A note on two expressions for the spatial power spectrum of the geomagnetic field

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Abstract
The spatial power spectrum of the geomagnetic field is usually assumed to be exponential in the spherical harmonic degree (n). An equally valid assumption, however, is that the spectrum at the surface of the Earth's core has a power-law dependence on n. Some implications of these two assumptions are discussed, noting that, in the first case, a value for the core radius can be derived from the slope of the spectrum and, in the second, that the spectrum for radii greater than the radius of the core is a mixture of the exponential and power-law types.

Key words *geomagnetic field – spatial power spectrum*

1. Introduction

The spatial power spectrum of the geomagnetic field is given, in terms of the coefficients (g_n^m, h_n^m) of the spherical harmonic expansion of the field, by the expression (Lowes, 1966)

$$R_n(r) = (a/r)^{2n+4}(n+1)\sum_{m=0}^{n}[(g_n^m)^2 + (h_n^m)^2], \tag{1.1}$$

where $R_n(r)$ is the mean square value of the terms of degree n in the spherical harmonic expansion of the geomagnetic field $\boldsymbol{B}$ over the surface of a sphere of radius r; a is the radius of a reference sphere, usually taken to be the mean radius of the Earth; and m is the spherical harmonic order. A plot of $R_n(r)$ against n is a representation of the spatial power spectrum of the geomagnetic field at radius r.

Analytical expressions for the power spectrum are often (*e.g.*, Langel and Estes, 1982) derived by fitting a function depending exponentially on n to the values of $R_n(r)$ derived from a spherical harmonic model for n greater than 1. We have recently shown (De Santis and Barraclough, 1994), however, that, when the term $n = 1$ is also included, a better fit to the spectrum at the Core-Mantle Boundary (CMB) is given by a power-law expression. These two fits to the power spectrum, exponential and power-law, are discussed in Sections 2 and 3, respectively.

2. Exponential dependence of the spectrum on spherical harmonic degree

Equation (1.1) shows that, in terms of the power spectrum $R_n(a)$ at the Earth's surface $r = a$, the spectrum at radius r is

$$R_n(r) = (a/r)^{2n+4}R_n(a). \tag{2.1}$$

If the dependence of the power spectrum on n is exponential, as is usually assumed, $R_n(a)$ can be expressed as

$$R_n(a) = K\alpha^n, \qquad (2.2)$$

where K and α are constant parameters.

At an arbitrary radius r, eqs. (2.1) and (2.2) give

$$R_n(r) = K\alpha^n(a/r)^{2n+4},$$

or, taking logarithms,

$$\log[R_n(r)] = [\log K + 4\log(a/r)] + \\ +[\log\alpha + 2\log(a/r)]n,$$

which is of the form

$$\log[R_n(r)] = A(r) + B(r)n,$$

where

$$A(r) = \log K + 4\log(a/r),$$

and

$$B(r) = \log\alpha + 2\log(a/r). \qquad (2.3)$$

The power spectrum at any radius is thus linear on a semi-logarithmic plot of $\log[R_n(r)]$ against n.

If we assume, as is often done (*e.g.* by Meyer, 1985 and by Nevanlinna, 1987), that the spectrum at the surface of the core ($r = b$) is white, the slope $B(b)$ is zero and eq. (2.3) gives

$$\log\alpha = -2\log(a/b).$$

Substituting for $\log\alpha$ in eq. (2.3) gives the slope of the spectrum at radius r:

$$B(r) = 2[\log a/r) - \log(a/b)] = -2\log(r/b).$$

The slope at the Earth's surface ($r=a$) is therefore

$$B(a) = -2\log(a/b). \qquad (2.4)$$

Thus, if the assumption of a white spectrum at the core surface is valid, a knowledge of the slope of the spectrum at the Earth's surface gives an estimate of the radius of the core (b).

3. Power-law dependence of spectrum on spherical harmonic degree

In this case, the power spectrum at the CMB ($r=b$) is assumed to be of the form

$$R_n(b) = kn^{-\beta}, \qquad (3.1)$$

where k and β are constant parameters.

Equation (2.1), with $r=b$, gives

$$R_n(b) = (a/b)^{2n+4}R_n(a),$$

or

$$R_n(a) = (b/a)^{2n+4}R_n(b) = kn^{-\beta}(b/a)^{2n+4},$$

substituting for $R_n(b)$ from eq. (3.1).

At an arbitrary radius r ($b < r \leq a$), this becomes

$$R_n(r) = kn^{-\beta}(b/r)^{2n+4},$$

or, taking logarithms,

$$\log[R_n(r)] = [\log k + 4\log(b/r)] + \\ +[2\log(b/r)]n - \beta\log n,$$

which is in the form

$$\log[R_n(r)] = A'(r) + B'(r)n + C'\log n. \qquad (3.2)$$

Thus, for a value of r between the radius of the core and that of the Earth's surface, the dependence of $R_n(r)$ on n is a mixture of exponential and power-law. At the Earth's surface B' is related to the the radius of the core by the same expression (2.4) as for the exponential case. Thus, supposing a power-law spectrum at the CMB, from a fit at the Earth's surface to an expression such as (3.2) we can estimate from $B'(a)$ a value of the radius b which is usually greater than that found using the white noise assumption, but a little closer to the value of 3485 km estimated seismologically (*e.g.*, Jordan and Anderson, 1974). For instance, considering the power spectra of IGRF (*e.g.*, Langel, 1992) from 1965 to 1985 we find that $b = 3660 \pm 60$ km, instead of $b = 3140 \pm 20$ km (or $b = 3290 \pm 20$ km, if we exclude the point $n = 1$) on the assumption of an exponential spectrum.

While the «exponential» hypothesis for the spatial spectrum comes from supposing a white noise source distribution at the CMB, the power-law form characterises the so-called *coloured* noise (*e.g.*, Schroeder, 1991, chap. 5) and is found also for some other planets, such as Jupiter and Saturn, for which there is strong evidence that a self-exciting dynamo is the source of their magnetic fields. According to Stevenson (1983) these observed spectra are consistent with those derived from theories of homogeneous turbulence.

However, nothing can exclude the coexistence of both (white and coloured) spatial phenomena at the CMB. It is worth noting that the seismologically determined value of b is almost half way between the values given by the two (exponential and power-law) assumptions. In some papers in progress we will exploit this and other related topics.

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