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The direct estimation of the potential coefficients by biorthogonal sequences

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Abstract
The modelling of the earth gravity field is fundamental for all the sciences related to surveying, to the study of the internal structure of the earth and of its behaviour. For this reason one of the priorities of the International Geoid Service is collecting existing models and developing new models. In the present work a new method is introduced to compute the geopotential coefficients starting from the integral problem of altimetry-gravimetry by means of biorthogonal series. The theoretical discussion of the method is followed by the description of the software program (BOSALT) implemented for the computation and by numerical tests (with simulated data). This paper was originally presented by the authors as a seminar at the International Summer School of Theoretical Geodesy «Satellite Altimetry in Geodesy and Oceanography» held in Trieste (Italy) from May 25 to June 6, 1992.

Key words *altimetry-gravimetry problems – biorthogonal sequences – global geopotential models*

1. Introduction

The knowledge of the gravity field of the earth and its anomalies provides fundamental information for the construction of physical models of its internal structure and dynamics. Up to now, the techniques developed and the methods used to determine the parameters describing the spatial variations of the Earth's gravity field have produced models having different resolutions and precisions.

The classical methods to determine global gravity models consist in combining, in the least squares sense, different sets of normal equations coming both from the perturbation analysis of different satellite arcs and from data collected on the earth surface. Therefore, the data used to estimate the geopotential coefficients are:

1) on the earth surface: gravimetric and astrogeodetic (vertical deflections) measurements, observations of the torsion balance (locally);
2) analysis of the perturbations of satellite arcs;
3) radaraltimetric measurements.

The first set of coefficients were determined by the analysis of the perturbations of satellite arcs by Kaula (1966); since then several models have been computed and afterwards improved by different research groups in Europe and in the USA (the French-German GRIM models, the GSFC models, the Hannover University models and the OSU models).

The different solutions are due to the fact that the various groups make use of different data sets, different procedures and different numerical methods. It has been proved (Migliaccio and Sansó, 1989) that this may give rise to relative differences between the models already amounting to 100% at harmonic degree $l=25$.

Besides it has to be remembered that the classical satellite methods only provide the low frequency part of the field, while the data at ground level provide the high resolution part, thus leaving a lack of information at medium wavelengths (typically in the range of 50 to 500 km): this makes very interesting the development of new space techniques (like Satellite to Satellite Tracking or Satellite Gravity Gradiometry) with satellites at low altitude (about 200 km).

The work presented in this paper can be considered within the limits of classical methods for global model estimation.

The original part of our paper is represented by a new approach to determine the gravity model: starting from the boundary value problem of altimetry-gravimetry, the geopotential coefficients are directly computed from the observations by means of a proper biorthogonal series.

This idea is already suggested, although not presented in the same mathematical frame, in a paper by Mainville (1987).

The first part of this report is devoted to introducing and discussing the method adopted from a theoretical point of view, while in the second part a numerical example is presented (with data simulated from the model OSU91A), in order to test the reliability of the procedure.

2. The altimetry-gravimetry problems

The problems of altimetry-gravimetry (Sacerdote and Sansó, 1987) are part of the integral methods allowing for the estimate of the geoid starting from geodetic measurements referring to the earth surface, considered as a boundary surface for the gravity potential. The problem was firstly formulated and tackled by Stokes (1849) who, at the end of the 19th century, found an explicit solution on the sphere. Significant developments of this theory were subsequently due to Molodensky (Molodensky *et al.*, 1962) by setting a more direct problem where the reference surface is the unknown earth's surface.

According to this problem, the gravity potential w and the gravity vector $\underline{g}$ are determinable (and therefore known) quantities on the earth's surface S, which is unknown; starting from these assumptions it is possible to solve for the surface S and the potential w outside it.

In recent times, the development of geodesy (in particular satellite geodesy) has given further impulse to integral methods leading to the introduction of new problems, namely the altimetry-gravimetry problems. In fact, while satellite radaraltimetry allows to determine the shape of the ocean surface, by means of marine gravimetry techniques, high precision measures of gravity or of gravity disturbances δg on sea can be obtained. The surface S can be therefore suitably divided into two parts: S_S (on sea), which is supposed to be known, and S_L (on land), which is unknown.

On the surface S two different kinds of boundary data are given, depending on the point, which may belong either to S_S or to S_L.

Besides, two altimetry-gravimetry boundary problems are given, depending on the kind of data given on S_S.

In the first problem (Sansó, 1981) the boundary data are: the gravity potential w_L and the gravity vector $\underline{g}_L = \nabla w$ on S_L (unknown), and the gravity potential $\overline{w}_S$ = constant on S_S (known).

From these data one wants to determine S_L and $u = w - \frac{1}{2}\,\omega^2(x^2 + y^2)$ such that:

$$\begin{cases} \nabla^2 u = 0 & \text{outside } S \\ u = 0\left(\dfrac{1}{|\underline{x}|}\right) & |\underline{x}| \to \infty \\ \left.\begin{array}{l} w = w_L \\ \nabla w = \underline{g}_L \end{array}\right\} & \text{on } S_L \\ w = \overline{w}_S & \text{on } S_s \end{cases} \tag{2.1}$$

Now if (2.1) is linearized taking as reference surface a telluroid Σ such that $\Sigma_S = S_S$ and if the gravity potential w is approximated with the normal potential U ($w = U + T$), on Σ_L the following boundary condition is obtained:

$$(T + \underline{m} \cdot \nabla T)|_{\Sigma_L} = \underline{m} \cdot \Delta \underline{g} \qquad (2.2)$$

where $\underline{m}$ is the iso-zenithal field defined by:

$$\underline{m} = - [\nabla \underline{\gamma}]^{-1} \cdot \underline{\gamma} \qquad (2.3)$$

and $\underline{\gamma} = \nabla U$ is the normal gravity vector.

Instead, on Σ_S the problem is already linear and we have:

$$T|_{\Sigma_S} = \delta w_S \qquad (2.4)$$

Let's now introduce a simplification (spherical approximation) supposing the normal gravity vector being of the type:

$$\underline{\gamma} = - \frac{K}{r^2} \frac{\underline{r}}{r} \qquad (2.5)$$

with K = constant;

$\frac{\underline{r}}{r}$ = radial direction with respect to the center of mass of the telluroid.

The first problem of altimetry-gravimetry in spherical approximation can be therefore expressed in the simplified form:

$$\begin{cases} \nabla^2 T = 0 & \text{outside } \Sigma \\ T = 0\left(\frac{1}{r}\right) & r \to \infty \\ -\frac{2T}{r} - \frac{\partial T}{\partial r}|_{\Sigma_L} = \Delta g_L & \text{on } \Sigma_L \\ T|_{\Sigma_S} = \delta w_S & \text{on } \Sigma_S \end{cases} \qquad (2.6)$$

In the second problem of altimetry-gravimetry (Holota, 1980; Barzaghi *et al.*, 1990; Sacerdote and Sansó, 1983) the boundary data are the same as in the first one (w_L and $\underline{g}_L$) on S_L, while on S_S the known quantity is $g_S = |\nabla w|$. After linearization, in the hypothesis of spherical approximation, this problem is represented by:

$$\begin{cases} \nabla^2 T = 0 & \text{outside } \Sigma \\ T = 0\left(\frac{1}{r}\right) & r \to \infty \\ -\frac{2T}{r} - \frac{\partial T}{\partial r}|_{\Sigma_L} = \Delta g_L & \text{on } \Sigma_L \\ -\frac{\partial T}{\partial r}|_{\Sigma_S} = \delta g_S & \text{on } \Sigma_S \end{cases} \qquad (2.7)$$

The two altimetry-gravimetry problems have different applications depending on the actual available data on the surface.

The second problem is typically used for the local estimate of the geoid (particularly in the case of islands of small or medium dimensions; for example, regarding the Mediterranean Sea, Sicily and Sardinia).

Instead, the first problem can have more general applications; in fact the boundary datum on the sea surface is the anomalous potential T, which can be obtained by treating radaraltimetric observations.

As these observations are (by their nature) omogeneous and equally distributed, it is possible, as we will see, to utilize the first problem for the estimate of global models of the potential.

3. Estimate of the geopotential coefficients with the first altimetry-gravimetry problem

Let's reconsider the altimetry-gravimetry problem in the simplified form obtained in the case of a spherical approximation. We must refer to expression (2.6).

The boundary conditions can be reorganized by introducing the boundary operator B defined as:

$$BT = \chi_S T + \frac{\sigma_T}{\sigma_{\Delta g}} \left(- \frac{\partial T}{\partial r} - \frac{2}{R} T\right) \chi_L \qquad (3.1)$$

where: χ_S = characteristic function of sea, that is:

$$\chi_S = \begin{cases} 1 & \text{if } P \in \Sigma_S \\ 0 & \text{if } P \notin \Sigma_S \end{cases} \quad (3.2)$$

χ_L = characteristic function of land;

R = average radius of the earth;

$\dfrac{\sigma_T}{\sigma_{\Delta g}}$ = regularization factor, accounting for the two different kinds of boundary data.

As for the $\dfrac{\sigma_T}{\sigma_{\Delta g}}$ factor let us note that this is «regularizing» only in the sense that by introducing it we can define the known term of our problem so as to avoid very large discontinuities across the shore lines (*e.g.* discontinuities of 5 orders of magnitude). Such huge discontinuities in fact can generate very large degree variances at high frequencies, degrading the approximation we are going to use by discretizing integrals.

We decompose T on the basis of the spherical harmonics

$$T = \sum_{lm} T_{lm} Y_{lm} \quad (3.3)$$

where Y_{lm} are normalized according to the norm

$$\frac{1}{4\pi}\int_\sigma Y_{lm}^2 d\sigma = 1$$

and we use the definition of the operator B to get

$$BT = \sum_{lm} T_{lm} BY_{lm} = \sum_{lm} T_{lm}\left\{\chi_S + \frac{\sigma_T}{\sigma_{\Delta g}}\,\frac{1-1}{R}\,\chi_L\right\} \quad (3.4)$$

Let's indicate the boundary data with f:

$$f = \chi_S T_o + \frac{\sigma_T}{\sigma_{\Delta g}}\,\Delta g_o \chi_L \quad (3.5)$$

where T_o and Δg_o are the observations (on sea and on land respectively) of the anomalous potential and of the gravity anomaly.

We consider now the boundary equation:

$$BT = f \quad (3.6)$$

in order to directly estimate the geopotential coefficients T_{lm} from the observations f. This is possible, as we will see, if it exists and if it is possible to construct a sequence biorthogonal to the sequence $\{BY_{lm}\}$.

Let's then briefly introduce the biorthogonal sequences and their main properties related to our purposes.

The sequences $\{a_i\}$ and $\{b_i\}$ form a biorthogonal system in a Hilbert space $\mathcal{H}$ if:

$$<a_i,b_j> = \delta_{ij} \quad (3.7)$$

here $<a,b>$ denotes the natural internal product (scalar product) of the space $\mathcal{H}$, which in this paper will be mostly $L^2(\sigma)$, (*i.e.* $<a,b> = \dfrac{1}{4\pi}\displaystyle\int_\sigma a(\phi,\lambda)n(\phi,\lambda)d\sigma$), but can have a more general expression, compatibly with the definition of a Hilbert space.

If $\{a_i\}$ is a complete sequence in $\mathcal{H}$, also $\{b_i\}$ is complete; every element h of $\mathcal{H}$ can therefore be expressed either by:

$$h = \sum_i <h,a_i>\, b_i \quad (3.8)$$

or by:

$$h = \sum_i <h,b_i>\, a_i \quad (3.9)$$

Let's now consider (3.9) and let's take as space $\mathcal{H}$ the one spanned by the Y_{lm}, that is a space of functions which are square integrable on the sphere.

If we can construct a sequence $\{Z_{pq}\}$ biorthogonal to $\{BY_{lm}\}$, it will be:

$$f = \sum_{lm} <f,Z_{lm}>\, BY_{lm} \quad (3.10)$$

and consequently:

$$T_{lm}(P) = <f,Z_{lm}> = \frac{1}{4\pi}\int_\sigma f(P)Z_{lm}d\sigma \qquad (3.11)$$

Before constructing the biorthogonal series $\{Z_{pq}\}$, its existence must be studied. This implies studying the characteristics of the operator B. In Sansó (1993) it is shown that the operator B possesses the property of Fredholm's alternative, so that, if $\{Y_{lm}\}$ is an orthonormal basis in the space $\mathcal{H}$, a series biorthogonal to $\{BY_{lm}\}$ must exist.

Such a series is a solution of system:

$$<Z_{pq}, BY_{lm}> = \delta_{pl}\delta_{qm} \qquad (3.12)$$

However the problem, formulated in this way, is not solvable because there are infinite unknown elements Z_{pq}.

If we want to practically determine the sequence $\{Z_{pq}\}$ the system (3.12) must truncated; this implies the hypothesis that T is of finite degree:

$$T = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} T_{lm}Y_{lm} \qquad (3.13)$$

so we look for the biorthogonal sequence $\{Z_{pq}\}$ with $p = 1,...,l_{\max}$ and $-l \le q \le 1$ which satisfies the condition:

$$\frac{1}{4\pi}\int_\sigma Z_{pq}BY_{lm}d\sigma = \delta_{pl}\delta_{qm} \qquad (3.14)$$

In order to simplify the mathematical notation, let's also rearrange the harmonics associating to the indices l and m the index j and let's introduce the new symbols:

$$BY_{lm} = W_j$$

$$Z_{pq} = b_k$$

which take into account the previous correspondence.

In this case (3.14) simply becomes:

$$\frac{1}{4\pi}\int_\sigma b_k(P)\mathrm{W}_j(P)d\sigma = \delta_{kj} \qquad (3.15)$$

For what concerns the construction of the sequence, we first of all observe that, once the set $\{W_j\}$, $j = 1,..., L$, is given, $\{b_k\}$ isn't univocally determined.

In fact, if W is the space spanned by the $\{W_j\}$ and $\mathcal{W}^\perp$ its orthogonal complement, and if the following condition holds:

$$<b_k,W_j> = \delta_{kj} \qquad (3.16)$$

we have also:

$$<b_k + w^\perp,W_j> = \delta_{kj} \qquad \forall w^\perp \in \mathcal{W}^\perp \qquad (3.17)$$

It is therefore necessary to define a criterion in order to make the b_k unique.

A possible criterion is for example the minimum norm: from it comes that the biorthogonal sequence b_k necessarily belongs to the space $\mathcal{W}$. In fact if we consider any $b_k' = b_k + b_k^\perp$, with $b_k \in \mathcal{W}$ and $b_k^\perp \in \mathcal{W}^\perp$, condition (3.16) is still satisfied, but we have:

$$\| b' \| = \int_\sigma b_k'^2 d\sigma > \int_\sigma b_k^2 d\sigma = \| b \| \qquad (3.18)$$

As a consequence the b_k can be obtained applying the minimum norm criterion as those linear combinations of the W_j

$$b_k = \sum_{j=1}^{L} \mu_j^k W_j \qquad (3.19)$$

which satisfy equations (3.15). In this case they are given by:

$$\begin{aligned}\frac{1}{4\pi}\int_\sigma b_k W_j d\sigma &= \frac{1}{4\pi}\int_\sigma \sum_{l=1}^{L} \mu_l^k W_l W_j d\sigma = \\ &= \sum_{l=1}^{L} \frac{\mu_l^k}{4\pi}\int_\sigma W_l W_j d\sigma = \delta_{kj}\end{aligned} \qquad (3.20)$$

and for them holds:

$$\int_\sigma b_k^2\, d\sigma = \min$$

Although the problem of constructing the

b_k is correctly set from an analytical point of view, it does not possess an easily derivable solution. A better procedure in the determination of the biorthogonal sequence may be employed, taking into account the fact that our goal is to use the sequence in order to estimate the $\hat{T}_k$, given by the relation:

$$\hat{T}_k = \frac{1}{4\pi}\int_\sigma b_k f(P) d\sigma_P$$

(where the symbol $\wedge$ denotes the estimated value) or better, if we want to practically compute the coefficients, by the discretized expression:

$$\hat{T}_k = \frac{1}{4\pi}\sum_{i=1}^{M} b_k(P_i)\bar{f}(P_i)S_i \tag{3.21}$$

in which S_i are the areas, of center P_i, forming the grid used to discretize the earth's surface and M is the number of areas. In (3.21) we have introduced the approximation:

$$f(P_i) \cong \frac{1}{S_i}\int_{S_i} f(P) d\sigma = \bar{f}(P_i) = \bar{f}_i \tag{3.22}$$

Starting from (3.22) is then more convenient to represent the b_k as:

$$b_k(P) = \sum_{i=1}^{M} \lambda_i^k \chi_i(P) \tag{3.23}$$

with:

$$\chi_i(P) = \begin{cases} 1 & \text{if } P \in S_i \\ 0 & \text{if } P \notin S_i \end{cases}$$

In this way we directly get:

$$\frac{1}{4\pi}\int_\sigma b_k f(P) d\sigma_P = \frac{1}{4\pi}\sum_{i=1}^{M} \lambda_i^k \int_{S_i} f(P) d\sigma = \tag{3.24}$$

$$= \frac{1}{4\pi}\sum_{i=1}^{M} \lambda_i^k S_i \bar{f}(P_i)$$

so that the estimate of the T_k is directly expressed as a function of the observable.

Yet if we put, as it is natural, $M > L$ (that is the number of the areas, and then of the observations, larger than the number of the coefficients to be estimated) the λ_i^k coefficients are still not unique and the problem consists also in this case in applying a minimum norm criterion, that is:

$$\int_\sigma b_k^2 d\sigma = \frac{1}{4\pi}\sum_{i=1}^{M} (\lambda_i^k)^2 S_i = \sum_{i=1}^{M} (\lambda_i^k)^2 p_i = \min \tag{3.25}$$

with $p_i = \dfrac{S_i}{4\pi}$.

Let's minimize the previous relation by means of the Lagrange multipliers, considering as bond conditions the biorthogonality ones, given (for k fixed and j=1,..., L) by:

$$\frac{1}{4\pi}\int_\sigma \sum_{i=1}^{M} \lambda_i^k \chi_i W_j d\sigma = \sum_{i=1}^{M} \lambda_i^k p_i \frac{1}{S_i}\int_{S_i} W_j d\sigma = \delta_{kj} \tag{3.26}$$

Setting:

$$\frac{1}{S_i}\int_{S_i} W_j d\sigma = A_{ij} \tag{3.27}$$

we have:

$$\frac{1}{2}\sum_{i=1}^{M} (\lambda_i^k)^2 p_i - \sum_{j=1}^{L} \gamma_j^k \left(\sum_{i=1}^{M} \lambda_i^k p_i A_{ij}\right) = \min \tag{3.28}$$

in which the γ_j^k are the Lagrange multipliers.

Minimizing (3.28) we have, $\forall\ i$:

$$\lambda_i^k p_i - p_i \sum_{j=1}^{L} \gamma_j^k A_{ij} = 0$$

that is:

$$\lambda_i^k = \sum_{j=1}^{L} \gamma_j^k A_{ij} \tag{3.29}$$

Substitution of (3.28) into the orthogonality conditions gives

$$\sum_{l=1}^{L} \gamma_l^k \left(\sum_{i=1}^{M} p_i A_{il} A_{ij} \right) = \delta_{kj} \qquad (3.30)$$

which can be written in the compact form:

$$\mathbb{G} \cdot \mathbb{N} = \mathbb{I} \qquad (3.31)$$

by introducing the matrices:

$$\mathbb{G} = [\gamma_l^k]$$

$$\mathbb{N} = \mathbb{A}^{+}\mathbb{P}\mathbb{A},\ \mathbb{A} = [A_{ij}] \text{ and } \mathbb{P} = [p_i]$$

where $\mathbb{A}^{+}$ is the transpose of the matrix $\mathbb{A}$.

The solution of (3.31) gives the estimate of the Lagrange multipliers. This passage, involving problems related to the inversion of matrix $\mathbb{N}$, in general full and of large dimensions (see section 4), can be avoided. In fact let's suppose for the moment that the Lagrange multipliers are known; the sequence biorthogonal to the $\{W_j\}$ is given by:

$$\hat{b}_k = \sum_{i=1}^{M} \frac{1}{p_i} \sum_{l=1}^{L} \hat{\gamma}_l^k A_{il} \qquad (3.32)$$

Since the $\{b_k\}$ is biorthogonal to the $\{W_j\}$ we know that:

$$\hat{T}_k = <\hat{b}_k, f> = \frac{1}{4\pi} \int_\sigma \sum_{i=1}^{M} \hat{\lambda}_i^k \chi_i(P) f(P) d\sigma =$$

$$= \sum_{i=1}^{M} \hat{\lambda}_i^k p_i \bar{f}(P_i) \qquad (3.33)$$

substituting in the previous expression the $\hat{\lambda}_i^k$ given by (3.29) we obtain:

$$\hat{T}_k = \sum_{l=1}^{L} \hat{\gamma}_l^k \sum_{i=1}^{M} p_i A_{il} \bar{f}(P_i) \qquad (3.34)$$

Then the geopotential coefficients are obtained from

$$\underline{T} = \mathbb{G}\, d \qquad (3.35)$$

with $\underline{T} = [T_1 \dots T_L]$ and $\underline{d} = \mathbb{A}^{+}\mathbb{P}\underline{\bar{f}}$, $\underline{\bar{f}} = [\bar{f}(P_1) \dots \bar{f}(P_M)]$, which simply implies the solution of:

$$\mathbb{G}^{-1}\underline{T} = \underline{d}$$

and therefore:

$$\mathbb{N}\underline{T} = \underline{d} \qquad (3.36)$$

4. A numerical experiment

In this section we shall present a numerical experiment which was performed in order to assess the validity of the equations obtained in the former part of the paper.

The theoretical formulas were converted into suitable algorithms and implemented in a Fortran program running on a Unisys 2200 computer.

As this was just a first test of the theory, no real data were used. Instead, the observations were simulated according to a scheme which, although simplified, was a quite realistic one.

The earth was covered with a regular geographical grid having block size equal to $5^\circ \times 5^\circ$: as a consequence the coast-lines were approximated and square-shaped. The blocks were superimposed on a planisphere obtained with a Mercator cylindrical projection.

An observation was supposed to be performed at the center of each block, representing the mean value of all observation performed on that block area according to the formula (3.22).

Introducing the characteristic functions of sea and land the expression of the observation at the point P_i can be written, for the purpose of programming, in the following way:

$$\bar{f}(P_i) = \chi_S T_o + \frac{\sigma_T}{\sigma_{\Delta g}} \cdot \Delta g_o \cdot \chi \qquad (4.1)$$

The factor $\sigma_T/\sigma_{\Delta g}$ must be introduced in order to obtain a homogeneous set of data starting from the original T and Δg values.

In fact one must remember that the differences between T values at the sea level and Δg values on land are of the order of 10^6. In particular, the values we simulated on sea and land (by means of software for harmonic synthesis on the sphere and using the OSU91A model (Rapp *et al.*, 1991) with coefficients of degree 10 to 35) had the following characteristics:

$$\overline{T} = 0.2025 \cdot 10^{-7} \cdot \frac{GM}{R}$$

$$\sigma_T = 0.6034 \cdot 10^{-6} \cdot \frac{GM}{R}$$

$$\overline{\Delta g} = 0.3013 \cdot 10^{-13} \cdot \frac{GM}{R}$$

$$\sigma_{\Delta g} = 0.1284 \cdot 10^{-11} \cdot \frac{GM}{R}$$

GM = gravitational constant x earth's mass = $3.986009 \cdot 10^{14}$ m^3/s^2; R = mean radius of the earth = 6371000 m.

The $\overline{f}\ (P_i)$ values obtained according to the formula (4.1) had mean value:

$$\overline{f} = 0.2145 \cdot 10^{-7}\ GM/R$$

and mean square error:

$$\sigma_f = 0.6001 \times 10^{-6}\ GM/R$$

The $\overline{f}\ (P_i)$ values represent the «known term» of our problem. Though of course it is not a least squares problem, we shall use the same notations and terminology of least squares as they fit our purpose equally well.

Now let's turn to the algorithms for the solution of the altimetry-gravimetry problem with the use of biorthogonal sequences. What we want to achieve is the estimate of a geopotential model which, under our simplified assumptions, has coefficients ranging from degree 10 to degree 35.

The first step we must consider is how to form the elements of the «design matrix» $\mathbb{A}$. The A_{ij}, given by (3.27), represent the mean values of the harmonics up to degree and order L: in our case $L = 35$ (remember that we have also $L_{\min} = 10$).

Using again the characteristic functions χ_S and χ_L, the value of W_j is given by:

$$W_j = \left\{ \chi_S + \frac{\sigma_T}{\sigma_{\Delta g}} \cdot \frac{l-1}{R} \chi_L \right\} Y_j \qquad (4.2)$$

where Y_j is the harmonic function j. Note that indexes l,m usually denoting a harmonic function have now conveyed into the single j index.

Index i refers to the S_i area on the sphere. A simple way (which of course introduces an approximation in the computations) to obtain the mean values of the harmonics is to use the Pellinen β_l factors (Pellinen, 1966):

$$\beta_l = \frac{2}{1 - \cos\psi} \frac{1}{2l+1} \left[\overline{P}_{l+1}(\cos\psi) - \overline{P}_{l-1}(\cos\psi) \right] \qquad (4.3)$$

They represent the eigenvalues of a moving average operator: the field is averaged over a cap of radius ψ (Sacerdote and Sansó, 1991). The $\overline{P}_l$ (cos ψ) are the fully normalized Legendre polynomials.

Multiplying each harmonic by the corresponding β_j coefficient, one obtains the A_{ij} element of the matrix $\mathbb{A}$, according to the formula:

$$A_{ij} \cong p_i\ [\beta_j\ W_j(P_i)] \qquad (4.4)$$

where p_i, as already mentioned, is a weight accounting for the fact that the area of the blocks varys with the latitude.

The «almost equal» sign $\cong$ is used because we have approximated an integral of Y_{lm} over a rectangular block by an integral over a cap with the same area.

Equation (4.4) directly allows for the calculation of the design matrix needed for the solution of the problem.

Nevertheless, one should remember what are the dimensions of that matrix: they are fixed by two parameters, the block size and the maximum degree of the model. In our case, with block size 5° × 5°, the grid is di-

Table I. Memory requirements (Mbytes) for matrices $\mathbb{A}$ and $\mathbb{N}$.

Block size	N. of rows	L	N. of coeff.	Memory requirements	
				$\mathbb{A}$	$\mathbb{N}$
$5° \times 5°$	2592	35	1369	28	15
$3° \times 3°$	7200	60	3721	215	111
$2° \times 2°$	16200	90	8281	1080	550
$1° \times 1°$	64800	180	32761	17000	8600
$0.5° \times 0.5°$	259200	360	130321	270000	136000

vided into $36 \times 72 = 2592$ blocks represented by the corresponding 2592 rows of matrix $\mathbb{A}$. Its columns correspond to the (sine and cosine) harmonics: 1196 in our case. The memory requirement for such a matrix is 28 Mbytes (in double precision), although nearly half this storage area can be saved if the «normal matrix» $\mathbb{N}$ is directly computed. For larger problems this can be of great importance, as one can see from table I, where the dimensions of matrix $\mathbb{A}$ are reported, with the subsequent memory requirements (in Mbytes) for $\mathbb{A}$ and $\mathbb{N}$.

To directly compute both the normal and the known normal term is quite easy, realizing that the elements needed can be achieved summing the contributions coming from each row of matrix $\mathbb{A}$: in this way only one row at a time of $\mathbb{A}$ has to be computed and what's more it doesn't need to be kept in core memory. The formulas allowing for such computations are the following:

$$n_{ik} = \sum_j a_{ji}\, a_{jk} \tag{4.5}$$

$$d_i = \sum_j a_{ji} f_j \tag{4.6}$$

where:

n_{ik} = element (i,k) of the normal matrix $\mathbb{N}$;

d_i = element (i) of the normal term $\underline{d}$;

a_{ji} = element (j,i) of the design matrix $\mathbb{A}$;

f_j = element (j) of the known term.

For a better understanding of how these algorithms work, one may refer to fig. 1 and 2.

Formulas (4.5) and (4.6) correspond, from the point of view of the software programmer, to algorithms which allow for the computation of the single contribution «j» to the element

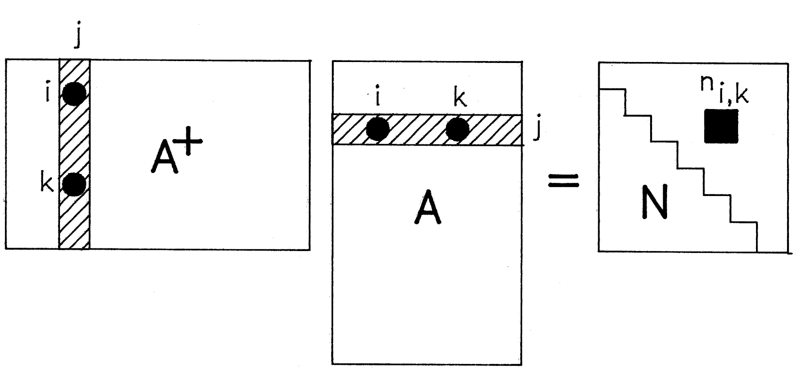


Fig. 1. Forming the elements of $\mathbb{N}$ adding the contributions of the single rows of $\mathbb{A}$.

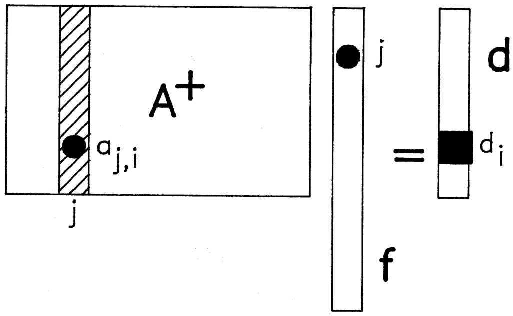


Fig. 2. Forming the elements of $\boldsymbol{d}$ adding the contributions of the single rows of $\mathbb{A}$.

required:

$$[n_{ik}]_{«j»} = p_j [\beta_i W_i(P_j) \beta_k W_k(P_j)] \qquad (4.7)$$

$$[d_i]_{«j»} = p_j [\beta_i W_i(P_j)] \overline{f}_j \qquad (4.8)$$

As one can see, the contributions «j» directly come from the point P_j on the grid.

Before discussing the results obtained from the numerical experiment, it is worth remembering the problem of the compatibility of the grid dimensions and the maximum estimable degree of the model, which is not of secondary importance.

The data we are using represent a sampling of data on the sphere, so the whole matter is settled by the Nyquist frequency rule, stating that the Nyquist frequency must not be exceeded. For a problem on the sphere, things are more complicated than in the unidimensional case: anyway, from a practical point of view, one can say that having an equiangular grid on the sphere of $2M^2$ points, the Nyquist frequency is given by the number of parallels (or rows), that are M or, to be more precise, all the coefficients of degree M can be estimated, except those of order M, as for them the Nyquist frequency is reached.

Exceeding the Nyquist frequency means giving rise to a folding of the spectrum with power from higher frequencies entering into lower frequencies and introducing an aliasing effect.

In our case, with a $5^\circ \times 5^\circ$ grid, $M = 36$. Coefficients $C_{36,36}$ and $S_{36,36}$ cannot be estimated and, in order to be far from the Nyquist frequency, we decided to deal with a model with maximum degree equal to 35.

Now coming to the test and its results, the main features are given in table II.

As one can see, it is a quite burdensome problem from the computational point of view, even if the model is not a large one. The data were simulated by means of a proper software for harmonic synthesis on the sphere: coefficients of model OSU91A were used. No noise was added to the Δg (on land) and T (on sea) values obtained.

A new set of coefficients was reconstructed using the Fortran program BOSALT, implementing the algorithms for the solution of the altimetry-gravimetry problem by biorthogonal sequences.

The estimated set was compared with the original OSU91A set of coefficients. First of all, relative differences were computed according to the formula:

$$d_r = \frac{C_{\mathrm{OSU91A}} - \hat{C}}{C_{\mathrm{OSU91A}}} \qquad (4.9)$$

where:

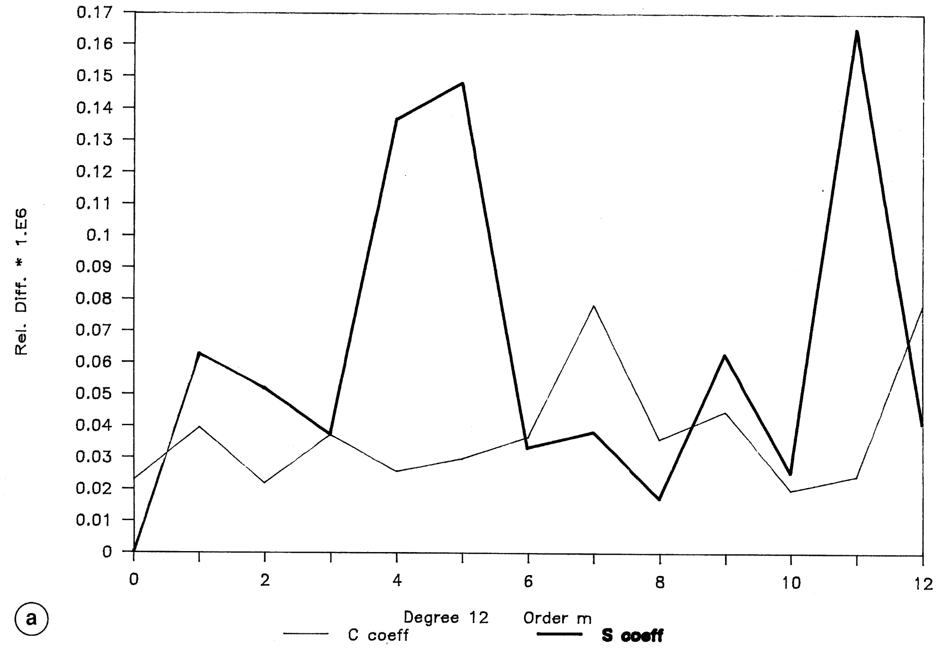
C_{OSU91A} = coefficient of model OSU91A;

$\hat{C}$ = stimated coefficient.

The results were very good: examples are

Table II. Characteristics of the test performed.

Block size	N. of blocks on the sphere	Model to be estimated	N. of coeff. to be estimated	CPU time hh:mm:ss
$5^\circ \times 5^\circ$	2592	$L_{min} = 10$ $L = 35$	1196	01:46:44



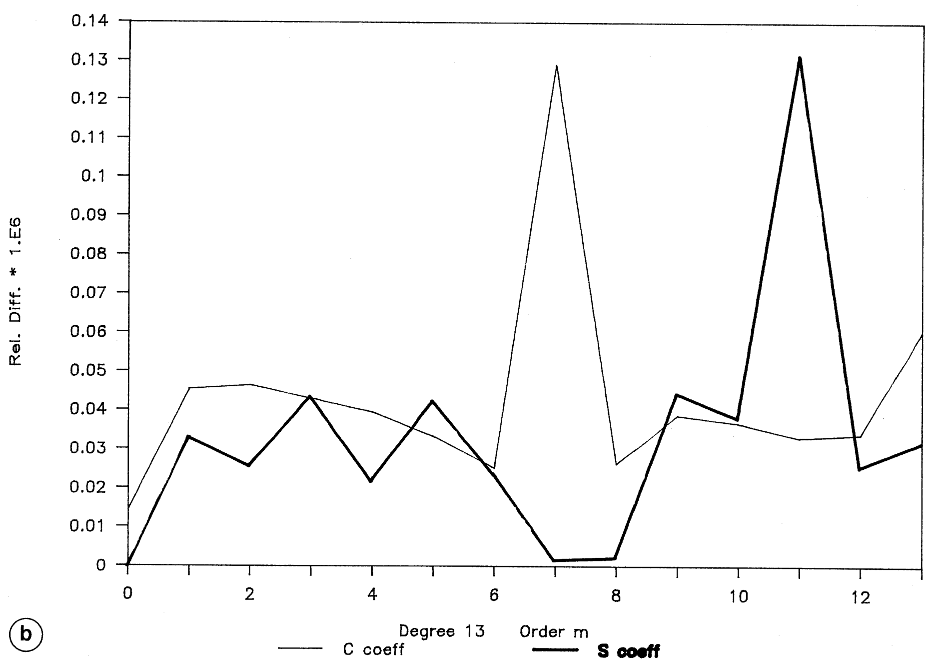


Fig. 3a,b. Relative differences for degree 12 (a) and degree 13 (b), computed according to formula (4.9).

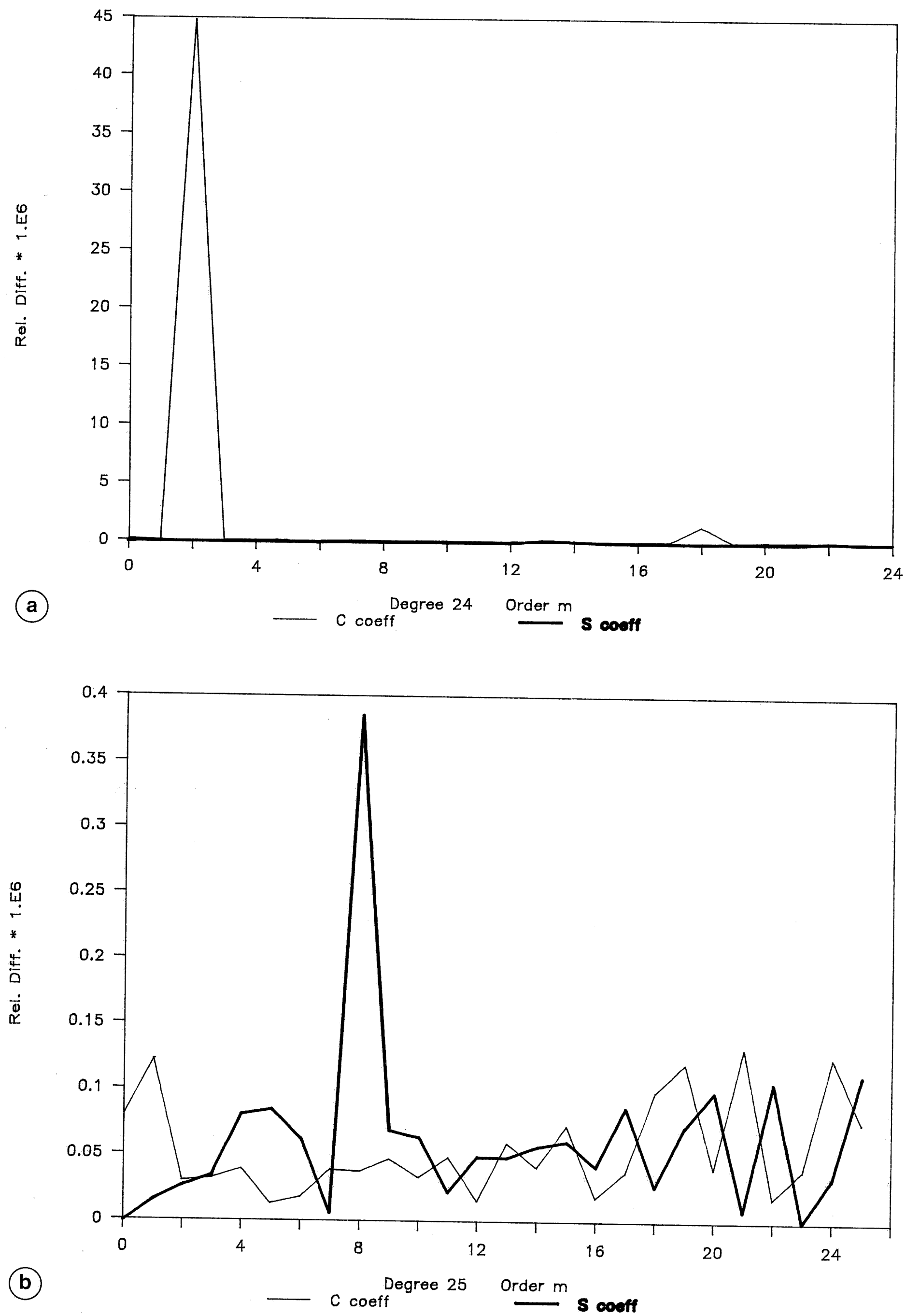


Fig. 4a,b. Relative differences for degree 24 (a) and degree 25 (b), computed according to formula (4.9).

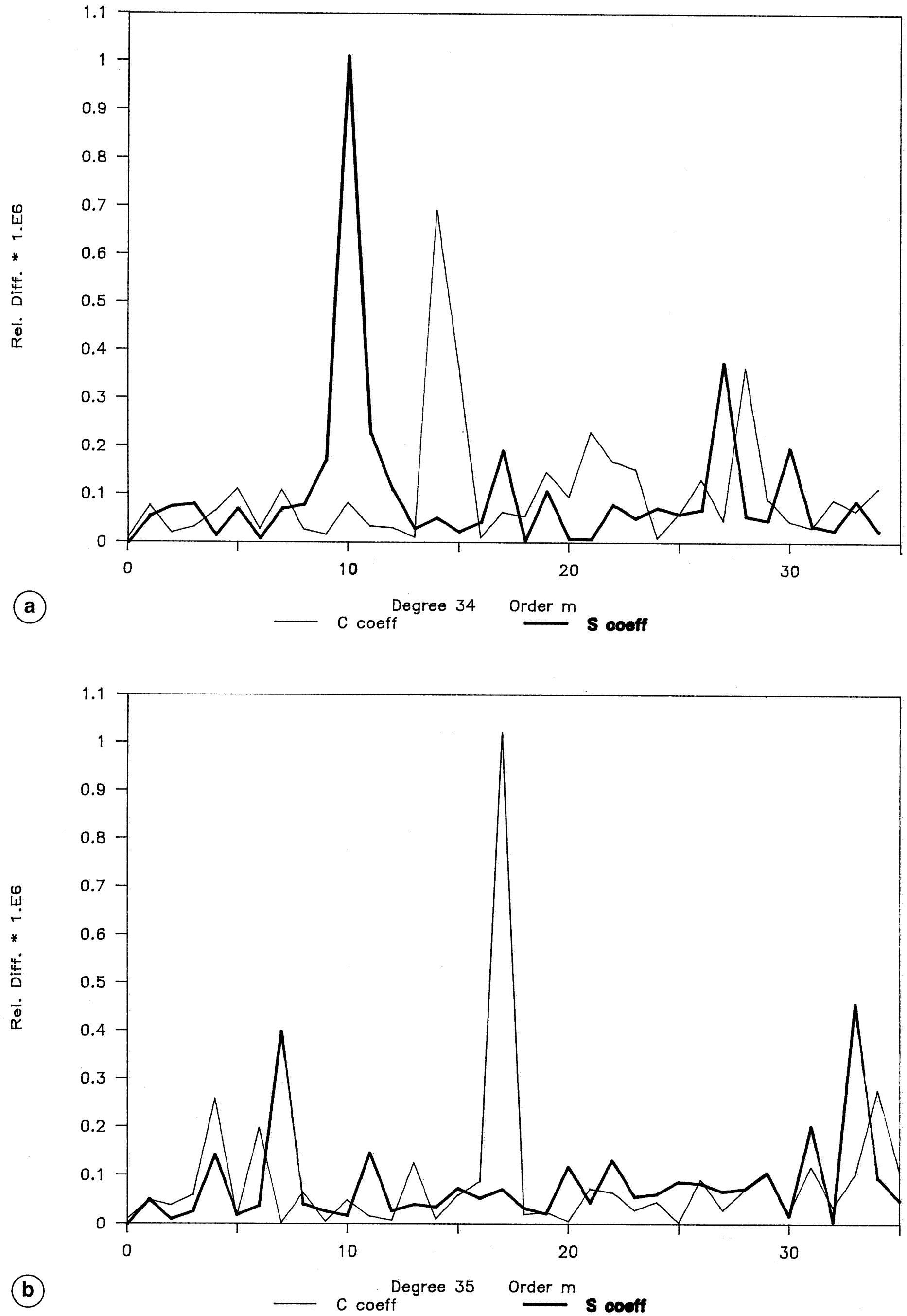


Fig. 5a,b. Relative differences for degree 34 (a) and degree 35 (b), computed according to formula (4.9).

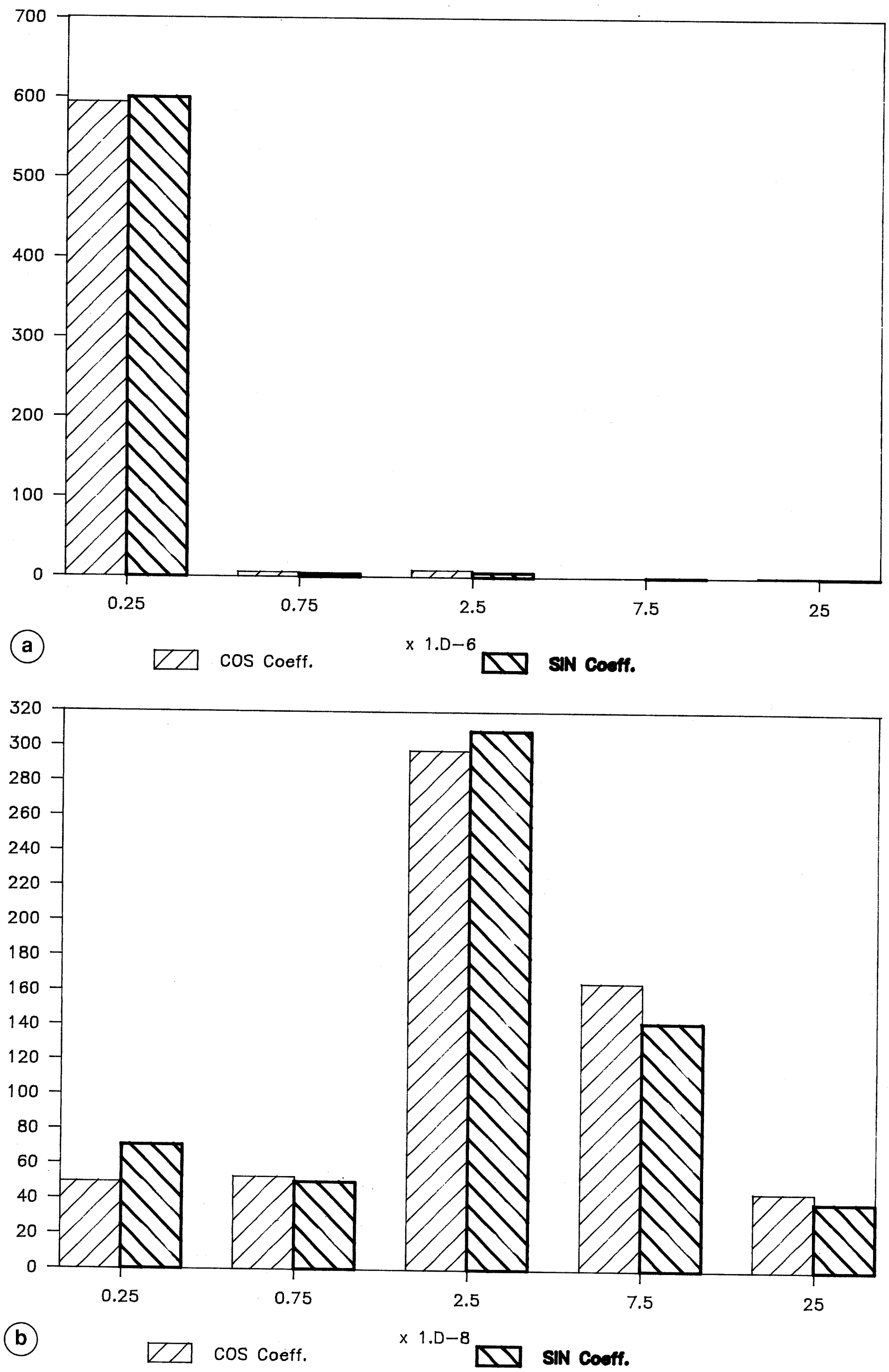


Fig. 6a,b. Histogram of relative differences obtained according to equation (4.9) (a) and equation (4.10) (b).

given in fig. 3, 4 and 5. Attention must be paid to the y-axis, where relative differences are multiplied by a 10^6 factor. Both the lower and higher degrees of the model were reconstructed equally well.

Another graphic evidence of the goodness of the estimation is given in fig. 6a,b.

In fig. 6a the histogram of relative differences calculated with equation (4.9) is shown: nearly all differences are smaller than $0.5 \cdot 10^{-6}$.

Another kind of relative differences was obtained with respect to the mean square values of the coefficients of each degree, using the following formula:

$$d_r' = \frac{C_{\mathrm{OSU91A}} - \hat{C}}{\sqrt{\Sigma C^2_{\mathrm{OSU91A}}}} \tag{4.10}$$

The histogram of such differences is shown in fig. 6b, giving another proof of the precision attainable by the method described.

Of course this was just the very first test performed. More refined computations need to be done, in order to accurately check the proposed method. The next steps to be taken are:

– adding a noise to the data, to see how it propagates to the estimated model;
– treating real data of Δg on land and T on sea, to deal with a real estimation problem.

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