Non - linear laminar flow of fluid into an open bottom well

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ABSTRACT

In steady state condition, non - linear laminar flow of fluid into an open bottom well just penetrating the semi-infinite porous aquifer is considered. The influence of non-linear laminar flow on discharge and its dependance on related physical quantities is examined. It is found that an open bottom well actually behaves like a hemispherical well, which is an obvious practical phenomenon.

RIASSUNTO

Si considera il flusso laminare non-lineare di un fluido in un pozzo a fondo aperto che penetri un acquifero poroso semi infinito.

Si esamina poi l'influenza del flusso laminare non-lineare sulla discarica e la sua dipendenza da quantità fisiche correlate. Si trova che un pozzo a fondo aperto si comporta come un pozzo emisferico che è un fenomeno pratico ovvio.

1. INTRODUCTION

In the study of flow through porous media, there exists a large number of investigations seeking steady state solutions of Darcy's

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law with respect to certain applications. One of the most important application is the study of flow of fluid from strata into wells. Many investigators obtained a series of such solutions (cf. [1], [2], [3]). The well penetrating fully the fluid bearing strata of finite thickness is termed as fully penetrating well otherwise partially penetrating well. But in a semi-infinite stratum, a well is always considered to be partially penetrating. In such cases, the flow in the region of sand not penetrated by the well will have an upward component of velocity tending to bring the fluid into the well, while the flow in the upper portion of the sand will mainly be radial as it will have a small negligible vertical component. The most interesting limiting case of a partially penetrating well is an 'open bottom well', which corresponds to a well just tapping a uniform sand of great thickness [1].

Beside the nature of a well the other important aspect is the type of flow which mainly depends upon the fluid velocity and the structural constitution of the porous matrix through which it flows. On the basis of fluid velocity, however, the flow [3] can be characterised into three different regimes — laminar, nonlinear laminar and turbulent, Reynolds number being the criteria for such demarcation. But, the intricacy of the nature of porous media does not always justify the natural flow of fluid to be purely laminar and it appears more desirable to be either non-linear laminar or turbulent. Consequently, UCHIDA, 1952, ENGELUND, 1953, ANANDKRISHAN and VARADARAJULU, 1963, WRIGHT, 1968, AHMAD and SUNADA, 1969, KHAN and RAZA, 1972, JAIN and UPADHYAY, 1976, UPADHYAY, 1977a b, and others obtained solutions of certain specific non-linear laminar flow problems.

In the present paper, we consider non-linear laminar radially symmetrical flow of an incompressible fluid into an open bottom well in steady state condition. The influence of non-linear laminar flow on discharge and its dependance on related physical quantities is examined. It is found that an open bottom well actually behaves like a hemispherical well, as in the former case where the well just penetrates the upper surface of semi-infinite aquifer, an hemispherical cavity is automatically formed to provide a surface for discharge. This is an obvious practical phenomenon.

2. Basic equations of fluid flow in porous medium

The Darcy's law governing the laminar flow of fluid in porous media is [2]

$$v = K \frac{dh}{ds}, \qquad [1]$$

v, k and $\frac{dh}{ds}$ denote the seepage velocity, seepage coefficient and hydraulic gradient respectively; flow being in the opposite direction of increasing h.

In case of laminar flow, the total head H_c at an infinitely large distance from the axis of the circular perforation of radius r_ω in the plane impervious boundary of semi-infinite homogeneous aquifer is

$$H_C = H_\omega + \frac{Q}{4 K r_\omega}, \qquad [2]$$

where Q denotes the flow rate and H_ω the head at the well surface.

Besides, relations [1] and [2], the law for non-linear laminar flow is [2]

$$\frac{dh}{ds} = av + bv^2, \qquad [3]$$

a and b being constants, which according to Engelund are

$$a = \frac{2000\,\mu}{\rho g d^2},\ b = \frac{35}{gd}, \qquad [4a. b]$$

where ρ and μ being density and viscosity of the fluid respectively and d the grain size of the medium.

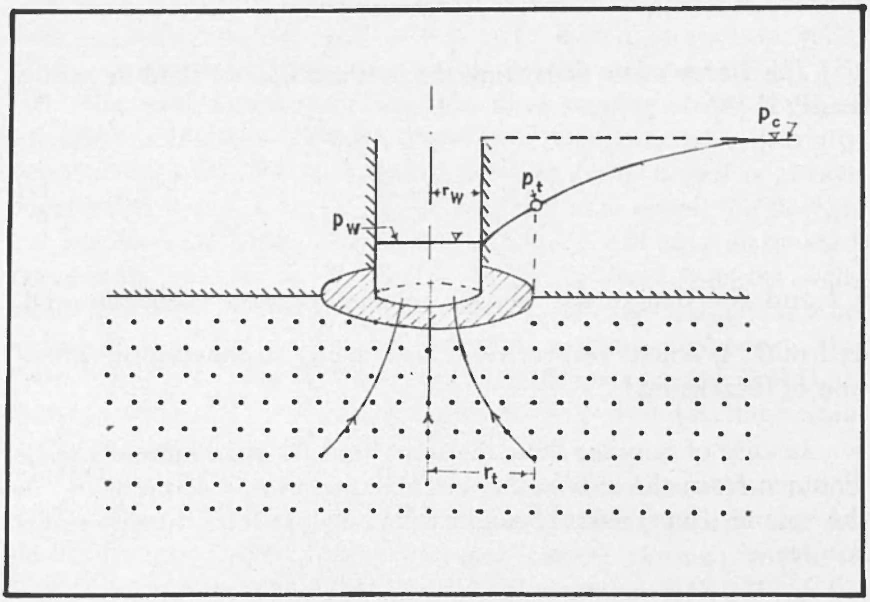

Fig. 1

3. Statement of the problem

In steady state condition, we consider radially symmetrical flow of an incompressible fluid into an open bottom well, which is a circular perforation with radius r_ω of an impervious boundary of semi-infinite porous aquifer (Fig. 1). The aquifer is assumed to be homogeneous and isotropic. The pressure at the contour of well and at the contour of intake are prescribed as p_ω and p_c respectively.

Let r be the radial distance measured from the axis of well. It is assumed that the flow is (i) non-linear laminar within the circular zone $r_\omega \leqslant r \leqslant r_t$, and (ii) laminar in the region $r_t < r \leqslant \infty$. Let p_t be the pressure at the transition boundary $r = r_t$.

The problem is to examine the influence of non-linear laminar flow on discharge, its dependance on the grain size of the medium and the viscosity of the fluid.

4. Solution

Since $p = \rho g h$, we write [2] as

$$p_c - p_\omega = \frac{Q \rho g}{4 K r_\omega} = \frac{Q \rho g}{4 K} \left(\frac{1}{\infty} - \frac{1}{r_\omega} \right), \qquad [5]$$

$$\longrightarrow \int_{p_\omega}^{p_c} \mathrm{d}p = - \frac{Q \rho g}{4 K} \left| \frac{1}{r} \right|_{r_\omega}^{\infty}, \qquad [6]$$

$$\longrightarrow p = C - \frac{Q \rho g}{4 K r}, \quad r_\omega \leqslant r \leqslant \infty, \qquad [7]$$

where C is the constant of integration determinable by boundary conditions. Thus, equation [7] describes the pressure distribution p for any arbitrary radius r ($r_\omega \leqslant r \leqslant \infty$) in the system for which equation [2] holds good.

Hence, from [7] the expression for pressure distribution at a radial distance r in the laminar zone $r_t < r \leqslant \infty$ is obtainable in the form

$$p_t = p_c - \frac{Q \rho g}{4 K r_t} \qquad [8]$$

For radially symmetrical flow, equation [1] becomes

$$v = \frac{K}{\rho g} \frac{\mathrm{d}p}{\mathrm{d}r} \qquad [9]$$

Substituting the value of $\frac{\mathrm{d}p}{\mathrm{d}r}$ from [7], equation [9] reduces to

$$v = \frac{Q}{4 r^2} \qquad [10]$$

where v is the seepage velocity at a radial distance r. Thus, v is independent of the well radius r_ω and z — the vertical coordinate.

Using $p = \rho g h$ in equation [3] and then combining it with relation [10], the expression for pressure distribution in the non-linear laminar zone is obtainable in the form

$$\frac{dp}{dr} = \rho g \left(\frac{a\,Q}{4\,r^2} + \frac{b\,Q^2}{16\,r^4} \right), \quad r_\omega \leq r < r_t \qquad [11]$$

Integrating [11] and evaluating the constant of integration with the help of boundary conditions

$$p = \begin{cases} p_\omega & \text{at} \quad r = r_\omega, \\ p_t & \text{at} \quad r - r_t, \end{cases} \qquad [12]$$

we obtain

$$p_t = p_\omega + \rho g \left[\frac{a\,Q}{4} \left(\frac{1}{r_\omega} - \frac{1}{r_t} \right) + \frac{b\,Q^2}{48} \left(\frac{1}{r_\omega^3} - \frac{1}{r_t^3} \right) \right] \qquad [13]$$

At the boundary of transition from laminar to non-linear laminar flow, the relation between critical Reynold's number ξ_c = 0.07 and critical velocity v_c is given by

$$v_c = \frac{Q}{4\,r_t^2} = \frac{a}{b}\,\xi_c \qquad [14]$$

Since at this boundary $\frac{dh}{ds}$ as given [1] and [3] yield the same value, it follows from [14] that

$$1.07\,a\,K = 1 \qquad [15]$$

Using [15] in [8] and then equating it to [13], we get

$$\frac{p_c - p_\omega}{\rho g} = \left[\frac{a\,Q}{4} \left(\frac{1}{r_\omega} + 0.07\,\frac{1}{r_t} \right) + \frac{b\,Q^2}{48} \left(\frac{1}{r_{\omega}^3} - \frac{1}{r^3} \right) \right] \qquad [16]$$

Combining equations [4 a, b] and [14] with [16], we obtain

$$\frac{\rho d^3 (p_c - p_\omega)}{\mu^2 r_\omega} = 8000 \left(\frac{r_t}{r_\omega}\right) \left[\frac{r_t}{r_\omega} + 0.0233 \left(\frac{r_t}{r_\omega}\right)^3 + 0.0467 \right] \quad [17]$$

where

$$r_t = \sqrt{\frac{Q \rho d}{16 \mu}} \quad [18]$$

If we assume purely laminar flow in the region $r_\omega \leq r \leq \infty$, then the flow rate Q_{lam} is obtainable from [7] in the form

$$Q_{lam} = \frac{4 K r}{\rho g} (p_c - p_\omega) \quad [19]$$

Hence from [14] and [19], we obtain the ratio

$$\frac{Q}{Q_{lam}} = 8560 \left| \frac{\mu^2 r_\omega}{\rho d^3 (p_c - p_\omega)} \right| \left(\frac{r_t}{r_\omega}\right)^2 \quad [20]$$

Introducing dimensionless quantity X and ratio Y such that

$$X = \frac{\rho d^3 (p_c - p_\omega)}{\mu^2 r_\omega}, \quad Y = \frac{Q}{Q_{lam}} \quad [21\ a, b]$$

and combining [20] with [17], we obtain an implicit relation

$$X = 8000 \left[\frac{XY}{8560} + 0.0233 \left(\frac{XY}{8560}\right)^2 + 0.0467 \left(\frac{XY}{8560}\right)^{1/2} \right] \quad (22]$$

This expression is the same as that obtained by the author [JAIN, 1981], which describes the non-linear laminar flow into a hemispherical well.

5. Conclusion

It may thus be concluded that an open bottom well actually behaves like a hemispherical well, as in the former case where the well just penetrates the upper surface of a semi-infinite porous aquifer, a hemispherical cavity is automatically formed to provide a surface for discharge. This is an obvious practical phenomenon, as our conclusion based on theoretical discussion justifies the statement: "... that is a practical case when well just taps the sand, the well surface is really a hemisphere".

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