l\,h_s \qquad [20b]$$

[18], [19] and [20], substituted in [17], give:

$$\frac{\partial x_i}{\partial S} = \varrho^{-1}\, d_{ijkl}\, h_l \,\frac{D_{jk}}{D} \qquad [21a]$$

$$\frac{\partial h_i}{\partial S} = -\frac{1}{2}\, d_{lkjs,i}\, h_l\, h_s \,\frac{D_{jk}}{D} \qquad [21b]$$

Equations [21] are the final system of ordinary differential equations for the characteristics of [15]. [21] are also the equations for seismic rays in a pre-stressed medium. In order to solve [15], we must know six initial conditions for x_i and h_i at time $t = 0$, namely:

$$x_i(0) = \bar{x}_i \qquad [22a]$$

$$h_i(0) = \bar{h}_i \qquad [22b]$$

$\bar{x}_i$ and $\bar{h}_i$ must satisfy the relation:

$$H_m\left(\bar{h}_i, \bar{x}_i\right) = 1 \qquad m = 1, 2, 3. \qquad [23]$$

The parameter along the ray is $S = t$, and, for each t, h_i and x_i must satisfy [21].

3. Amplitudes of first ray term

Let us now investigate about the amplitude of $\vec{A}^{(o)}$. For sake of simplicity we assume that the three eigenvalues H_m of the matrix Γ_{jk} are distinct. $\vec{A}^{(o)}$ must be in the direction of one of the $\vec{g}^{(m)}$, thus we may write, dropping the subscript m,

$$\vec{A}^{(o)} = \varphi^{(o)}\, \vec{g} \qquad [24]$$

where $\varphi^{(o)}$ is the amplitude of $\vec{A}^{(o)}$ that we now want to calculate.

Equation [9], for $n = 1$, gives:

$$\vec{N}\left(\vec{A}^{(1)}\right) - \vec{M}\left(\vec{A}^{(o)}\right) = 0$$

or

$$(d_{ijkl}\, h_j\, h_l - \varrho\, \delta_{ik})\, A_k^{(1)} - \left\{ \varrho^{-1} h_j\, d_{ijkl}\, A^{(o)}_{k,l} + \right.$$
$$\left. + \varrho^{-1} d_{ijkl,j}\, h_l\, A_k^{(o)} + \varrho^{-1} d_{ijkl}\, h_{l,j}\, A_k^{(o)} + \varrho^{-1} d_{ijkl}\, h_j\, A^{(o)}_{k,l} \right\} = 0 \qquad [25]$$

If we now contract this equation with g_j and replace $\vec{A}^{(o)}$ with $\varphi^{(o)} \vec{g}$ we obtain:

$$2\, d_{ijkl}\, g_j\, g_k\, h_l\, \varphi^{(o)}{}_{,i} + \varphi^{(o)} \left(d_{ijkl}\, g_i\, g_k\, h_{l,j} \right) = 0 \qquad [26]$$

were use has been made of the symmetry properties of d_{ijkl}. Now we shall simplify this equation showing that the direction of differentiation is along a ray whose equations are [17]. If we consider the two equations:

$$g_i \left(d_{ijkl}\, h_j\, h_l - \varrho\, \delta_{ik} \right) g_k = 0$$
$$H\left(\vec{h}, \vec{x}\right) = 1 \qquad [27]$$

we see that both represent the same surface in h-space, thus we can obtain two expressions for the normal to this surface at the point $\vec{h}$; hence, for some scalar quantity σ:

$$\sigma \frac{\partial H_m}{\partial h_j} = 2\, d_{ijkl}\, g_i\, g_k\, h_l \qquad [28]$$

By contracting this equation with h_l, the homogeneity of $H\left(\vec{h}, \vec{x}\right)$ leads to:

$$\sigma = d_{ijkl}\, g_j\, g_k\, h_i\, h_l = \varrho \qquad [29]$$

The ray derivative can be written as:

$$\frac{d}{dS} = \vec{v} \cdot \vec{\nabla} \qquad [30]$$

where

$$\varrho\, v_i = \frac{dx_i}{dS} = d_{ijkl}\, h_l\, g_j\, g_k \qquad [31]$$

We have already identified S, the parameter along the ray, as t; $\vec{v}$ is then the velocity along the ray. Equation [24] now reads:

$$\frac{d\varphi^{(o)}}{dS} + \frac{\varphi^{(o)}}{2\varrho} \left(\vec{\nabla} \cdot \varrho\, \vec{v} \right) = 0 \qquad [32]$$

which can be directly integrated to give:

$$\varphi^{(o)}(S) = \varphi^{(o)}(S_o) \exp \left[- \int \frac{1}{2\varrho} \left(\vec{\nabla} \cdot \varrho\, \vec{v} \right) dS \right] \qquad [33]$$

Thus we have obtained the time-dependence of the amplitude of the first term in the ray expansion of $u_i(\vec{x},t)$. A similar procedure can be worked out for higher order amplitudes and, doing this, several complications may occur in the calculations. In most cases, indeed, the first term is the only one we need to consider in detail since it is the most relevant.

Equation [32] is often referred to as the transport equation and can be interpreted, in the linear theory of elasticity, in terms of conservation of energy.

First of all [30] tells us that [32] is equivalent to:

$$\mathrm{div}\left[\varrho\,(\varphi^{(o)})^2\,\vec{v}\right] = 0 \qquad [34]$$

and hence:

$$\int_{\Sigma} \varrho\,(\varphi^{(o)})^2\,\vec{v}\cdot\vec{n}\,d\Sigma = 0 \qquad [35]$$

for any surface Σ with normal $\vec{n}$. If Σ is the surface of a ray tube we conclude that $[(\varrho\,(\varphi^{(o)})^2\,\vec{v}\cdot\vec{n})\,d\Sigma]$ is constant along an elementary ray tube with cross section $d\Sigma$. Let us now consider a volume V_o in the *primary* state. This volume is V when the pre-stress is applied and then, during the action of a perturbing force, the total strain energy is:

$$Q = \int_{V_o} W\left(A_{iA} + \frac{\partial u_i}{\partial X_A}\right) d V_o \qquad [36]$$

where A_{iA} are the deformation gradients and $W = W(I_1,I_2,I_3)$ is the strain energy function. From [36] we obtain:

$$\dot{Q} = \int_{V_o} \frac{\partial W}{\partial A_{iA}}\left(A_{jB} + \frac{\partial u_i}{\partial X_B}\right) \frac{\partial \dot{u}_i}{\partial X_A}\, d V_o \qquad [37]$$

The integrand of [37] can be expanded to give:

$$\dot{Q} = \int_{V_o} \left\{ \frac{\partial W}{\partial A_{iA}}\,(A_{jB}) + \frac{\partial u_k}{\partial X_G}\,\frac{\partial^2 W}{\partial A_{iA}\,\partial A_{kG}}\,(A_{jB}) \right\} \frac{\partial \dot{u}_i}{\partial X_A}\, d V_o \qquad [38]$$

Remembering that I_3 in our case is the Jacobian of the transformation $x_i = x_i(X_A, t)$, we can utilize in [38] the well-known relations:

$$
\begin{aligned}
I_3^{-1/2}\, \mathrm{d}V_o &= \mathrm{d}V \\
u_{i,A} &= u_{i,j}\, A_{jA} \\
t_{ij} &= I_3^{-1/2}\, A_{jA}\, \frac{\partial W}{\partial A_{iA}}\, (A_{kB})
\end{aligned}
\qquad [39]
$$

to obtain:

$$
\dot{\Omega} = \int_V \left\{ t_{ij} + I_3^{-1/2}\, u_{k,l}\, A_{jA}\, A_{lB}\, \frac{\partial^2 W}{\partial A_{iA}\, \partial A_{kB}} \right\} \dot{u}_{i,j}\, \mathrm{d}V \qquad [40]
$$

Finally we observe that the second term in the integrand is an alternative expression (12) for the tensor d_{ijkl} and hence the last equation becomes:

$$
\dot{\Omega} = \int_V (t_{ij} + d_{ijkl}\, u_{k,l})\, \dot{u}_{i,j}\, \mathrm{d}V \qquad [41]
$$

If

$$
T = \frac{1}{2} \int_V \varrho\, \dot{u}^2\, \mathrm{d}V \qquad [42]
$$

is the kinetic energy, the total change in energy is:

$$
\dot{E} = \int_V (t_{ij}\, \dot{u}_{i,j} + d_{ijkl}\, u_{k,l}\, \dot{u}_{i,j} + \varrho\, \dot{u}_i\, \ddot{u}_i)\, \mathrm{d}V \qquad [43]
$$

Equation [1] and Gauss theorem transform [43] into the

$$
\dot{E} = \int_S t_{ij}\, \dot{u}_i\, n_j\, \mathrm{d}S + \int_V \varrho\, \dot{u}_i\, (F^*_i - F_i)\, \mathrm{d}V + \int_S d_{ijkl}\, u_{k,l}\, \dot{u}_i\, n_j\, \mathrm{d}S \qquad [44]
$$

Each term of [44] can be easily interpreted: the first integral is the rate of working of the surface forces on S due to pre-stress; the second term refers to the work of the difference $(F^*_i - F_i)$, which is the only meaningful quantity in the theory because it is hard to imagine a realistic method which could give the absolute value of F^*_i or F_i separately. The third term represents the rate of change of the incremental energy flux.

4. A PROPERTY OF RAYS IN PRE-STRESSED MEDIA

We want to show that the rays, whose equations are [17], are extremals of a certain line integral, i.e. that:

$$I = \int_{x_0}^{x_1} \frac{dS}{v} \qquad [45]$$

is minimum when the path of integration is the ray [17] connecting the two fixed points x_0 and x_1. We denote by $\vec{t}(\vec{y})$ the tangent at the point $\vec{y}$ of L. To calculate $v(\vec{x}, \vec{t})$ we consider the ray along $\vec{t}(\vec{y})$; equation [17a] gives the direction of the corresponding $\vec{h}(\vec{y})$; the calculation of $v(\vec{x}, \vec{t})$ follows then from [31]. If we write $\vec{x} = (x,y,z)$ and if we consider x as the ray variable we have:

$$I = \int_{x_0}^{x_1} \left[v\,(\vec{x}, \vec{t}) \right]^{-1} (1 + y'^2 + z'^2)^{1/2}\, dx \qquad [46]$$

Euler's equations must be satisfied for the integrand to be a minimum along the path of integration. These equations can be combined and written in an elegant form as:

$$\frac{d}{dS} \left\{ \frac{1}{v} \vec{t} - \frac{1}{v^2} \frac{\delta v}{\delta t} \right\} = \vec{\nabla} \cdot \left(\frac{1}{v} \right) \qquad [47]$$

where:

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - \vec{t} \cdot \left(\vec{t} \frac{\partial}{\partial t} \right)$$

is the normal derivative. We want to write down [47] in terms of $\vec{h}$ and H instead of v and $\vec{t}$. Now $\vec{h}(\vec{x}, \vec{t})$ and $v(\vec{x}, \vec{t})$ are defined by equations [15] and [17a], i.e.:

$$2\, \vec{v} \cdot \vec{t} = \frac{\partial H}{\partial \vec{h}} \qquad [48]$$

$$H\left(\vec{h}, \vec{x}\right) = 1 \qquad [49]$$

and, for the homogeneity of $H\left(\vec{x}, \vec{h}\right)$, we get:

$$\left|\vec{t} \cdot \vec{h}\right| v = 1 \qquad [50]$$

Since these equations are valid for all $\vec{x}$ and for all unit vectors $\vec{t}$, the differentiation of equation [49] with $\delta/\delta t$ gives:

$$t_j \frac{\delta h_j}{\delta t_i} = 0 \qquad [51]$$

and of equation [50]:

$$\frac{\delta v}{\delta \vec{t}} = v\,\vec{t} - v^2\,\vec{h} \qquad [52]$$

[51] and [52] can be used to write equation [47] in the form:

$$\frac{d\vec{h}}{ds} = \nabla \cdot \left(\frac{1}{v}\right) \qquad [53]$$

But, again from equation [50], we can get:

$$\nabla_i \left(\frac{1}{v}\right) = \left[\frac{\partial}{\partial x_i}\left(\frac{1}{v}\right)\right]_t = t_j \left(\frac{\partial h_j}{\partial x_i}\right)_t \qquad [54]$$

and, from [49],

$$\left(\frac{\partial H}{\partial x_i}\right)_t = \left(\frac{\partial H}{\partial x_i}\right)_h + 2\,v\,t_j \left(\frac{\partial h_j}{\partial x_i}\right)_t = 0 \qquad [55]$$

Combining now [53] and [55] and remembering that $ds = v dS$ we obtain:

$$\frac{d\vec{h}}{dS} = -\frac{1}{2}\left(\frac{\partial H}{\partial \vec{x}}\right)_h \qquad [56]$$

i.e. we obtain equation [17b], which is satisfied only if L is a ray. Thus we have shown that I, the travel time between x_o and x_1, is minimum only on a path whose equations are [17].

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APPENDIX

We give here the explicit expressions for each of D_{jk}, as a function of the elasticity tensor. From [2] and [11] it follows that:

$$D_{11} = 1 + \varrho^{-2} \left\{ (t_{jl} h_j h_l)^2 + t_{jl} h_j h_l h_r h_s (c_{2r2s} + c_{3r3s}) + c_{2j2l}\, c_{3r3s}\, h_j h_l h_r h_s + \right.$$
$$\left. - (c_{2j3l}\, h_j h_l)^2 \right\} - \varrho^{-1} \left\{ (2t_{jl} + c_{2j2l} + c_{3j3l})\, h_j h_l \right\}$$

$$D_{22} = 1 + \varrho^{-2} \left\{ (t_{jl} h_j h_l)^2 + t_{jl} h_j h_l h_r h_s (c_{1r1s} + c_{3r3s}) + c_{1j1l}\, c_{3r3s}\, h_j h_l h_r h_s + \right.$$
$$\left. - (c_{1j3l}\, h_j h_l)^2 \right\} - \varrho^{-1} \left\{ (2t_{jl} + c_{1j1l} + c_{3j3l})\, h_j h_l \right\}$$

$$D_{33} = 1 + \varrho^{-2} \left\{ (t_{jl} h_j h_l)^2 + t_{jl} h_j h_l h_r h_s (c_{1r1s} + c_{2r2s}) + c_{1j1l}\, c_{2r2s}\, h_j h_l h_r h_s + \right.$$
$$\left. - (c_{1j2l}\, h_j h_l)^2 \right\} - \varrho^{-1} \left\{ (2t_{jl} + c_{1j1l} + c_{2j2l})\, h_j h_l \right\}$$

$$D_{12} = \varrho^{-2} \left\{ h_j h_l h_r h_s (c_{1j3l}\, c_{2r3s} - t_{jl}\, c_{1r2s} - c_{3j3l}\, c_{1r2s}) \right\} + \varrho^{-1}\, c_{1j2l}\, h_j h_l$$

$$D_{13} = \varrho^{-2} \left\{ h_j h_l h_r h_s (c_{1j2l}\, c_{2r3s} - t_{jl}\, c_{1r3s} - c_{2j2l}\, c_{1r3s}) \right\} + \varrho^{-1}\, c_{1j3l}\, h_j h_l$$

$$D_{23} = \varrho^{-2} \left\{ h_j h_l h_r h_s (c_{1j2l}\, c_{1r3s} - t_{jl}\, c_{2r3s} - c_{1j1l}\, c_{2r3s}) \right\} + \varrho^{-1}\, c_{2j3l}\, h_j h_l$$

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