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A study of function space topologies for multifunctions

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Abstract

Function space topologies are investigated for the class of continuous multifunctions. Using the notion of continuous convergence, splittingness and admissibility are discussed for the topologies on continuous multifunctions. The theory of net of sets is further developed for this purpose. The (τ, μ)-topology on the class of continuous multifunctions is found to be upper admissible, while the compact-open topology is upper splitting. The point-open topology is the coarsest topology which is coordinately admissible, it is also the finest topology which is coordinately splitting.

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1. Introduction

The interplay of properties of the topological spaces X and Y and those of the function space $\mathcal{C}(X, Y)$ of continuous functions from X to Y has been an area of active research in topology. Several different sets of conditions under which the compact-open, Isbell or natural topologies on the set of continuous

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real-valued functions on a space may coincide, have been studied in [13]. A unified theory of function spaces and hyperspaces has been developed in [3]. In [4], it is shown that the intersection of all admissible topologies on $\mathcal{C}(X, Y)$ is admissible under certain conditions. These and many other research papers published in the recent years are the testimony to the keen interest of the researchers in the study of function spaces.

In [13], while discussing coincidence of the function space topologies, a natural topology on the set of upper semi continuous set-valued functions has been constructed. Apart from this, the continuous multifunctions in the study of function spaces have been investigated by several researchers [9, 10, 12, 15, 20, 21, 22]. At the same time, the multifunctions are being extensively used now-a-days in several areas of mathematics such as Optimization theory, Frame theory, Approximation theory etc., to name a few.

In this paper, we further develop the topological aspects of the function spaces for multifunctions. Starting from the basic level, we provide discussions for several new as well as already existing topologies for continuous multifunctions. We have adopted the net theoretic approach to discuss continuous convergence for the topology of multifunctions. The net theory for sets is further developed for the same purpose. Here it may be mentioned that characterizations of upper semi-continuity and lower semi-continuity are provided in [14] using net convergence. In our paper, we show that under certain conditions, continuity of multifunctions implies a net-theoretic result which is similar to its counterpart for single-valued functions. Our study here is purely topological, unlike [11], where metric spaces and normed spaces are considered for similar results. Similarly, the continuous convergence introduced in our paper is different from that of [2] and [18]. In [2] and [18], upper and lower topologies, defined on the second space, are used for defining continuous convergence. However our definition is more straight forward and appears similar to its counterpart of single-valued functions. Conditions for splittingness (resp. upper and lower splittingness) and admissibility (resp. upper and lower admissibility) are obtained by using the concept of continuous convergence. The characterizations of admissibility and splittingness using net theory as shown in Arens and Dugundji [1] do not hold for multifunctions. Their variants are investigated in our paper. Several examples are provided to explain the intrinsic differences between the topologies of continuous functions and topologies of continuous multifunctions. In the last section, several topologies on $\mathcal{C}_{\mathcal{M}}(Y, Z)$, the class of continuous multifunctions are studied in the light of splittingness and admissibility. While the (τ, μ)-topology is found to be upper admissible, the compact-open topology on $\mathcal{C}_{\mathcal{M}}(Y, Z)$ is upper splitting. The point-open topology is found to be the coarsest topology which is coordinately admissible, it is also the finest topology which is coordinately splitting.

During our investigation, we have also noticed that for a multifunction $F : (X, \tau) \to (Y, \mu)$ and $U \in \mu$, the two different types of inverse images, that is, $F^+(U)$ and $F^-(U)$ types give rise to two families of open sets of τ. This leads to the possibility of having more than one dual topology for a given function space topology on $\mathcal{C}_{\mathcal{M}}(Y, Z)$. However, further in depth research is required in this regard, which is beyond the scope of this paper.

2. Preliminaries

Definition 2.1. A *multifunction* $F : X \to Y$ is a point-to-set correspondence from X to Y.

We always assume that $F(x) \neq \varnothing$ for all $x \in X$. For each $B \subseteq Y$, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \varnothing\}$. For each $A \subseteq X$, $F(A) = \bigcup\limits_{x \in A} F(x)$.

The collection of all the multifunctions from X to Y is denoted by $Y^X_{\mathcal{M}}$.
The following definitions and results are taken from the available literature.

Definition 2.2. Let (X, τ) and (Y, μ) be two topological spaces. Then $F : X \to Y$ is called

(i) *upper semi continuous* (or *u.s.c.*, in brief) at $x \in X$ if for each open set $V \subseteq Y$ with $F(x) \subseteq V$, there exists an open set U of X such that $x \in U$ and $F(U) \subseteq V$;
(ii) *lower semi continuous* (or *l.s.c*, in brief) at $x \in X$ if for each open set $V \subseteq Y$ with $F(x) \cap V \neq \varnothing$, there exists an open set U of X such that $x \in U$ and $F(u) \cap V \neq \varnothing$ for every $u \in U$;
(iii) *continuous* at $x \in X$, if it is both *u.s.c.* and *l.s.c.* at x;
(iv) *continuous* (resp. *u.s.c.*, *l.s.c.*) if it is continuous (resp. *u.s.c.*, *l.s.c.*) at each point of X.

If (X, τ) and (Y, μ) are two topological spaces and $F : X \to Y$ is a multifunction, then the following conditions are equivalent:

(i) F is *l.s.c.* (resp. *u.s.c.*);
(ii) $F^-(U)$ (resp. $F^+(U)$) is open in X for each open subset U of Y;
(iii) $F^+(A)$ (resp. $F^-(A)$) is closed in X for each closed subset A of Y.

Definition 2.3. A multifunction $F : X \to Y$ is called a closed map if $F(A)$ is closed in Y whenever A is closed in X.

3. A topology on $\mathcal{C}_{\mathcal{M}}(Y, Z)$

Now we proceed to define a topology on $Z^Y_{\mathcal{M}}$ in the following way:
Let (Y, τ) and (Z, μ) be two topological spaces. For $U \in \tau$ and $V \in \mu$, we define

$$(U, V) = \{F \in Z^Y_{\mathcal{M}} \mid F(U) \subseteq V\}$$

Let $\mathcal{S}^{\mathcal{M}}_{\tau,\mu} = \{(U, V) \mid U \in \tau, V \in \mu\}$.

Lemma 3.1. *$\mathcal{S}^{\mathcal{M}}_{\tau,\mu}$ is a subbasis for a topology on $Z^Y_{\mathcal{M}}$.*

Proof. For $F \in Z^Y_{\mathcal{M}}$, we have $F(\varnothing) \subseteq V$, for each $V \in \mu$. Hence, we get $F \in (\varnothing, V)$ for each $V \in \mu$. Therefore $\bigcup \mathcal{S}^{\mathcal{M}}_{\tau,\mu} = Z^Y_{\mathcal{M}}$. □

In our discussion, we take $\mathcal{F} = \mathcal{C}_{\mathcal{M}}(Y,Z)$, the collection of all continuous multifunctions from Y to Z. The topology on $\mathcal{F}$ obtained in the above manner is denoted by $\mathfrak{T}^{\mathcal{M}}_{\tau,\mu}$, and is called the (τ,μ)-topology on $\mathcal{C}_{\mathcal{M}}(Y,Z)$. This topology reduces to open-open topology τ_{oo}[17] if the multifunctions are replaced by functions. In fact, τ_{oo} turns out to be the relative topology of $\mathfrak{T}^{\mathcal{M}}_{\tau,\mu}$, when we consider the subspace $\mathcal{C}(Y,Z)$ of $\mathcal{C}_{\mathcal{M}}(Y,Z)$.

Lemma 3.1 ensures the existence of a topology on $\mathcal{C}_{\mathcal{M}}(Y,Z)$. In fact, several other interesting topologies exist on $\mathcal{C}_{\mathcal{M}}(Y,Z)$. We will discuss some of them in the last section.

Definition 3.2. Let (Y,τ) and (Z,μ) be two topological spaces. Let (X,λ) be another topological space. For a multifunction $G : X \times Y \to Z$, we define a map $G^* : X \to \mathcal{C}_{\mathcal{M}}(Y,Z)$ by $G^*(x)(y) = G(x,y)$.

The mappings G and G^* related in this way are called *associated maps.*

Definition 3.3. Let (Y,τ) and (Z,μ) be two topological spaces. A topology $\mathfrak{T}$ on $\mathcal{C}_{\mathcal{M}}(Y,Z)$ is called

- (i) *admissible* (resp. *upper admissible*, *lower admissible*) if the evaluation mapping $E : \mathcal{C}_{\mathcal{M}}(Y,Z) \times Y \to Z$ defined by $E(F,y) = F(y)$ is continuous (resp. $u.s.c.$, $l.s.c.$).
- (ii) *splitting* (resp. *upper splitting*, *lower splitting*) if for each topological space X, continuity (resp. $u.s.c.$, $l.s.c.$) of $G : X \times Y \to Z$ implies the continuity of $G^* : X \to \mathcal{C}_{\mathcal{M}}(Y,Z)$, where G^* is the associated map of G.

In [5], Georgiou, Iliadis and Papadopoulos have introduced one more variation of admissibility and splittingness for function spaces. We extend those definitions for multifunctions as follow:

Definition 3.4. Let (Y,τ) and (Z,μ) be two topological spaces. A topology $\mathfrak{T}$ on $\mathcal{C}_{\mathcal{M}}(Y,Z)$ is called

- (i) *coordinately admissible* (resp. *upper coordinately admissible*, *lower coordinately admissible*) if the evaluation mapping $E : \mathcal{C}_{\mathcal{M}}(Y,Z) \times Y \to Z$ defined by $E(F,y) = F(y)$ is coordinately continuous (resp. coordinately $u.s.c.$, coordinately $l.s.c.$).
- (ii) *coordinately splitting* (resp. *coordinately upper splitting*, *coordinately lower splitting*) if for each topological space X, coordinately continuity (resp. coordinately $u.s.c.$, coordinately $l.s.c.$) of $G : X \times Y \to Z$ implies the continuity of $G^* : X \to \mathcal{C}_{\mathcal{M}}(Y,Z)$, where G^* is the associated map of G.

Here, a map $F : X \times Y \to Z$ is said to be *coordinately continuous* (resp. coordinately $u.s.c.$, coordinately $l.s.c.$) if the maps $F_x : Y \to Z$ and $F_y : X \to Z$ defined by $F_x(y) = F(x,y)$ and $F_y(x) = F(x,y)$ are continuous (resp. $u.s.c.$, $l.s.c.$) for every $x \in X$ and for every $y \in Y$.

The following two observations will be used at several places in the paper:

- (i) Let $f : X \to Y$ and $G : Y \to Z$ be a continuous function and a continuous (resp. $u.s.c.$, $l.s.c.$) multifunction respectively. Then Gof is continuous (resp. $u.s.c.$, $l.s.c.$).

(*ii*) Let (Y, τ) and (Z, μ) be two topological spaces. A topology $\mathfrak{T}$ on $\mathcal{C}_{\mathcal{M}}(Y, Z)$ is admissible (resp. upper admissible, lower admissible) if and only if for each topological space X, continuity of $G^* : X \to \mathcal{C}_{\mathcal{M}}(Y, Z)$ implies continuity (resp. *u.s.c.*, *l.s.c.*) of $G : X \times Y \to Z$, where G is the associated map of G^*.

4. Continuous convergence of multifunctions

In this section, we first investigate the relationships between the net convergence criteria and the continuity of multifunctions. Convergence of net of sets is required for this purpose.

Definition 4.1 ([2, 16])**.** Let $S = \{A_n\}_{n\in\Delta}$ be a net of sets in a topological space (X, τ). Then for any $x \in X$, we say

(*i*) $x \in Lim\ Inf(A_n)$ (or, $x \in LI(A_n)$, in brief) if S eventually intersects every open neighbourhood of x, that is, given an open neighbourhood U of x, there exists $m \in \Delta$, such that $A_n \cap U \neq \varnothing$ for all $n \geq m$.

(*ii*) $x \in Lim\ Sup(A_n)$ (or, $x \in LS(A_n)$, in brief) if S frequently intersects every open neighbourhood of x, that is, given an open neighbourhood U of x and any $m \in \Delta$, there exists an $n \geq m$ such that $A_n \cap U \neq \varnothing$.

(*iii*) The net of sets $S = \{A_n\}_{n\in\Delta}$ is said to *converge* to A and we write $Lim(A_n) = A$ if $LI(A_n) = LS(A_n) = A$.

Lemma 4.2. *For any net of sets, we have* $LI(A_n) \subseteq LS(A_n)$.

Theorem 4.3. *Let F be a multifunction from a topological space (X, τ) to a regular topological space (Y, μ). Let $\{x_n\}_{n\in\Delta}$ be a net in X, which converges to x in X. Then the net $\{F(x_n)\}_{n\in\Delta}$ converges to $F(x)$, if F is continuous at $x \in X$ and $F(x)$ is closed in (Y, μ).*

Proof. Let $y \in F(x)$ and V be any open neighbourhood of y in Y. Then $V \cap F(x) \neq \varnothing$. Thus $x \in F^-(V)$. As F is continuous, $F^-(V)$ is open. Therefore, there exists an open neighbourhood U of x such that $x \in U \subseteq F^-(V)$. Since $\{x_n\}_{n\in\Delta}$ converges to x, we have $x_n \in U \subseteq F^-(V)$ eventually. Therefore, $F(x_n) \cap V \neq \varnothing$ eventually and hence $y \in LI(F(x_n)_{n\in\Delta})$. Thus, we have $F(x) \subseteq LI\,(F(x_n)_{n\in\Delta})$.
Now we claim that $LS(F(x_n)_{n\in\Delta}) \subseteq F(x)$. Let $y \notin F(x)$. Since (Y, μ) is regular and $F(x)$ is a closed set not containing y, therefore, there exist disjoint open sets U and V such that $y \in U$ and $F(x) \subseteq V$. As multifunction F is given to be continuous at $x \in X$, there exists an open neighbourhood W of x such that $F(W) \subseteq V$. Again, since the given net $\{x_n\}_{n\in\Delta}$ converges to x, therefore $x_n \in W$ eventually. Then we have $F(x_n) \subseteq V$ for all $n \geq n_0$, for some $n_0 \in \Delta$. This implies that $F(x_n) \cap U = \varnothing$ for all $n \geq n_0$. Hence, $y \notin LS(F(x_n)_{n\in\Delta})$. Thus, we have $LS(F(x_n)_{n\in\Delta}) \subseteq F(x)$. Hence, $F(x) = LI(F(x_n)_{n\in\Delta}) = LS(F(x_n)_{n\in\Delta})$. Therefore $\{F(x_n)\}_{n\in\Delta}$ converges to $F(x)$. □

In the above theorem, the condition of regularity of the space (Y, μ) and closedness of $F(x)$ can not be relaxed. Here we provide examples to demonstrate this.

Example 4.4. Let $X = \mathbb{R}$ be the set of real numbers with the usual topology $\mathcal{U}$ and μ be the irrational slope topology defined on $Y = \{(x, y) \mid y \geq 0, x, y \in \mathbb{Q}\}$. We fix some irrational number θ. The *irrational slope topology*[23] μ on Y is generated by neighbourhoods of the form $N_\epsilon((x, y)) = \{(x, y)\} \cup B_\epsilon(x + y/\theta) \cup B_\epsilon(x - y/\theta)$ where $B_\epsilon(a) = \{(r, 0) \in Y \mid |r - a| < \epsilon\}$ is the collection of all rationals in $(a - \epsilon, a + \epsilon)$. This irrational slope topology is Hausdorff but not regular[23].
Let $F : (X, \mathcal{U}) \to (Y, \mu)$ be a multifunction defined by

$$F(x) = \begin{cases} \{(1, 5)\}, & x = 1 \\ B_\epsilon(1 - 5/\theta), & \text{for all } x \in (0, 1) \\ B_\epsilon(1 + 5/\theta), & \text{for all } x \in (1, 2) \\ \{(5, 10)\} & \text{otherwise.} \end{cases}$$

We claim that

(i) $F(1) = \{(1, 5)\}$ is closed in (Y, μ);
(ii) F is continuous at $x = 1$;
(iii) $\{F(1 - 1/n)\}_{n \in \mathbb{N}}$ does not converge to $F(1)$ in (Y, μ), while $\{1 - 1/n\}_{n \in \mathbb{N}}$ converges to 1 in $(X, \mathcal{U})$.

Below we provide justifications for claim (i), (ii) and (iii):

(i) Since (Y, τ) is a Hausdorff space, there does not exist any $y \in Y$ distinct from the point $(1, 5)$ such that every neighbourhood U of y intersects $(1, 5)$. Thus $cl\{(1, 5)\} = \{(1, 5)\}$. Thus $F(1) = \{(1, 5)\}$ is closed;
(ii) First we show that F is *u.s.c.* at $x = 1$.
Let V be any open set in Y such that $F(1) \subseteq V$. Since V is open in Y, we have $\{(1, 5)\} \cup B_\epsilon(1 + 5/\theta) \cup B_\epsilon(1 - 5/\theta) \subseteq V$, for some $\epsilon > 0$. Therefore $F((0, 2)) \subseteq V$, where $(0, 2)$ is neighbourhood of 1. Hence F is *u.s.c.* at $x = 1$.
Now, we show that F is *l.s.c.* at $x = 1$.
Let V be any open set in Y such that $F(1) \cap V \neq \varnothing$. Any open set containing $(1, 5)$ must contain $B_\epsilon(1 + 5/\theta) \cup B_\epsilon(1 - 5/\theta)$ also. Thus, again we have $F(x) \cap V \neq \varnothing$ for all $x \in (0, 2)$. Therefore F is *l.s.c.* at $x = 1$. Hence F is continuous at $x = 1$;
(iii) Next, we prove that $\{F(1 - 1/n)\}_{n \in \mathbb{N}}$ does not converge to $F(1)$, that is $F(1) \neq LI(F(x_n))$, where $x_n = 1 - 1/n$.
Let $y \in B_\epsilon(1 - 5/\theta)$. Then every neighbourhood of y intersects $F(1 - 1/n)$ for all n. Thus $y \in LI(F(x_n))$. But $y \notin F(1)$. Hence $\{F(1 - 1/n)\}_{n \in \mathbb{N}}$ does not converge to $F(1)$ in (Y, μ) while $\{1 - 1/n\}_{n \in \mathbb{N}}$ converges to 1 in $(X, \mathcal{U})$.

In the next example, we show that the condition of closedness can not be relaxed.

Example 4.5. Let $X = \mathbb{R}$ be the set of real numbers with the usual topology $\mathcal{U}$ and let $Y = \{a, b, c, d\}$ with topology $\tau = \{\varnothing, Y, \{a, b\}, \{c, d\}\}$. Let $F : (X,\mathcal{U}) \to (Y, \tau)$ be a multifunction defined by

$$F(x) = \begin{cases} \{a\} & \text{for } x = 1, \\ \{a, b\} & \text{for all } x \in (1 - 1/n, 1) \cup (1, 1 + 1/n) \quad \text{for all } n \geq m, \text{ for some } m \in \mathbb{N}, \\ \{c\} & \text{otherwise.} \end{cases}$$

Here (Y, τ) is regular and $F(1) = \{a\}$ is not closed in Y. Now, we claim that

(i) F is continuous at $x = 1$;
(ii) $\{F(1\text{-}1/n)\}_{n\in\mathbb{N}}$ does not converge to $F(1)$ in (Y, τ), while $\{1\text{-}1/n\}_{n\in\mathbb{N}}$ converges to 1 in $(X,\mathcal{U})$.

We have

(i) First we show that F is $u.s.c.$ at $x = 1$.
Let V be any open set in Y such that $F(1) \subseteq V$. Since V is open in Y, we have $\{a, b\} \subseteq V$. Therefore there exists $(1\text{-}1/m, 1 + 1/m)$, an open neighbourhood of 1 in $(X,\mathcal{U})$ such that $F((1\text{-}1/m, 1+1/m)) \subseteq V$. Hence F is $u.s.c.$ at $x = 1$.
Now, we show that F is $l.s.c.$ at $x = 1$.
Let V be any open set in Y such that $F(1) \cap V \neq \varnothing$. Any open set containing $\{a\}$ must contains $\{a, b\}$ also. Thus, again we have $F(x) \cap V \neq \varnothing$ for all $x \in (1 - 1/m, 1 + 1/m)$. Therefore F is $l.s.c.$ at $x = 1$. Hence F is continuous at $x = 1$.

(ii) Now, we show that $\{F(1\text{-}1/n)\}_{n\in\mathbb{N}}$ does not converge to $F(1)$, that is $F(1) \neq LI(F(x_n))$, where $x_n = 1\text{-}1/n$.
Let $y = b$, then every neighbourhood of y intersects $F(1 - 1/n)$ for all $n \geq m$. Thus $y \in LI(F(x_n))$. But $y \notin F(1)$. Hence $\{F(1 - 1/n)\}_{n\in\mathbb{N}}$ does not converge to $F(1)$ in (Y, τ), while $\{1 - 1/n\}_{n\in\mathbb{N}}$ converges to 1 in $(X,\mathcal{U})$.

The condition which we imposed on F in Theorem 4.3, that is, $F(x)$ is closed for every netwise limit $x \in X$ can not be replaced by the condition that F is a closed map. (Here, x is a *netwise limit* means x is a limit of some non-trivial net in X.) For this, first we define a topology on $\mathbb{R}$, which is the *Countable Complement Extension Topology* [23], as given below.

Definition 4.6. Let $X = \mathbb{R}$ be the set of real numbers and τ_1 be the Euclidean topology and τ_2 be the topology of countable complements on X. We define τ to be the smallest topology generated by $\tau_1 \cup \tau_2$. Here a set U is open in τ if and only if $U = O \setminus A$, where $O \in \tau_1$ and A is countable. Then τ is the *Countable Complement Extension Topology* on $\mathbb{R}$.

Example 4.7. Let X and Y be the set of real numbers having the co-countable topology and the countable complement extension topology respectively. Let $F : X \to Y$ be a multifunction defined as $F(x) = (-\infty, -2 - 1/x] \cup \{x\}$ for all

$x \in X$. Here $F(\mathbb{N}) = (-\infty, -2) \cup \mathbb{N}$, which is not a closed set in Y. Hence F is not a closed map, although $F(x)$ is closed for all $x \in X$.

The following result available in [14], may be treated as a partial converse of Theorem 4.3.

Theorem 4.8. *Let (X, τ) and (Y, μ) be two topological spaces. Let $F : X \to Y$ be a multifunction. Then F is lower semi continuous at $x \in X$ if for any net $\{x_n\}_{n \in \Delta}$ in X converging to $x \in X$, the image net $\{F(x_n)\}_{n \in \Delta}$ converges to $F(x)$.*

Here we also mention to our readers that in [14], it is proved that if (X, τ) is a compact Hausdroff space and $\{A_n\}_{n \in \Delta}$ is a net of sets, then $Lim(A_n) = A$ if and only if $\{A_n\}_{n \in \Delta}$ converges to A under Vietoris Topology $\tau_{\mathcal{V}}$ of X.

Generalized nets, defined in [19], were used in [7] to introduce the notion of continuous convergence for function spaces on generalized topologies. Here, we use net theory to the introduce the concept of continuous convergence for multifunctions. We shall use this concept extensively in our paper for classifying various topologies on $\mathcal{C}_{\mathcal{M}}(Y, Z)$. However, before coming to that, below we provide a small discussion related to directed sets and convergence of nets.
Let Δ be a directed set. We add a point ∞ to Δ satisfying $\infty \geq n$ for all $n \in \Delta$ and write $\Delta_0 = \Delta \cup \{\infty\}$. Then a topology τ_0 may be generated on Δ_0 by declaring every singleton of Δ is open and neighbourhood of ∞ to be all the sets of the form $U_{n_0} = \{n : n \geq n_0\}$, where n_0 is any arbitrary member of Δ.

Lemma 4.9 ([1])**.** *Let (Y, μ) be a topological space and $\{y_n\}_{n \in \Delta}$ be a net in Y. Then $\{y_n\}_{n \in \Delta}$ converges to y if and only if the function $s : \Delta_0 \to Y$ defined by $s(n) = y_n$ for $n \in \Delta$ and $s(\infty) = y$ is continuous at ∞.*

In the next set of theorems, we will provide some characterizations of splittingness and admissibility of topologies on $\mathcal{C}_{\mathcal{M}}(Y, Z)$. First we define continuous convergence for multifunctions.

Definition 4.10. Let $\{F_n\}_{n \in \Delta}$ be a net in $\mathcal{C}_{\mathcal{M}}(Y, Z)$. Then $\{F_n\}_{n \in \Delta}$ is said to *continuously converge* to F if for each net $\{y_m\}_{m \in \sigma}$ in Y converging to y, $\{F_n(y_m)\}_{(n,m) \in \Delta \times \sigma}$ converges to $F(y)$ in Z.

If we take functions in place of multifunctions, the above definition coincides with that of continuous convergence of functions defined in [1].

Theorem 4.11. *Let (Y, τ) and (Z, μ) be two topological spaces in which (Z, μ) is regular. Let $\mathfrak{T}$ be a topology on $\mathcal{C}_{\mathcal{M}}(Y, Z)$ such that for each net $\{F_n\}_{n \in \Delta}$ in $\mathcal{C}_{\mathcal{M}}(Y, Z)$, continuous convergence of $\{F_n\}_{n \in \Delta}$ to F implies $\{F_n\}_{n \in \Delta}$ converges to F under $\mathfrak{T}$, provided $G(x, y)$ is closed for every continuous map $G : X \times Y \to Z$ and for every netwise limit $(x, y) \in X \times Y$. Then $\mathfrak{T}$ is splitting.*

Proof. Suppose, continuous convergence implies convergence. Let $G : X \times Y \to Z$ be continuous. Let $\{x_n\}_{n \in \Delta}$ be a convergent net which converges to x in X. We need to show that $\{G^*(x_n)\}_{n \in \Delta}$ converges to $G^*(x)$ in $\mathcal{C}_{\mathcal{M}}(Y, Z)$.

Let $\{y_m\}_{m\in\sigma}$ be a net in Y converging to y. Then $\{(x_n, y_m)\}_{(n,m)\in\Delta\times\sigma}$ converges to (x, y) in $X \times Y$. Hence $\{G(x_n, y_m)\}$ converges to $G(x, y)$, in view of Theorem 4.3 because G is continuous and Z is regular. Let us define that $G^*(x_n) = F_n$ and $G^*(x) = F$. Then $G(x_n, y_m) = G^*(x_n)(y_m) = F_n(y_m)$ and $G(x, y) = G^*(x)(y) = F(y)$. That is, $\{F_n(y_m)\}_{(n,m)\in\Delta\times\sigma}$ converges to $F(y)$. Thus $\{F_n\}_{n\in\Delta}$ continuously converges to F. Therefore by the given condition $\{F_n\}_{n\in\Delta}$ converges to F in $\mathcal{C}_{\mathcal{M}}(Y, Z)$. That is, $\{G^*(x_n)\}_{n\in\Delta}$ converges to $G^*(x)$. Hence G^* is continuous, that is, $\mathfrak{T}$ is splitting. □

Our next result provides a partial converse of the above theorem.

Theorem 4.12. *Let (Y, τ) and (Z, μ) be two topological spaces such that $\mathfrak{T}$ on $\mathcal{C}_{\mathcal{M}}(Y, Z)$ is lower splitting. Then for each net $\{F_n\}_{n\in\Delta}$ in $\mathcal{C}_{\mathcal{M}}(Y, Z)$, continuous convergence of $\{F_n\}_{n\in\Delta}$ to F implies $\{F_n\}_{n\in\Delta}$ converges to F under $\mathfrak{T}$.*

Proof. Let $\mathfrak{T}$ be lower splitting and $\{F_n\}_{n\in\Delta}$ converge continuously to F. Let $\Delta_0 = \Delta \cup \{\infty\}$ be the topological space generated from Δ. We define $G : \Delta_0 \times Y \to Z$ by $G(n, y) = F_n(y)$ and $G(\infty, y) = F(y)$.
Now, we claim that G is *l.s.c.*. Let W be an open set in Z such that $G(n, y) \cap W \neq \varnothing$, that is, $F_n(y) \cap W \neq \varnothing$. Since $F_n \in \mathcal{C}_{\mathcal{M}}(Y, Z)$, therefore there exists an open neighbourhood V of y such that $F_n(v) \cap W \neq \varnothing$ for each $v \in V$. Thus, we have $G(n, v) \cap W \neq \varnothing$ for all $(n, v) \in n \times V$. Therefore G is *l.s.c.* at (n, y). Now, we show that G is *l.s.c.* at (∞, y). It is clear that the only non-constant net in Δ_0 is $\{n\}$, which converges to ∞. Hence if $\mathcal{S}$ is a convergent net in $\Delta_0 \times Y$, then we have, $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, where $\mathcal{S}_1 = \{n\}$ and $\mathcal{S}_2 = \{y_m\}_{m\in\sigma}$ is some convergent net in Y. Then $\mathcal{S}$ converges to $\{\infty\} \times \{y\}$, for some $y \in Y$ such that $\{y_m\}_{m\in\sigma}$ converges to y. Then we have, $G(\mathcal{S}) = \{F_n(y_m)\}_{(n,m)\in\Delta\times\sigma}$. By continuous convergence of $\{F_n\}$, $G(\mathcal{S})$ converges to $F(y) = G(\infty, y)$. Hence G is lower semi continuous at (∞, y). Thus G is *l.s.c.* on $\Delta_0 \times Y$. As $\mathfrak{T}$ is lower splitting, this implies that G^* is continuous. As $\{n\}$ converges to ∞ in Δ_0, we have , $\{G^*(n)\}_{n\in\Delta}$ converges to $G^*(\infty)$. Now, $G^*(n)(y) = G(n, y) = F_n(y)$ and $G^*(\infty)(y) = G(\infty, y) = F(y)$. That is, $G^*(n) = F_n$ and $G^*(\infty) = F$, hence $\{F_n\}$ converges to F in $\mathcal{C}_{\mathcal{M}}(Y, Z)$. □

Remark 4.13. If we take the subfamily $\mathcal{C}(Y, Z)$, instead of $\mathcal{C}_{\mathcal{M}}(Y, Z)$, then Theorem 4.11 and Theorem 4.12 reduce to the following, which was proved in [1].

Corollary 4.14. *Let (Y, μ) and (Z, τ) be two topological spaces. A topology $\mathfrak{T}$ in $\mathcal{C}(Y, Z)$ is splitting if and only if for any net $\{f_n\}_{n\in\Delta}$ in $\mathcal{C}(Y, Z)$, continuous convergence of $\{f_n\}_{n\in\Delta}$ to f implies convergence of $\{f_n\}_{n\in\Delta}$ to f in $\mathfrak{T}$.*

Proof. It holds in view of the fact that net-theoretic characterization of continuity of functions does not need regularity, nor closedness of $f(x)$. □

In the following two results, we investigate the relationship between admissibility and continuous convergence.

Theorem 4.15. *Let (Y, τ) and (Z, μ) be two topological spaces. A topology $\mathfrak{T}$ on $\mathcal{C}_{\mathcal{M}}(Y, Z)$ is lower admissible if for any net $\{F_n\}_{n\in\Delta}$ in $\mathcal{C}_{\mathcal{M}}(Y, Z)$, $\{F_n\}_{n\in\Delta}$ converges to F in $\mathfrak{T}$ implies continuous convergence of $\{F_n\}_{n\in\Delta}$ to F.*

Proof. Let $G^* : X \to \mathcal{C}_{\mathcal{M}}(Y, Z)$ be continuous. We have to show that the associated map G is lower semi continuous. Let $\{x_n, y_n\}_{n\in\Delta}$ be a convergent net which converges to $(x, y) \in X \times Y$. Then $\{x_n\}_{n\in\Delta}$ converges to x in X and $\{y_n\}_{n\in\Delta}$ converges to y in Y. Since $\{x_n\}_{n\in\Delta}$ converges to x and G^* is continuous, therefore $\{G^*(x_n)\}_{n\in\Delta}$ converges to $G^*(x)$, that is, $\{F_{x_n}\}_{n\in\Delta}$ converges to F_x in $\mathcal{C}_{\mathcal{M}}(Y, Z)$, where $F_{x_n} = G^*(x_n)$ and $F_x = G^*(x)$ respectively. Then, by the given hypothesis, $\{F_{x_n}\}_{n\in\Delta}$ continuously converges to F_x. Hence for the convergent net $\{y_n\}_{n\in\Delta}$ which converges to y in Y, we have $\{F_{x_n}(y_n)\}_{n\in\Delta}$ converges to $F_x(y)$, that is, $\{G(x_n, y_n)\}_{n\in\Delta}$ converges to $G(x, y)$. Hence G is lower semi continuous. Therefore $\mathfrak{T}$ is lower admissible. □

For the converse part, we have the following result:

Theorem 4.16. *Let (Y, τ) and (Z, μ) be two topological spaces in which (Z, μ) is regular. Suppose a topology $\mathfrak{T}$ on $\mathcal{C}_{\mathcal{M}}(Y, Z)$ is admissible. Then for each net $\{F_n\}_{n\in\Delta}$ in $\mathcal{C}_{\mathcal{M}}(Y, Z)$, convergence of $\{F_n\}_{n\in\Delta}$ to F implies $\{F_n\}_{n\in\Delta}$ continuously converges to F under $\mathfrak{T}$, provided $G(x, y)$ is closed for every netwise limit $(x, y) \in X \times Y$, for every map $G : X \times Y \to Z$ and for any topological space (X, λ).*

Proof. Let $\mathfrak{T}$ be admissible and $\{y_m\}_{m\in\sigma}$ be any net in Y such that $\{y_m\}_{m\in\sigma}$ converges to y in Y. Let $\{F_n\}_{n\in\Delta}$ be any net in $\mathcal{C}_{\mathcal{M}}(Y, Z)$ such that $\{F_n\}_{n\in\Delta}$ converges to F. Let us define $G^* : \Delta_0 \to \mathcal{C}_{\mathcal{M}}(Y, Z)$ as $G^*(n) = F_n$ and $G^*(\infty) = F$, where Δ_0 is generated by Δ. Now the only non-constant net in Δ_0 is $\{n\}$ which converges to ∞. Also, $\{G^*(n)\}_{n\in\Delta} = \{F_n\}_{n\in\Delta}$ converges to $F = G^*(\infty)$. Hence, G^* is continuous. Therefore $G : \Delta_0 \times Y \to Z$ is continuous as $\mathfrak{T}$ is admissible. Now $\{n, y_m\}_{(n,m)\in\Delta\times\sigma}$ is a convergent net in $\Delta_0 \times Y$ which converges to (∞, y). Therefore, in view of Theorem 4.3, $\{G(n, y_m)\}_{(n,m)\in\Delta\times\sigma}$ converges to $G(\infty, y)$, that is, $\{G^*(n)(y_m)\}_{(n,m)\in\Delta\times\sigma}$ converges to $G^*(\infty)(y)$. This implies $\{F_n(y_m)\}_{(n,m)\in\Delta\times\sigma}$ converges to $F(y)$. Hence $\{F_n\}$ continuously converges to F. □

As a corollary of Theorem 4.15 and Theorem 4.16, we get the following result for $\mathcal{C}(Y, Z)$ [1]:

Corollary 4.17. *Let (Y, τ) and (Z, μ) be two topological spaces. A topology $\mathfrak{T}$ on $\mathcal{C}(Y, Z)$ is admissible if and only if for any net $\{f_n\}_{n\in\Delta}$ in $\mathcal{C}(Y, Z)$, $\{f_n\}_{n\in\Delta}$ converges to f in $\mathfrak{T}$ implies continuous convergence of $\{f_n\}_{n\in\Delta}$ to f.*

In [1], Arens and Dugundji provided few lemmas for function spaces which are valid for function spaces for multifunctions as well. Below we mention them without proof. Here $\mu \geq \tau$, means $\tau \subseteq \mu$.

Lemma 4.18. *Let τ and μ be two topologies on $\mathcal{C}_{\mathcal{M}}(Y, Z)$. If τ is admissible (resp. upper admissible, lower admissible) and $\mu \geq \tau$, then μ is admissible* *(resp. upper admissible, lower admissible). If μ is splitting (resp. upper splitting, lower splitting) and $\mu \geq \tau$, then τ is splitting (resp. upper splitting, lower splitting).*

Lemma 4.19. *If μ is splitting (resp. upper splitting, lower splitting) and τ is admissible (resp. upper admissible, lower admissible) on $\mathcal{C}_{\mathcal{M}}(Y,Z)$, then $\mu \leq \tau$.*

The following theorem, provided by Georgiou, Iliadis and Papadopoulos in [5] for function space is also valid for the function spaces for multifunctions. That is,

Theorem 4.20. *The following hold good:*

(i) *Every coordinately splitting (resp. coordinately upper splitting, coordinately lower splitting) topology on $\mathcal{C}_{\mathcal{M}}(Y,Z)$ is splitting (resp. upper splitting, lower splitting).*

(ii) *Every admissible (resp. upper admissible, lower admissible) topology on $\mathcal{C}_{\mathcal{M}}(Y,Z)$ is coordinately admissible (resp. coordinately upper admissible, coordinately lower admissible).*

5. Various topologies over $\mathcal{C}_{\mathcal{M}}(Y,Z)$

In Section 3, we introduce the (τ,μ)-topology, denoted by $\mathfrak{T}^{\mathcal{M}}_{\tau,\mu}$, on $\mathcal{C}_{\mathcal{M}}(Y,Z)$. In this section, we study several other topologies over $\mathcal{C}_{\mathcal{M}}(Y,Z)$ in the light of splittingness and admissibility. We also study the interrelationship between these topologies. First, we come back to the topology $\mathfrak{T}^{\mathcal{M}}_{\tau,\mu}$ on $\mathcal{C}_{\mathcal{M}}(Y,Z)$ defined in Section 3. We show that $\mathfrak{T}^{\mathcal{M}}_{\tau,\mu}$ is upper admissible on $\mathcal{C}_{\mathcal{M}}(Y,Z)$.

Theorem 5.1. *Let (Y,τ) and (Z,μ) be two topological spaces. Then the topology $\mathfrak{T}^{\mathcal{M}}_{\tau,\mu}$ is upper admissible for $\mathcal{C}_{\mathcal{M}}(Y,Z)$.*

Proof. Let $(F,y) \in \mathcal{C}_{\mathcal{M}}(Y,Z) \times Y$ and let $V \in \mu$ such that $E(F,y) \subseteq V$. This implies $F(y) \subseteq V$. Since F is continuous, there exists an open set W containing y such that $F(W) \subseteq V$, that is, $F \in (W,V)$. Thus $E((W,V) \times W) \subseteq V$. Thus the evaluation map E is $u.s.c.$. Hence $\mathcal{C}_{\mathcal{M}}(Y,Z)$ is upper admissible. □

Below we provide definitions for the compact-open topology and the point-open topology for multifunctions, and investigate some of their properties.
Let (Y,τ) and (Z,μ) be two topological spaces. Then we define

$$(C,V) = \{F \in \mathcal{C}_{\mathcal{M}}(Y,Z) \mid F(C) \subseteq V\}$$
$$(y,V) = \{F \in \mathcal{C}_{\mathcal{M}}(Y,Z) \mid F(y) \subseteq V\}$$

where C is a compact subset of Y, $y \in Y$ and $V \in \mu$.
Let $S^{\mathcal{M}}_{co} = \{(C,V) \mid C \text{ is compact in } Y \text{ and } V \in \mu\}$ and $S^{\mathcal{M}}_{po} = \{(y,V) \mid y \in Y, V \in \mu\}$.
It can be shown that $S^{\mathcal{M}}_{co}$ and $S^{\mathcal{M}}_{po}$ form subbasis for two topologies on $\mathcal{C}_{\mathcal{M}}(Y,Z)$. These topologies are called the *compact-open topology* and the *point-open topology* respectively. They are denoted by $\mathfrak{T}^{\mathcal{M}}_{co}$ and $\mathfrak{T}^{\mathcal{M}}_{po}$ respectively. Clearly $\mathfrak{T}^{\mathcal{M}}_{co}$ is finer than $\mathfrak{T}^{\mathcal{M}}_{po}$. For further study on compact-open topology and compact convergence, one may refer to [9, 10, 12].

Theorem 5.2. *The compact-open topology over $\mathcal{C}_{\mathcal{M}}(Y,Z)$ is upper splitting.*

Proof. Left for the readers. □

In our next result, we prove that the point-open topology for multifunctions is coordinately admissible.

Theorem 5.3. *Let (Y,τ) and (Z,μ) be two topological spaces. Then the topology $\mathfrak{T}_{po}^{\mathcal{M}}$ on $\mathcal{C}_{\mathcal{M}}(Y,Z)$ is coordinately admissible. Also, this is the coarsest topology on $\mathcal{C}_{\mathcal{M}}(Y,Z)$ which is coordinately admissible.*

Proof. First we show that $\mathfrak{T}_{po}^{\mathcal{M}}$ is coordinately upper admissible. For this, let $F \in \mathcal{C}_{\mathcal{M}}(Y,Z)$ and $V \in \mu$ such that $E_y(F) \subseteq V$. This implies $F(y) \subseteq V$. Consider, $(y,V) \in \mathfrak{T}_{po}^{\mathcal{M}}$. We have $E_y((y,V)) \subseteq V$. Hence the evaluation map E_y is coordinately *u.s.c.*
Now, let $y \in Y$ and $V \in \mu$ such that $E_F(y) \subseteq V$. This implies $F(y) \subseteq V$. Since $F \in \mathcal{C}_{\mathcal{M}}(Y,Z)$, then there exists an open neighbourhood W of y such that $F(W) \subseteq V$, then we have $E_F(w) \subseteq V$ for every $w \in W$. Hence the evaluation map E_F is coordinately *u.s.c.* Therefore, $\mathfrak{T}_{po}^{\mathcal{M}}$ is coordinately upper admissible.
Now, consider $F \in \mathcal{C}_{\mathcal{M}}(Y,Z)$ and $V \in \mu$ such that $E_y(F) \cap V \neq \varnothing$, that is, $F(y) \cap V \neq \varnothing$. Then again, we have $(y,V) \in \mathfrak{T}_{po}^{\mathcal{M}}$ such that $E_y(F) \cap V \neq \varnothing$ for each $F \in (y,V)$. Hence the evaluation map E_y is coordinately *l.s.c.* Similarly, we can show that E_F is coordinately *l.s.c.*. Thus $\mathfrak{T}_{po}^{\mathcal{M}}$ is coordinately lower admissible as well.
Accordingly, $\mathfrak{T}_{po}^{\mathcal{M}}$ is coordinately admissible.

Now, we show that it is the coarsest topology on $\mathcal{C}_{\mathcal{M}}(Y,Z)$ having this property.
Let (y,U) be a subbasic open set in $\mathfrak{T}_{po}^{\mathcal{M}}$ and $\mathfrak{U}$ be another topology which is coordinately admissible. We show that (y,U) is open in $\mathfrak{U}$.
Consider $F \in (y,U)$, that is, $F(y) \subseteq U$. Therefore, $E_y(F) \subseteq U$. Since the topology $\mathfrak{U}$ is coordinately admissible, the evaluation map E_y is continuous. Therefore there exists an open set V in $\mathfrak{U}$ containing F such that $E_y(V) \subseteq U$. That is, $F \in V \subseteq (y,U)$. Hence (y,U) is open in $\mathfrak{U}$. □

Our next result shows that $\mathfrak{T}_{po}^{\mathcal{M}}$ is coordinately upper splitting and hence upper splitting.

Theorem 5.4. *Let (Y,τ) and (Z,μ) be two topological spaces. Then the topology $\mathfrak{T}_{po}^{\mathcal{M}}$ is coordinately upper splitting.*

Proof. Let (X,λ) be any topological space such that $G : X \times Y \to Z$ is coordinately upper semi continuous. We have to show that its associated mapping $G^* : X \to \mathcal{C}_{\mathcal{M}}(Y,Z)$ is continuous. Let $x \in X$ and (y,V) be a subbasic open set in $\mathfrak{T}_{po}^{\mathcal{M}}$ with $G^*(x) \in (y,V)$. That is, $G^*(x)(y) \subseteq V$. Then, $G(x,y) \subseteq V$. Since the map $G : X \times Y \to Z$ is coordinately upper semi continuous, therefore $G_y : X \to Z$ is upper semi continuous for each y. Hence there exists an open neighbourhood O_x of x such that $G_y(O_x) \subseteq V$. Consequently, we have

$G^*(O_x) \subseteq (y, V)$. Hence G^* is continuous. Accordingly $\mathfrak{T}_{po}^{\mathcal{M}}$ is coordinately upper splitting. □

In view of Lemma 4.18 and 4.19, the above two results lead us to the following interesting result:

Theorem 5.5. *The point-open topology $\mathfrak{T}_{po}^{\mathcal{M}}$ over $\mathcal{C}_{\mathcal{M}}(Y, Z)$ is the unique topology which is coordinately upper splitting as well as coordinately upper admissible. It is also the finest coordinately upper splitting as well as the coarsest coordinately upper admissible topology on $\mathcal{C}_{\mathcal{M}}(Y, Z)$.*

The Corollary 3.6 of [5] on $\mathcal{C}(Y, Z)$ can be viewed as a particular case of the above result.

Remark 5.6. The open sets of the domain space which can be realized as pre-images of continuous functions have been used to define the dual topologies for a given function space topology in [6] and [8]. Several interesting relationships are established between a function space topology and its dual topology in these studies. Investigations in this regard may also be carried out for multifunction as well. However, in case of multifunctions, the development is not straightforward. It is due to the fact that, for a continuous multifunction $F : (X, \tau) \to (Y, \mu)$, the inverse images of open sets form two different classes in (X, τ) formed by $F^+(U)$ and $F^-(U)$ types of sets where $U \in \mu$. A detailed discussion about the same is beyond the scope of the present paper and needs further investigation.

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