The effect of arbitrarily small rigidity on the free oscillations of the Earth

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Abstract
The system of propagator equations for an elastic solid becomes singular as the shear modulus becomes vanishingly small. In computational applications there is severe loss of precision as the limit of zero shear modulus is approached. The use of perturbation theory to address the effect of very small shear modulus, using the fluid state as a basis, is unsatisfactory because certain phenomena, *e.g.*, Rayleigh waves, cannot be represented. Two approximate methods are presented to account for the singular perturbation. Since most of the Earth is nearly neutrally stratified, in which case the motion is nearly irrotational, one can impose the irrotational constraint and obtain a modified and reduced system of propagator equations. This system does not have the singular perturbation. In the second method the transition zone between a fluid and a solid is represented as an infinitesimally thin, Massive, Elastic Interface (MEI). The boundary conditions across the MEI are dispersive and algebraic. The limit of zero shear modulus is non-singular.

Key words *boundary conditions – singular perturbation – massive elastic interface*

1. Introduction

There is some reason to believe that the top of the inner core might be a «mushy zone» with very low shear modulus and high attenuation (Loper and Roberts, 1983; Tromp, 1995). Also, there is recent evidence that the base of the mantle, region D″, has regions of very low shear modulus (Wen and Helmberger, 1998). The presence of a low shear modulus poses a challenging computational problem, for, as is well known, the system of governing ODE becomes singular as the shear modulus vanishes.

In the following theoretical development the notation of Dahlen and Tromp (1998, hereinafter referenced as DT) is used with only minor variations. According to DT (8.9.2) the sixth order system of governing ODE is

$$\begin{aligned}
\dot{U} &= -2C^{-1}Fr^{-1}U + \zeta C^{-1}Fr^{-1}V + C^{-1}R \\
\dot{V} &= -\zeta r^{-1}U + r^{-1}V + L^{-1}S \\
\dot{P} &= -4\pi G\rho U - (l+1)r^{-1}P + B \\
\dot{R} &= [-\omega^2\rho - 4\rho g r^{-1} + (A - N - C^{-1}F^2)r^{-2}]U + \\
&+ [\zeta\rho g r^{-1} - 2\zeta(A - N - C^{-1}F^2)r^{-2}]V - \\
&-2(1 - C^{-1}F)r^{-1}R + \zeta r^{-1}S - (l+1)\rho r^{-1}P + \rho B \\
\dot{S} &= [\zeta\rho g r^{-1} - 2\zeta(A - N - C^{-1}F^2)r^{-2}]U - \\
&-[\omega^2\rho + 2Nr^{-2} - \zeta^2(A - C^{-1}F^2)r^{-2}]V - \\
&-\zeta C^{-1}Fr^{-1}R - 3r^{-1}S + \zeta\rho r^{-1}P \\
\dot{B} &= -4\pi G(l+1)\rho r^{-1}U + 4\pi G\zeta\rho r^{-1}V + (l-1)r^{-1}B
\end{aligned} \tag{1.1}$$

where $\zeta^2 = l(l+1)$ (DT use $k^2 = l(l+1)$).

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In (1.1) the two moduli of rigidity are L and N. They are equal in an isotropic solid ($L = N = \mu$). The system (1.1) is unaffected by the limit $N \to 0$ but it becomes singular as $L \to 0$. In this case it is customary to consider that the solid has become an isotropic fluid ($L = N = \mu = 0$, $A = C = F = \kappa$). The sixth-order system (1.1) is replaced by a fourth-order system for the fluid.

In order to retain the effect of finite but very small rigidity one can use the fourth-order system for a fluid and resort to perturbation theory for the effect of the rigidity. Alternatively, one can approximate (1.1) directly and avoid the use of perturbation theory. In the subsequent development two approximate methods are presented; in the first, a transitional solid, one in which $\nabla \times s = 0$, is introduced; in the second, boundary conditions are derived for a massive, elastic interface in which the rigidity can become arbitrarily small.

2. The irrotational constraint

The equation for $\dot{V}$ in (1.1) is the site of the singularity as $L \to 0$. Given that the limit $L \to 0$ will almost always be attributed to a fluid, one can consider imposing the conditions $\nabla \times s = 0$ on the displacement s. For spheroidal motion the symbolic representation

$$s = U\hat{r}\, Y_{lm}(\theta, \varphi) + V\zeta^{-1}\nabla_1 Y_{lm}(\theta, \varphi) \tag{2.1}$$

is used. Then

$$\nabla \cdot s = [\dot{V} + r^{-1}(V - \zeta U)]\, \zeta^{-1}(\hat{r} \times \nabla_1)\, Y_{lm}(\theta, \varphi). \tag{2.2}$$

In a radially stratified fluid with bulk modulus κ the relation

$$\nabla \cdot s = -\frac{N^2}{\omega^2}\frac{\kappa}{\rho g r}(\hat{r} \times \nabla_1)\, \nabla \cdot s \tag{2.3}$$

shows that $\nabla \times s \neq 0$ unless $N^2 = 0$, where N^2 is the squared Brunt-Väisälä frequency. However in most of the fluid earth it is true that N^2 is very small. In addition, where $N^2 > 0$ the presence of ω^2 in the denominator of (2.3) ensures that $|\nabla \times s| \ll |\nabla \cdot s|$ in the fluid earth. Consequently, it is reasonable to impose the constraint $\nabla \times s = 0$ as $L \to 0$. In this case the expression for $\dot{V}$ in (1.1) is replaced by

$$\dot{V} = \zeta r^{-1} U - r^{-1} V \tag{2.4}$$

and the equation for $\dot{S}$ is omitted. The expression for S is (DT, 8.196)

$$S = L(\dot{V} - r^{-1} V + \zeta r^{-1} U) \tag{2.5}$$

which, with (2.4), becomes

$$S = 2Lr^{-1}(\zeta U - V). \tag{2.6}$$

Thus, the sixth order system (1.1) becomes a fifth order system. The equations for $\dot{U}$, $\dot{P}$ and $\dot{B}$ are unchanged. The term $\zeta r^{-1} S$ in the equation for $\dot{R}$ becomes

$$\zeta r^{-1} S = 2L\zeta r^{-2}(\zeta U - V) \tag{2.7}$$

and $\dot{V}$ is given by (2.4).

A solid in which L is much smaller than A, C and F and which is governed by the five, coupled, first order ODE for U, V, P, R and B, is called a transitional solid.

3. Boundary conditions and minors

The boundary between a transitional solid and a normal solid, governed by (1.1), has the usual conditions of continuity for the six scalars for displacement, traction and potential. The irrotational constraint in the transitional solid, which leads to the condition (2.6) for S, requires that the three independent solutions of (1.1) be combined to form two independent solutions of the fifth order system. That is, the condition

$$S - 2Lr^{-1}(\zeta U - V) = 0 \tag{3.1}$$

which must be met on both sides of the boundary, requires that the (6×3) system in the normal solid become a (5×2) system in the transitional solid. In terms of minors the 20 third order minors of the normal solid are translated into 10 second order minors of the transitional solid. See Gomberg and Masters (1988) and Woodhouse (1988) for discussion of minor.

Let UVS denote a third-order minor of the (6×3) system in which rows 1, 2, 3 are occupied by the scalars U, V, S respectively, and let UV be the like notation for the (5×2) system. Then propagation of the solution from the normal solid to the transitional solid leads to the following expressions at the boundary

$$UV = UVS,\ UP = UPS - \lambda UVP,$$

$$UR = URS - \lambda UVR,\ UB = -USB - \lambda UVB,$$

$$VP = VPS - \lambda\zeta UVP,\ VR = VRS - \lambda\zeta UVR,$$

$$VB = -VSB - \lambda\zeta UVB, \tag{3.2}$$

$$PR = PRS - \lambda\zeta UPR + \lambda VPR,$$

$$PB = -PSB - \lambda\zeta UPB + \lambda VPB,$$

$$RB = RSB - \lambda\zeta URB + \lambda VRB$$

where $\lambda = 2Lr^{-1}$.

The conditions (3.2) would apply, for example, in the inner core with a transitional solid beneath the ICB.

Propagation of the solution from the transitional solid to the normal solid requires that the (5×2) system become a (6×3) system as the boundary is crossed. Since $S = \lambda(\zeta U - V)$ in the transitional solid one cannot require continuity of all six scalars. One of them must be discontinuous. Since the transitional solid is nearly a fluid, it is reasonable to choose V to be discontinuous, as it is at a fluid-solid boundary. In this case the (5×2) system is augmented to become a (6×3) system as follows

$$\begin{bmatrix} U_1 & U_2 & 0 \\ V_1 & V_2 & 1 \\ P_1 & P_2 & 0 \\ R_1 & R_2 & 0 \\ \lambda(\zeta U_1 - V_1) & \lambda(\zeta U_2 - V_2) & -\lambda \\ B_1 & B_2 & 0 \end{bmatrix}. \tag{3.3}$$

The 20 third order minors of the (6×3) system (3.3) are related to the 10 second order minors of the (5×2) system as follows

$$UVP = -UP,\ UVT = -UR,\ UVS = 0,$$

$$UVB = -UB,\ UPR = 0,$$

$$UPS = -\lambda UP,\ UPB = -0,\ URS = -\lambda UR,$$

$$URB = 0,\ USB = \lambda UB,$$

$$VPR = PR,\ VPS = -\lambda\zeta UP,\ VPB = PB, \tag{3.4}$$

$$VRS = -\lambda\zeta UR,$$

$$VRB = RB,\ VSB = \lambda\zeta UB,\ PRS = -\lambda PR,$$

$$PRB = 0,$$

$$PSB = \lambda PB,\ RSB = \lambda RB.$$

The conditions (3.4) would apply, for example, at the base of the mantle with a transitional solid above the CMB.

Equations (3.2) and (3.4) permit the solution in terms of minors to be propagated in either direction across the boundary between a normal solid and a transitional solid.

Assume that $L = 0$ ($\lambda = 0$) in (3.2) and (3.4) denotes a fluid. Then the second order minors in (3.2), excluding those containing V, are the correct ones for a fluid, and the third order minors in (3.4) are those derived from a fluid. Note that in (3.4) the second order minors containing V are absent.

It remains to consider the boundary between a fluid and a transitional solid. If the condition $S = 0$ is imposed, then (2.6) shows that $\zeta U = V$ in the transitional solid at the boundary. To propagate the solution from the fluid to the transitional solid the minors are related as follows: minors in the transitional solid not containing V are equal to those in the fluid. Those containing V are related to the fluid minors:

$$UV = 0,\ VP = \zeta UP,\ VR = \zeta UR,\ VB = \zeta UB. \tag{3.5}$$

These conditions are valid, for example, at the CMB with a transitional solid above it.

For the final case, propagation from the transitional solid to the fluid, the V-minors are ignored and the rest are continuous. This would be the case, for example, at the ICB with a transitional solid below it.

4. The massive, elastic interface

If the thickness of the region where $L \ll (A, C, F)$ is small compared to wavelengths outside the region, then one can approximate the region as a very thin interface with surface mass density and surface elasticity. A very thin interface is one within which U, V and P are constant and across which there are jump discontinuities in R, S and B. In the case of toroidal motion W is constant through the interface and there is a jump in T.

The jump conditions are easily derived via Rayleigh's principle (DT, 8.6.4 and 8.9). Let $\mathcal{L}$ be the Lagrangian density. For a displacement or potential scalar X the radial Euler-Lagrange equation can be written

$$r^2 \partial\mathcal{L}/\partial X - \frac{d}{dr}(r^2 \partial\mathcal{L}/\partial \dot{X}) = 0.$$

The traction or potential gradient scalars are represented by $-\partial\mathcal{L}/\partial\dot{X}$. The jump condition in $\partial\mathcal{L}/\partial\dot{X}$ is

$$[\![r_0^2 \partial\mathcal{L}/\partial\dot{X}]\!] = \int_{r_0-\varepsilon/2}^{r_0+\varepsilon/2} r^2 \partial\mathcal{L}/\partial\dot{X}\, dr. \tag{4.1}$$

In performing the integration in (4.1) X is considered to be constant within the very thin interface.

For toroidal modes the Lagrangian density for the interface is (DT, 8.191)

$$\mathcal{L}_T = \frac{1}{2}(\omega^2\rho - Lr_0^{-2} - (\zeta^2 - z)Nr_0^{-2})\,W^2. \tag{4.2}$$

Let

$$\bar{\rho} = \rho\varepsilon,\ \ \bar{L} = L\varepsilon\ \ \bar{N} = N\varepsilon. \tag{4.3}$$

The overbar thus denotes parameters of the interface (mass per unit area, force per unit length).

In (4.1) for toroidal modes $\partial\mathcal{L}/\partial\dot{W} = -T$ (DT, 8.99) and the jump in T is

$$[\![T]\!] = [(\zeta^2 - 2)\,r_0^{-2}\bar{N} + \bar{L}r_0^{-2} - \omega^2\bar{\rho}]\,W. \tag{4.4}$$

The interpretation of (4.4) is that the external traction scalar T has a jump discontinuity across the massive, elastic interface (MEI) at r_0; the MEI has displacement scalar W, surface density $\bar{\rho}$ and surface elastic parameters $\bar{L}$ and $\bar{N}$.

The virtue of using (4.4) as a boundary condition for toroidal modes (along with $[\![W]\!] = 0$ at an internal boundary) is that (4.4) is well behaved as either $\bar{L} \to 0$ or $\bar{N} \to 0$. In the limit of a plane interface, $\zeta r_0^{-1} = k$, the horizontal wave number. As $r_0 \to \infty$ (4.4) becomes

$$[\![T]\!] = (\bar{N}k^2 - \omega^2\bar{\rho})\,W. \tag{4.5}$$

Equation (4.5) shows that very short waves can propagate on the massive, elastic interface when external tractions are absent. The speed of propagation is $\sqrt{\bar{N}/\bar{\rho}}$, which is the speed of shear waves.

For spheroidal modes the Lagrangian density for the interface is obtained from (DT, 8.198) by setting $\dot{U} = \dot{V} = \dot{P} = 0$

$$\mathcal{L}_s^I = \frac{1}{2}[\omega^2\bar{\rho}(U^2 + V^2) - (\bar{A} - \bar{N})\,r_0^{-2}(2U - \zeta V)^2 -$$
$$- \bar{L}(\zeta r_0^{-1}U - r_0^{-1}V)^2 - (\zeta^2 - 2)\,\bar{N}r_0^{-2}V^2 - \tag{4.6}$$
$$- 4\pi G\,(\overline{\rho^2})\,U^2 - 2\bar{\rho}\,(\zeta r_0^{-1}VP)].$$

In deriving (4.6) terms involving g have been discarded since $g = d\phi/dr$ and $[\![\phi]\!] = 0$. In addition, terms not involving ρ or the elastic parameters have been discarded since they vanish as $\varepsilon \to 0$. The external tractions (DT, 8.99) and the generalized potential gradient, $Q = (4\pi G)^{-1}\dot{P} + \rho U$, (DT, p. 257) are

$$R = \partial\mathcal{L}_s/\partial\dot{U},\ S = -\partial\mathcal{L}_s/\partial\dot{V},\ Q = -\partial\mathcal{L}_s/\partial\dot{P}.$$

From (4.1) and (4.6) these scalars have the jump conditions

$$\begin{aligned}
[[R]] &= [\bar{L}\zeta^2 r_0^{-2} + 4(\bar{A}-\bar{N})r_0^{-2} + 4\pi G(\overline{\rho^2}) - \\
&\quad - \omega^2\bar{\rho}]\,U - [\bar{L} + 2(\bar{A}-\bar{N})]\,\zeta r_0^{-2} V \\
[[S]] &= [\bar{A}\zeta^2 r_0^{-2} + (\bar{L}-2\bar{N})r_0^{-2} - \omega^2\bar{\rho}]\,V - \qquad (4.7)\\
&\quad - [\bar{L} + 2(\bar{A}-\bar{N})]\,\zeta r_0^{-2} U + \bar{\rho}\zeta r_0^{-1} P \\
[[Q]] &= \bar{\rho} r_0^{-1} V.
\end{aligned}$$

Since $B = 4\pi GQ + (l+1)r^{-1}P$ and $[[P]] = 0$

$$[[B]]\,4\pi G\bar{\rho} r_0^{-1} V. \qquad (4.8)$$

Note that (4.7) is well behaved as the rigidities $\bar{L}$ and $\bar{N}$ become arbitrarily small. Again, in the limit of a plane interface $\zeta r_0^{-1} = k$, the horizontal wave number. As $r_0 \to \infty$ (4.7) becomes

$$\begin{aligned}
[[R]] &= (\bar{L}k^2 - \omega^2\bar{\rho})\,U \\
[[S]] &= (\bar{A}k^2 - \omega^2\bar{\rho})\,V.
\end{aligned} \qquad (4.9)$$

Gravitational terms have been omitted in (4.9). Very short, high frequency waves can propagate on the MEI for vanishing external tractions. There are flexural waves that travel at the speed $\sqrt{\bar{L}/\bar{\rho}}$, which is the speed of shear waves, and there are areal waves that travel at the speed $\sqrt{\bar{A}/\bar{\rho}}$, which is the speed of compressional waves.

The jump conditions (4.7)-(4.8) lead to jump conditions for the second order minors and third order minors at a boundary. For applications to the CMB and the ICB the MEI would be located at a fluid-solid boundary.

At the ICB it is assumed that the solution is propagated from the solid inner core across the MEI to the fluid outer core.

In an obvious notation the jump conditions can be written

$$\begin{aligned}
[[R]] &= c_1 U + c_2 V, \\
[[S]] &= c_3 U + c_4 V + c_5 P, \qquad (4.10)\\
[[B]] &= c_6 U + c_7 V.
\end{aligned}$$

On the fluid side of the MEI $S = 0$ so (4.10) becomes

$$\begin{aligned}
R_f &= R_s + c_1 U + c_2 V_s \\
0 &= S_s + c_3 U + c_4 V_s + c_5 P \qquad (4.11)\\
B_f &= B_s + c_6 U + c_7 V_s.
\end{aligned}$$

In (4.11) the subscripts f and s refer to fluid and solid, respectively. The conditions (4.11) lead to the following expressions for the second order minors in the fluid in terms of the third order minors in the solid

$$\begin{aligned}
UP &= UPS - c_4 UVP \\
UR &= URS - c_4 UVR - c_5 UPR + c_2 UVS + c_2 c_5 UVP \\
UB &= USB - c_4 UVB - c_5 UPB + c_7 UVS + c_5 c_7 UVP \\
PR &= PRS + c_3 UPR + c_4 VPR - c_1 UPS + \\
&\quad + c_1 c_4 UVP - c_2 VPS - c_2 c_3 UVP \\
PB &= -PSB + c_3 UPB + c_4 VPB - c_6 UPS + \\
&\quad + c_4 c_6 UVP - c_7 VPS + c_3 c_7 UVP \qquad (4.12)\\
RB &= -RSB + c_3 URB + c_4 VRB + c_5 PRB - \\
&\quad - c_1 USB - c_1 c_4 UVB - c_1 c_5 UPB - \\
&\quad - c_2 VSB + c_2 c_3 UVB - c_2 c_5 VPB - c_6 URS + \\
&\quad + c_4 c_6 UVR + c_5 c_6 UPR - \\
&\quad - c_2 c_6 UVS - c_2 c_5 c_6 UVP - c_7 VRS - c_3 c_7 UVR + \\
&\quad + c_5 c_7 VPR + c_1 c_7 UVS + c_1 c_5 c_7 UVP.
\end{aligned}$$

At the CMB it is assumed that the solution is propagated from the fluid outer core across the

MEI to the solid mantle. In this case (4.10) can be written

$$R_s = R_f + c_1 U + c_2 V$$
$$S_s = c_3 U + c_4 V + c_5 P \qquad (4.13)$$
$$B_s = B_f + c_6 U + c_7 V$$

which leads to the following expressions for the third order minors in the solid in terms of the second order minors in the fluid

$$UVP = -UP,\ UVR = -UR,\ UVS = -c_5 UP,$$
$$UVB = -UB,$$
$$UPR = c_2 UP,\ UPS = c_4 UP,\ UPB = c_7 UP,$$
$$URS = c_4 UR - c_2 c_5 UP,$$
$$URB = c_7 UR - c_2 UB,\ USB = c_5 c_7 UP - c_4 UB,$$
$$VPR = PR - c_1 UP,$$
$$VPS = -c_3 UP,\ VPB = PB - c_6 UP,$$
$$VRS = c_1 c_5 UP + c_5 RP - c_3 UR, \qquad (4.14)$$
$$VRB = RB - c_6 UR + c_1 UB,$$
$$VSB = c_3 UB + c_5 PB - c_5 c_6 UP,$$
$$PRS = c_4 PR + (c_2 c_3 - c_1 c_4)\, UP,$$
$$PRB = c_7 PR - c_2 PB + (c_2 c_6 - c_1 c_7)\, UP,$$
$$PSB = (c_4 c_6 - c_3 c_7)\, UP - c_4 PB,$$
$$RSB = c_5\,(c_1 c_7 - c_2 c_6)\, UP + (c_4 c_6 - c_3 c_7)\, UR +$$
$$+(c_2 c_3 - c_1 c_4)\, UB - c_7^2 PR + c_2 c_5 PB - c_4 RB.$$

5. Discussion

Arbitrarily small rigidity can be incorporated into the computational methods used for elastic wave and vibration problems in two ways. The constraint, $\nabla \times \mathbf{s} = 0$, can be imposed and placed into the propagator equations. As the rigidity, L, becomes very small the equations and boundary conditions approach those of a neutrally stratified fluid. Alternatively, the zone of very small rigidity can be treated as a very thin, Massive, Elastic Interface (MEI) across which there are discontinuities, or jump conditions, in the tractions and the potential gradient. The MEI boundary conditions are well behaved as the rigidity becomes very small.

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