# An inversion model applied to DC soundings interpretation

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## ABSTRACT

The inversion technique is used in DC soundings intepretation to determine the thicknesses and the true resistivities of the layers, starting from the field apparent resistivities.

The relationship between the predicted apparent resistivities and the earth parameters is not linear. Starting from this relationship a methodology is described to obtain a wellposed system of $M$ linear equations. This system permits to calculate, by means of an iterative procedure, the earth parameters that minimize the differencies (error) between the field and the predicted apparent resistivities.

Three different iterative procedures are described. Practical examples have shown that all the iterative procedures are reliable and give comparable results in terms of minimum error reached and CPU time.

## RIASSUNTO

La tecnica dell'inversione è usata nella interpretazione di sondaggi elettrici verticali per determinare gli spessori e le resistività vere degli elettrostrati (parametri del terreno) partendo dalle resistività apparenti. La relazione tra le resistività apparenti ed i parametri del terreno non è lineare.

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Partendo da questa relazione viene descritta una metodologia già nota per ottenere un sistema ben posto di $M$ equazioni lineari.

Questo sistema permette di calcolare, per mezzo di una procedura iterativa, i parametri del terreno che minimizzano la differenza tra le resistività apparenti di campo e quelle calcolate. Vengono inoltre descritte tre diverse procedure iterative. Gli esempi pratici hanno mostrato che tutte le procedure iterative utilizzate sono attendibili e paragonabili tra loro per quanto riguarda i risultati del procedimento di minimizzazione e i tempi di calcolo utilizzati.

## 1. Statment of problem

The apparent resistivities in ohm·m versus the AB/2 distances in $M$ represent the field data. In terms of the voltage $V_i$, currents $I_i$ and geometric factor $K_i$ ($i = 1, N$) where $N$ is the number of samples, the apparent resistivity is:

$$\rho_{ai} = K_i \frac{\Delta V_i}{I_i}$$

We want to find an earth model, consisting of a distribution of true resistivities and thicknesses that minimize the error between the field apparent resistivities and the predicted ones.

The observed apparent resistivities are, in vectorial form:

$$\bar{\rho}_a = \begin{bmatrix} \rho_{a1} \\ \rho_{a2} \\ \cdot \\ \cdot \\ \cdot \\ \rho_{aN} \end{bmatrix}$$

If we call $\bar{\rho}'$ the resistivities predicted by a forward earth model $\bar{P}$ consisting of the true resistivities and the thicknesses of the $\frac{M-1}{2}$ layers

$$\bar{\rho}' = \begin{bmatrix} \rho_1' \\ \rho_2' \\ \cdot \\ \cdot \\ \cdot \\ \rho_N' \end{bmatrix}$$

The above values are functions of the particular earth model $\bar{P}_j$ ($j = 1, M$) where generally $M < N$.

$$\bar{P} = \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_M \end{bmatrix}$$

Then $$\bar{\rho}' = \bar{f}(\bar{P}) \qquad [1]$$

The relationship [1] between $\bar{P}$ and $\bar{\rho}'$ is not linear. By expanding the equation [1] in a Taylor series and by keeping only the linear terms, we obtain:

$$\bar{\rho}_i' = \bar{f}(\bar{P} + \bar{\Delta P}) = \bar{f}(\bar{P}) + \frac{\partial \bar{f}(\bar{P})}{\partial P} \bar{\Delta P}$$

where $\bar{\Delta P}$ is the model improvement.

We need the $\bar{\rho}'$ to fit the observed data:

$$\bar{\rho}_a = \bar{\rho}'$$

Or:

$$\bar{\rho}_a = \bar{f}(\bar{P}) + \frac{\partial \bar{f}(\bar{P})}{\partial P} \bar{\Delta P}$$

Calling $\bar{d} = \bar{p}_a - \bar{f}(\bar{P})$, this is:

$$\bar{d} = \frac{\partial \bar{f}(\bar{P})}{\partial P} \bar{\Delta P}$$

Or:

$$\bar{d} = [A] \bar{\Delta P} \qquad [2]$$

where [A] is the sensitivity matrix and $\bar{\Delta P}$ is the vector parameter correction (Jupp, Vozof 1975, Lanczos 1961, Marquardt 1963). Writing out the matrices of equation [2] we have:

$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\bar{P})}{\partial P_1} & \cdots & \frac{\partial f_1(\bar{P})}{\partial P_M} \\ \vdots & & \vdots \\ \frac{\partial f_N(\bar{P})}{\partial P_1} & & \frac{\partial f_M(\bar{P})}{\partial P_M} \end{bmatrix} \cdot \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \\ \vdots \\ \Delta P_M \end{bmatrix} \qquad [3]$$

This is a system of $N$ equations in $M$ unknowns with $N > M$. The above system of equations is then over-determined and generally ill-posed in the sense that small changes in the data lead to large changes in the solutions. To solve this redundant system of equations, we can apply the method of least squares.

Calling $\bar{e} = \bar{d} - [A] \bar{\Delta P}$ we have:

$$\bar{e}^2 = \sum_{i=1}^{N} e^2 = \bar{e}^T \bar{e}$$

where $\bar{e}^T$ is the transpose of $\bar{e}$ or:

$$\bar{e}^2 = [\bar{d} - [A]\ \overline{\Delta P}]^T \cdot [\bar{d} - [A]\ \overline{\Delta P}]$$

For $e^2$ to be a minimum; its derivative with respect to $\overline{\Delta P}$ must be zero.

Then differentiating the above equation we have:

$$[A]^T [A]\ \overline{\Delta P} = [A]^T \bar{d} \qquad [4]$$

We have transformed the old system [2] of $N$ equations in $M$ unknowns into an $M \times M$ system [4]. From the above expression [4] we have:

$$\overline{\Delta P} = [A^T A]^{-1} [A]^T \bar{d} \qquad [5]$$

## 2. Iterative procedures

In the iterative procedure we do not use the algorithm [5] but its modification made by Marquardt (Marquardt 1963):

$$\overline{\Delta P} = [A^T A + k^2 I]^{-1} [A]^T \bar{d} \qquad [6]$$

where $k^2$ is called «Marquardt parameter» and I is the unit matrix.

The system of equations [6] is well-posed.

The algorithm [6] has the advantage over [5] that the region of convergence is greater and the amplitude of the parameter correction $\overline{\Delta P}$ is smaller.

The general expression for the iterative procedure is:

$$P^{(m)} = \bar{P}^{(m-1)} + \Delta P^{(m)} \qquad [7]$$

where $m$ indicates the iteration number and:

$$\Delta P^{(m)} = [\ (A^T A)^{(m-1)} + (k^2 I)^{(m)}]^{-1} (A^T \bar{d})^{(m-1)} \quad [8]$$

In each iteration, we can use $i$ values of $k^2$ that give $i$ corresponding values of $\overline{\Delta P}$. We then select the $k^2$ that gives the mininum error between the apparent resistivities and the predicted ones. Then:

$$\bar{P}^{(m+i)} = \bar{P}^{(m-i)} + \overline{\Delta P}^{(m+i)} \quad [9]$$

To ascertain that a minimum has been found so far, we have to test neighbouring points, increasing the number of $k^2$ parameters and decreasing the range of variability of $k^2$. Thus the expression [9] becomes:

$$\bar{P}^{(m+i+n)} = \bar{P}^{(m-i-n)} + \overline{\Delta P}^{(m+i+n)} \quad [10]$$

where $n$ is the number of times that the "neighbourhood test" is performed. In expression [10] $\overline{\Delta P}^{(m+i+n)}$ is equal to:

$$\overline{\Delta P}^{(m+i+n)} = [(A^T A)^{(m-i-n)} + (k^2 I)^{(m+i+n)}]^{-1} (A^T \bar{d})^{(m-i-n)} \quad [11]$$

Figure 1 shows the chart of a computer program based on this first iterative approach.

In order to find a faster procedure two other approaches have been examined.

In the second approach, the equation [10] has been modified:

$$\bar{P}^{(m+i+n)} = \bar{P}^{(m-i+n)} + \overline{\Delta P}^{(m+i+n)} \quad [12]$$

Hence in the "neighbourhood test" we substitute for the vector $\bar{P}$ the current value of it that gives the minimum error $\bar{\rho}_a - \bar{\rho}'$ where for $\overline{\Delta P}$ we use the same expression [11].

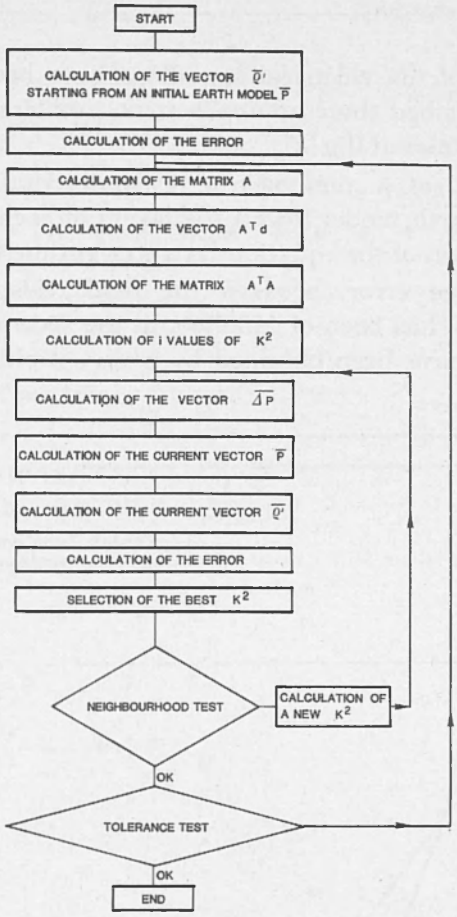


Figure 1 - First iterative approach flow chart

In the third approach, we use the expression [12] also modifying the $\overline{\Delta P}$ expression that becomes:

$$\overline{\Delta P}^{(m+i+n)} = [(A^T A)^{(m-i+n)} + (k^2 I)^{(m+i+n)}]^{-1} (A^T \bar{d})^{(m-i+n)} \quad [13]$$

## 3. Practical applications

Two sets of ten electrical soundings have been run utilizing the above described three approaches; no problem of convergence has been encountered (fig. 2).

In the first set, a common true resistivity value has been used as an initial earth model for all the layers of each VES obtaining as a first solution of the equation [1] a straightline (fig. 3).

The average error, between the field resistivities and the computed ones, has been of 113.91%. In the second set, the initial earth models have been obtained by a very rapid analysis of the

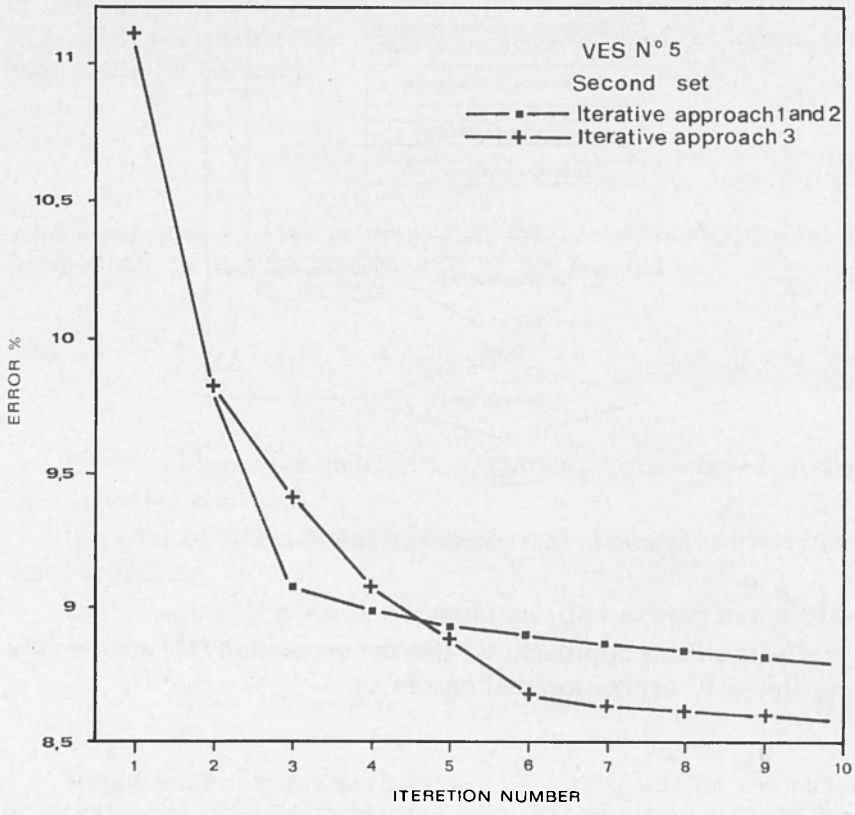


Figure 2 - Error's trend relative to the three iterative approaches

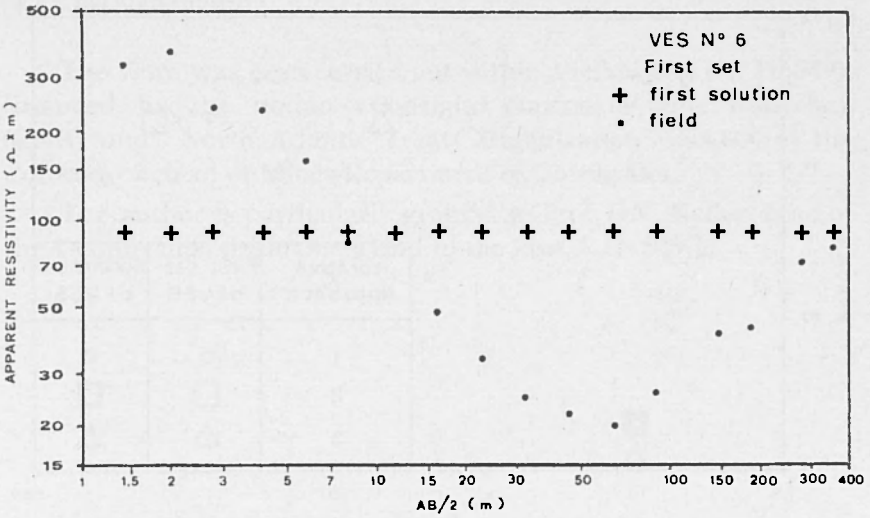


Figure 3 - First solution relative to an initial model apparteining to the first set

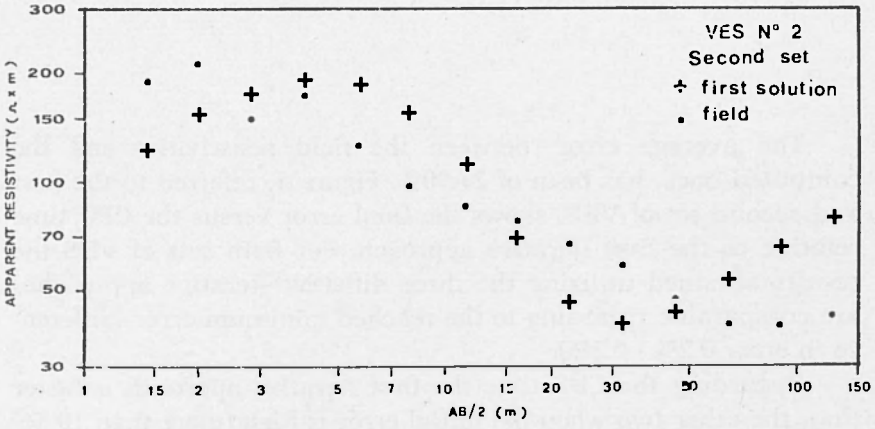


Figure 4 - First solution relative to an initial model apparteining to the second set

field curve, giving to the true resistivities an approximate asymptotic value and deducing the thicknesses from the abscissa of the inflection point (fig. 4).

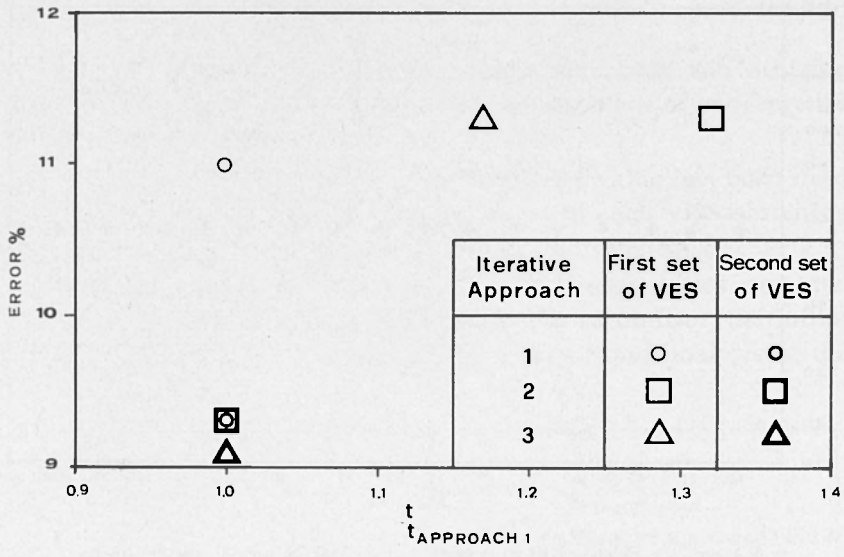


Figure 5 - Final error versus CPU time

The average error, between the field resistivities and the computed ones, has been of 27.90%. Figure 5, referred to the first and second set of VES, shows the final error versus the CPU time relative to the first iterative approach. For both sets of VES the results obtained utilizing the three different iterative approaches are comparable regarding to the reached minimum error (difference in error 0.2% - 0.3%).

Regarding the CUP time the first iterative approach is faster than the other two when the initial error is high (more than 100%) while the three iterative approaches are comparable when the initial error is small (20% - 30%). Results obtained by using the three iterative procedures applied to the above practical examples have shown that all the procedures are reliable and comparable among themselves regards to the minimum error reached and the CPU time.

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