Some considerations on the usual derivation of the Pockels (or Helmholtz) equation in steady two dimensional fluid dynamics

L. DELL'OSSO (*) – E. SALUSTI (**) – F. ZIRILLI (***)

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SUMMARY. — The simplest non-linear motion of a fluid is studied; i.e. the steady two dimensional motion of a perfect fluid. These equations have a remarkable practical importance because they describe the air-motion over mountains and a wake on an oceanic current. In particular the number of physical solutions is discussed in relation to the known boundary conditions.

RIASSUNTO. — Si studia il caso più semplice di moto non lineare di un fluido: come, ad esempio, il moto non viscoso stazionario bidimensionale di un fluido omogeneo. Queste equazioni hanno applicazioni pratiche notevoli, come moto sopra le montagne o scia di isole in correnti stazionarie. Si esaminano e discutono, in particolare, il numero di soluzioni fisiche in rapporto alle condizioni al contorno conosciute.

In the study of the problem of the steady two dimensional air-flow over a mountain or of the wake generated by an island on a steady-state flow, many people discussed the derivation of an equation for the stream function and studied the corresponding solutions.

(*) Istituto di Fisica dell'Atmosfera del C.N.R. – Roma.
(**) Istituto di Fisica dell'Università, I.N.F.N. – Roma.
(***) Rockfeller University, N.Y.C., 10021 New York, U.S.A.

One of these equations has been known as the Pockels (or Helmholtz) equation, it is an elliptic linear partial differential equation for the stream function.

In this paper, we derive this equation, as is usually done [3,6,7,8,9,10], and we point out some inaccuracies of the usual derivation.

The steady motion of a non viscous two dimensional fluid in the plane x, z can be described by the following equations:

$$\begin{cases} u \dfrac{\partial u}{\partial x} + w \dfrac{\partial u}{\partial z} = - \dfrac{\partial \pi}{\partial x} & [1] \\ u \dfrac{\partial w}{\partial x} + w \dfrac{\partial w}{\partial z} = - \dfrac{\partial \pi}{\partial z} + \lambda \theta & [2] \\ u \dfrac{\partial \theta}{\partial x} + w \dfrac{\partial \theta}{\partial z} = - S(z) w & [3] \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = \sigma w & [4] \end{cases}$$

where $\dfrac{\partial}{\partial x}$, $\dfrac{\partial}{\partial z}$ are partial derivatives respect to x and z respectively, u, w are the velocity components in the x and z directions respectively, λ = constant is the so called convection or *buoyancy* parameter, $S(z)$ is a given function, σ = constant (the term σw in the continuity equation [4] allows a reduction of the fluid density with the altitude if the z-axis is directed upward), π is a quantity connected with the pressure and θ is the potential temperature.

For a complete derivation of equations [1], [2], [3], [4] see the very interesting book of Gutman [6].

Introducing the stream function ψ by eq. [5]

$$u = e^{-\sigma z} \frac{\partial \psi}{\partial z}, \qquad w = - e^{-\sigma z} \frac{\partial \psi}{\partial x} \qquad [5]$$

we derive from eqs. [1], [2], [3], [4] an equation for ψ.

First of all we notice that eq. [4] is always satisfied by the choice of eq. [5].

Multiplying eq. [3] by $e^{-\sigma z}$ we have

$$e^{-\sigma z} u \frac{\partial \theta}{\partial x} + e^{-\sigma z} w \frac{\partial \theta}{\partial z} + e^{-\sigma z} S(z) w = 0$$

that means by eq. [5]

$$\frac{\partial \psi}{\partial z}\,\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\,\frac{\partial \theta}{\partial z} - S(z)\,\frac{\partial \psi}{\partial x} = 0$$

that is

$$\left\| \frac{\partial}{\partial\,(x,\,z)}\,\left(\psi,\ \theta + \int_0^z S(z')\,\mathrm{d}z'\right) \right\| = 0$$

the determinant of the Jacobian is zero.

Then we have

$$\theta + \int_0^z S(z')\,\mathrm{d}z' = f_1\,(\psi) \qquad [6]$$

where $f_1(\psi)$ is an arbitrary function (some requirement of regularity of $f_1(\psi)$ is needed).

Eliminating π from equations [1], [2] and making use of eq. [5] we have

$$\left\| \frac{\partial\,(\psi,\,\mathrm{L}\,\psi)}{\partial\,(x,\,z)} \right\| = \lambda\,\frac{\partial \theta}{\partial x} \qquad [7]$$

where

$$\mathrm{L}\,\psi = e^{2\sigma z}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + \sigma\,\frac{\partial \psi}{\partial z}\right)$$

Using eq. [6]

$$\frac{\partial \theta}{\partial x} = f'_1(\psi)\,\frac{\partial \psi}{\partial x} \qquad [8]$$

so that

$$\left\| \frac{\partial\,(\psi,\,\mathrm{L}\,\psi)}{\partial\,(x,\,z)} \right\| = \lambda\,f'_1(\psi)\,\frac{\partial \psi}{\partial x}$$

that is in Jacobian form

$$\left\| \frac{\partial}{\partial\,(x,\,z)}\,(\psi,\ \mathrm{L}\,\psi - \lambda\,f'_1(\psi)\,z) \right\| = 0$$

Then we have

$$L\,\psi = f_2(\psi) + \lambda\, f'_1(\psi)\, z \qquad [9]$$

where f_2 as f_1 is an arbitrary function.

Thus the system of eqs. [1], [2], [3], [4], has been reduced to eq. [9] for the stream function ψ that depends on the arbitrary functions f_1, f_2.

In order to have a "*well posed*" problem the domain where the equation has to be verified, the boundary conditions and the properties of the functions f_1, f_2 have to be specified.

The Pockels (or Helmholtz) equation is a particular form of eq. [9].

In particular the Pockels equation is commonly derived in this way.

The region where eq. [9] is studied is of the type $z \geqslant \delta\,(x)$ where $\delta(x)$ is a regular function of x (i.e. $\delta(x)$ is the profile of a mountain) see Fig. 1.

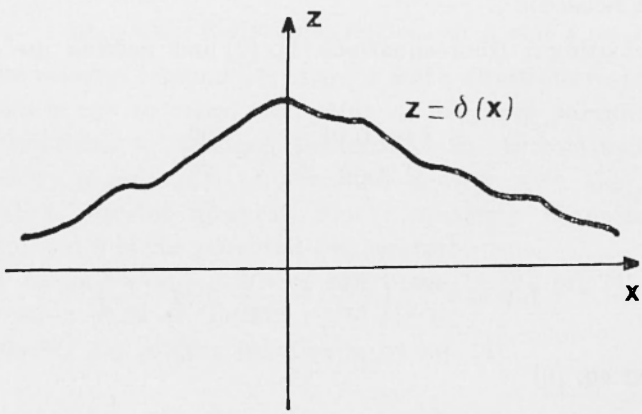

Fig. 1

(in order to simplify the problem we assume now $S(z) = S =$ const. $\sigma = 0$)

The boundary condition on the system of eqs. [1], [2], [3], [4] are the following [Gutman (6)]:

$$\left.\begin{aligned} &\text{a)} \quad u = V = \text{const} \qquad [10] \\ &\text{b)} \quad \theta = 0 \qquad [11] \end{aligned}\right\} \quad x = -\infty \qquad z \geqslant \delta\,(-\infty)$$

We can assume $\delta(-\infty) = 0$.

By eqs. [5] and [10] we have

$$\psi = \psi_\infty = Vz + c \qquad x = -\infty, \quad z \geqslant 0 \tag{12}$$

Assuming $c = 0$ by eqs. [5] [11] and [12] we have

$$f_1(\psi) = \frac{S}{V}\,\psi \qquad x = -\infty, \quad z \geqslant 0 \tag{13}$$

and finally from eqs. [9], [12] and [13] we obtain

$$f_2(\psi) = -\frac{\lambda S}{V^2}\,\psi \qquad x = -\infty, \quad z \geqslant 0. \tag{14}$$

Assuming that $f_1(\psi)$, $f_2(\psi)$ have everywhere the form given by eqs. [13] [14] we reach to the Pockels' equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\lambda S}{V^2}(\psi - Vz) = 0 \tag{15}$$

A rather similar discussion is usually done for the oceanographic case: the wake of an island on an oceanic stream. It has to be remarked that in this problem the Coriolis force is taken into account, as in many other meteorological cases.

From our point of view the derivation of eq. [15] is non satisfactory for two different reason:

1) The translation of the boundary conditions on u, ϑ in terms of ψ is not completely correct.

In fact $u = V$ where $x = -\infty$ $z > 0$ and $u = \dfrac{\partial \psi}{\partial z}$ does not imply $\psi = Vt$ in any finite region. The meaning of $u = V$ when $x = -\infty$, $z \geqslant 0$ is $\lim\limits_{x \to -\infty} u(x, z) = V$, $z > 0$.

So that what we can expect is that $\lim\limits_{x \to -\infty} \psi(x, z) = \psi_\infty = Vz$ if $z \geqslant 0$ that means $\psi = Vz$, $x = -\infty$, $z \geqslant 0$.

Moreover the boundary condition on θ and eq. [6] tell us that

$$\lim_{x \to -\infty} \theta(x, z) = \lim_{x \to -\infty} (S z - f_1(\psi(x, z))), \qquad z \geqslant 0$$

that is

$$0 = S z - f_1(Vz), \qquad z \geqslant 0.$$

So we have that f_1 is fixed only for positive arguments ($z \geqslant 0$) and not everywhere as people seems to believe.

Finally using eq. [9] in order to derive the form of f_2 we use the following fact:

$$\lim_{x \to -\infty} \Delta \psi(x, z) = \Delta \lim_{x \to -\infty} \psi(x, z) = \Delta \psi_\infty = 0, \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)$$

this is also incorrect if no special assumptions are done on the previous limit. However if we study the problem in a compact domain, as it is the case of a numerical computation, the far upstream part of the domain take the rule of ψ_∞. In this case the situation is that people hope to know from the conditions in part of the boundary not only the solutions of one well determined elliptic non linear differential equation but also the explicit shape of the non linear part $f_1(\psi)$ and $f_2(\psi)$.

This appears to us as an overstatement.

2) The boundary condition eqs. [10], [11] are not enough to determine an unique solution of eqs. [1], [2], [3], [4] and so also the equation [15] has only the boundary condition given by eq. [12] which is not enough to determine a unique solution.

Concluding the idea that the boundary conditions can determine the form of f_1, f_2 in eq. [9] seems due to other non rigorous reasons, perhaps of historical origin.

It has to be remarked, however, that the above derivation of eq. [15] is now a classical method in geophysics and that an enormous amount of practical work is done on it.

But it has to be said also that an enormous mathematical literature exists on eq. [9] under various assumptions of f_1, f_2 [see for example (7)]. The problem however of determining the physical form of f_1, f_2 and so the physical solutions of eq. [9] is in our opinion essentially open, so the use of the primitive equations [1], [2], [3], [4] seems to us the most reasonable way to handle these problems.

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