APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 18, no. 1 (2017), 75-90
doi:10.4995/agt.2017.5869

A new cardinal function on topological spaces

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Communicated by J. Galindo

Abstract

Using neighbourhood assignments, we introduce and study a new cardinal function, namely GCI(X), *for every topological space* X. *We shall mainly investigate the spaces* X *with finite* GCI(X). *Some properties of this cardinal in connection with special types of mappings are also proved.*

2010 MSC: *54A25; 54C08; 54D3.*

Keywords: *neighbourhood assignment; gauge compact space; gauge compact index;* M*-uniformly continuous mapping.*

1. Introduction

A neighbourhood assignment on a topological space is a function that assigns an open neighbourhood to each point in the space. A number of classes of topological spaces can be characterized by means of neighbourhood assignments (for example, compact spaces, D-spaces, connected spaces, metrizable spaces)[14, 3, 5, 9]. Neighbourhood assignments have also been employed to characterize special type of mappings, in particular those that are linked to Baire class one functions [1][13]. In [2], using neighbourhood assignments, a compactness type of topological property (called gauge compactness), was defined. This property is weaker than compactness in general and equivalent to compactness for Tychonoff spaces. In order to refine the classification of gauge compact spaces, here we define a cardinal GCI(X) for every topological space X, called the gauge compact index of X. As we shall see, this cardinal

Received 03 June 2016 – Accepted 18 October 2016

indeed reveals some subtle distinctions between certain spaces. An immediate and natural task is to determine the spaces X with $\mathrm{GCI}(X)$ equal to a given value. The layout of the paper is as follows. In Section 2, we define the gauge compact index and prove some general properties. Section 3 is mainly about T_1 spaces having gauge compact index 2. In Section 4, we study spaces having gauge compact index 3. Section 5 is devoted to the links between gauge compact indices and special types of mappings between topological spaces. One of the results is that every M-uniformly continuous mapping (in particular, every continuous mapping) from a space with gauge compact index 2 to a T_1 space is a constant function.

From the results proved in this paper, it appears that having a finite gauge compact index is a property more useful for non-Hausdorff spaces than to Hausdorff spaces. Although in classical topology, the main focus was on Hausdorff spaces, the interests in non-Hausdorff spaces had been growing for quite some time. This is particularly the case in domain theory which has a strong background in theoretical computer science. The most important topology in domain theory is the Scott topology of a poset, which is usually only T_0 (it is T_1 iff the poset is discrete). See [8] for a systematic treatment of non-Hausdorff spaces from the point of view of domain theory. This paper contains some results about the gauge compact indices of Scott spaces of algebraic posets. More investigation on gauge compact indices of Scott spaces (poset with their Scott topology) are expected.

2. The gauge compact index of a space

In what follows, in order to be consistent with the definition of gauge compactness, a neighbourhood assignment on a space will be simply called a gauge on the space and the symbol $\Delta(X)$ will be used to denote the collection of all gauges on the space X.

All topological spaces considered below will be assumed to be non-empty.

A space X is called gauge compact if for any $\delta \in \Delta(X)$ there exists a finite subset $A \subseteq X$ such that for every $x \in X$, there exists $a \in A$ so that either $x \in \delta(a)$ or $a \in \delta(x)$ holds [2]. Equivalently, X is gauge compact if and only if for any $\delta \in \Delta(X)$, there is a finite set $A \subseteq X$ such that $\delta(x) \cap A \neq \varnothing$ holds for every $x \in X - \bigcup_{a \in A} \delta(a)$.

Let A, B be non-empty subsets of space X and $\delta \in \Delta(X)$. We write $A \prec_\delta^M B$ if for any $x \in A$, there exists $y \in B$ such that $x \in \delta(y)$ or $y \in \delta(x)$. Also $\{x\} \prec_\delta^M \{y\}$ will be simply written as $x \prec_\delta^M y$ (thus $x \prec_\delta^M y$ iff either $x \in \delta(y)$ or $y \in \delta(x)$). So a space X is gauge compact if and only if for any $\delta \in \Delta(X)$, there is a finite set $A \subseteq X$ satisfying $X \prec_\delta^M A$.

Definition 2.1. The gauge compact index of a space X, denoted by $\mathrm{GCI}(X)$, is defined as

$$\mathrm{GCI}(X) = \inf\{\beta : \forall \delta \in \Delta(X), \exists A \subseteq X \text{ so that } |A| < \beta \text{ and } X \prec_\delta^M A\},$$

where β is a cardinal number and $|A|$ is the cardinality of set A.

Remark 2.2.

(1) A space X is gauge compact if and only if $\mathrm{GCI}(X) \leq \aleph_0$.
(2) Since all topological spaces considered here are non-empty, the gauge compact index of a space is at least 2.
(3) If τ_1 and τ_2 are two topologies on X such that $\tau_1 \subseteq \tau_2$, then $\mathrm{GCI}(X, \tau_1) \leq \mathrm{GCI}(X, \tau_2)$.
(4) If X is a discrete finite space with $|X| = n$, then $\mathrm{GCI}(X) = n + 1$. Conversely, if $|X| = n$ and $\mathrm{GCI}(X) = n + 1$ with n finite, then X is a discrete space.
(5) If there exists $\delta \in \Delta(X)$ such that for any $A \subseteq X$, $X \prec_\delta^M A$ implies $|A| \geq m$, then $\mathrm{GCI}(X) \geq m + 1$.

Let $\mathcal{B}$ be a base of the topology of space X and $\Delta_{\mathcal{B}}(X)$ be the collection of all gauges $\delta \in \Delta(X)$ satisfying $\delta(x) \in \mathcal{B}$ for every $x \in X$. The following lemma shows that to determine the gauge compact indices we can only consider $\delta \in \Delta_{\mathcal{B}}(X)$.

Lemma 2.3. *Let $\mathcal{B}$ be a base of the topology of space X. Then*

$$\mathrm{GCI}(X) = \inf\{\beta : \forall \lambda \in \Delta_{\mathcal{B}}(X), \exists A \subseteq X, |A| < \beta \text{ and } X \prec_\lambda^M A\}.$$

Proof. Let $\alpha_0 = \inf\{\beta : \forall \lambda \in \Delta_{\mathcal{B}}(X), \exists A \subseteq X, |A| < \beta \text{ and } X \prec_\lambda^M A\}$. We need to show that $\mathrm{GCI}(X) = \alpha_0$. For any $\delta \in \Delta_{\mathcal{B}}(X)$, δ is also in $\Delta(X)$, so there is $A \subseteq X$ satisfying $|A| < \mathrm{GCI}(X)$ and $X \prec_\delta^M A$. This implies that $\alpha_0 \leq \mathrm{GCI}(X)$.

Now, for any $\delta \in \Delta(X)$, we can construct a $\lambda \in \Delta_{\mathcal{B}}(X)$ so that $\lambda(x) \subseteq \delta(x)$ for each $x \in X$. Then there exists $A \subseteq X$ such that $|A| < \alpha_0$ and $X \prec_\lambda^M A$. But then $X \prec_\delta^M A$ also holds. Hence $\mathrm{GCI}(X) \leq \alpha_0$. All these show that $\mathrm{GCI}(X) = \alpha_0$. □

Example 2.4. Let $X = (\mathbb{R}, \tau_{up})$, where τ_{up} is the upper topology on $\mathbb{R}$ ($U \in \tau_{up}$ iff $U = \varnothing$, $U = \mathbb{R}$, or $U = (a, +\infty)$ for some $a \in \mathbb{R}$). Then for any $\delta \in \Delta(X)$ and $x \in \mathbb{R}$, $0 \in \delta(x)$ if $x < 0$ and $x \in \delta(0)$ if $0 \leq x$. Hence $X \prec_\delta^M \{0\}$ holds and so $\mathrm{GCI}(X) = 2$. Note that X is not compact.

We first prove some general results on gauge compact indices.

Theorem 2.5. *The gauge compact index of a Hausdorff space X equals a finite integer n if and only if $|X| = n - 1$.*

Proof. Let $\mathrm{GCI}(X) = n$. Firstly, as every finite Hausdorff space is discrete, so by Remark 2.2(4), $|X| \not< n - 1$.

Now assume that $|X| > n - 1$. Take n distinct elements $x_1, x_2, \cdots, x_n$ in X. Since X is Hausdorff, we can choose disjoint open sets $\gamma(x_i)$ with $x_i \in \gamma(x_i)$ ($i = 1, 2, \cdots, n$). Define the gauge δ on X as follows:

$$\delta(x) = \begin{cases} \gamma(x_i), & \text{if } x \in \gamma(x_i); \\ X - \{x_1, x_2, \cdots, x_n\}, & \text{if } x \in X - \bigcup\limits_{i=1}^{n} \gamma(x_i). \end{cases}$$

Then $x \prec_\delta^M x_i$ if and only if $x \in \gamma(x_i)$. Also, as $\gamma(x_i)$'s are n disjoint sets, for any $n-1$ distinct elements $y_1, y_2, \cdots, y_{n-1}$ in X, there must be an i_0 such that

$$\{y_1, y_2, \cdots, y_{n-1}\} \cap \gamma(x_{i_0}) = \varnothing.$$

So, $x_{i_0} \prec_\delta^M y_i$ does not hold for each i. Hence $X \prec_\delta^M \{y_1, y_2, \cdots, y_{n-1}\}$ fails, which contradicts the assumption that $\mathrm{GCI}(X) = n$. It thus follows that $|X| = n-1$.

Conversely, if X is a Hausdorff space such that $|X| = n-1$, then X is a discrete space. By Remark 2.2(4), $\mathrm{GCI}(X) = n$. □

From Theorem 2.5 and Remark 2.2(1), we obtain the following.

Corollary 2.6.

(1) *If X is Hausdorff, then every subspace $Y \subseteq X$ with a finite gauge compact index is discrete.*
(2) *Every Hausdorff space with a finite gauge compact index is finite and therefore compact.*
(3) *If X is a gauge compact Hausdorff space and X is an infinite set, then* $\mathrm{GCI}(X) = \aleph_0$.

The example below shows that the converse of Corollary 2.6(1) is not always true.

By Proposition 9 of [2], a topological space X is gauge compact if and only if for any net $\{x_n\}$ in X, either the net is gauge clustered or it has a cluster point. Here, a net $\{x_n\}$ is called gauge clustered if for any $\delta \in \Delta(X)$, there is a subnet $\{x_{n_k}\}$ such that $\bigcap\limits_k \delta(x_{n_k}) \neq \varnothing$.

Example 2.7. Let X be the set of all real numbers equipped with the topology generated by the Euclidean open sets and the co-countable sets. Let $Y = X \times \{0,1\}$ and $\mathcal{B} = \{(U \times \{0,1\}) - F : U \text{ is open in X and } F \subseteq Y \text{ is finite}\}$. It is easy to see that the space Y equipped with the topology generated by the base $\mathcal{B}$ is T_1. However the two points $(0,0)$ and $(0,1)$ do not have disjoint neighbourhoods in Y, hence Y is not Hausdorff.

Let Z be an infinite subset of Y. Without loss of generality, we assume that the set $\{(x,0) : (x,0) \in Z\}$ is infinite (otherwise we consider $\{(x,1) : (x,1) \in Z\}$), and we further assume that there is a sequence $\{(x_k,0) : k = 1,2,\cdots\} \subseteq Z$ satisfying $x_{k+1} > x_k$ for all k (otherwise we consider a decreasing sequence).

For any $(x,i) \in Z$, there exists n_x such that $x \neq x_k$ for all $k \geq n_x$. Then

$$U_{(x,i)} = (((x-1, x+1) \setminus \{(x_k,0)\}_{k \geq n_x}) \times \{0,1\}) \cap Z$$

is an open neighbourhood of (x, i) in Z. Since $(x_k, 0) \notin U_{(x,i)}$ holds for all large enough k, (x, i) is not a cluster point of the sequence $\{(x_k, 0) : k = 1, 2, \cdots\}$ in Z.

Now, define $\delta \in \Delta(Z)$ by

$$\delta(z) = \begin{cases} (\frac{x_n + x_{n-1}}{2}, \frac{x_n + x_{n+1}}{2}) \times \{0, 1\}, & \text{if } z = (x_n, 0); \\ Z, & \text{if } z \neq (x_n, 0) \text{ for any } n. \end{cases}$$

Since the sets $\delta(x_k, 0)$ are pairwise disjoint, $\bigcap_{k_j} \delta(x_{k_j}, 0) = \varnothing$ for every subnet $\{(x_{k_j}, 0)\}$. Thus $\{(x_k, 0)\}$ is not gauge clustered. By Proposition 9 of [2], the subspace Z is not gauge compact, thus $\mathrm{GCI}(Z)$ is not finite. Hence every subspace Z of Y with a finite gauge compact index is a finite set and hence discrete because Y is a T_1 space.

Proposition 2.8. *Let X and Y be two disjoint spaces whose gauge compact indices are finite. Then $\mathrm{GCI}(X \oplus Y) = \mathrm{GCI}(X) + \mathrm{GCI}(Y) - 1$, where $X \oplus Y$ is the sum of X and Y.*

Proof. Assume that $\mathrm{GCI}(X) = n$ and $\mathrm{GCI}(Y) = m$. Let $\lambda \in \Delta(X \oplus Y)$. For any $x \in X$ and $y \in Y$, let $\lambda_X(x) = \lambda(x) \cap X$ and $\lambda_Y(y) = \lambda(y) \cap Y$. Then $\lambda_X \in \Delta(X)$ and $\lambda_Y \in \Delta(Y)$. Thus, there exist sets $\{x_1, x_2, \cdots, x_{n-1}\} \subseteq X$ and $\{y_1, y_2, \cdots, y_{m-1}\} \subseteq Y$ such that

$$X \prec^M_{\lambda_X} \{x_1, x_2, \cdots, x_{n-1}\} \text{ and } Y \prec^M_{\lambda_Y} \{y_1, y_2, \cdots, y_{m-1}\}.$$

Let $C = \{x_1, x_2, \cdots, x_{n-1}\} \cup \{y_1, y_2, \cdots, y_{m-1}\}$. Then $X \oplus Y \prec^M_\lambda C$. Therefore, $\mathrm{GCI}(X \oplus Y) \leq n + m - 1$.

Next, as $\mathrm{GCI}(X) = n$ and $\mathrm{GCI}(Y) = m$, there exist $\delta_X \in \Delta(X)$ and $\delta_Y \in \Delta(Y)$ such that there are no $A \subseteq X$, $B \subseteq Y$ with $|A| < n - 1$ and $|B| < m - 1$ satisfying $X \prec^M_{\delta_X} A$ and $Y \prec^M_{\delta_Y} B$. Let $\delta_X \oplus \delta_Y$ be the gauge on $X \oplus Y$ such that $\delta_X \oplus \delta_Y | X = \delta_X$ and $\delta_X \oplus \delta_Y | Y = \delta_Y$. For any $C \subseteq X \oplus Y$, if $X \oplus Y \prec^M_{\delta_X \oplus \delta_Y} C$, then $X \prec^M_{\delta_X} X \cap C$ and $Y \prec^M_{\delta_Y} Y \cap C$. Thus by the assumption on δ_X and δ_Y, we have $|X \cap C| \geq n - 1, |Y \cap C| \geq m - 1$. So,

$$|C| \geq (n - 1) + (m - 1) = (m + n) - 2.$$

By Remark 2.2(5), $\mathrm{GCI}(X \oplus Y) \geq (m + n - 2) + 1 = m + n - 1$. It follows that $\mathrm{GCI}(X \oplus Y) = n + m - 1 = \mathrm{GCI}(X) + \mathrm{GCI}(Y) - 1$. □

Corollary 2.9.

(1) *Let X_i $(i = 1, 2, \cdots, m)$ be pairwise disjoint spaces such that each $\mathrm{GCI}(X_i)$ is finite. Then*

$$\mathrm{GCI}(\bigoplus_{i=1}^m X_i) = \sum_{i=1}^m \mathrm{GCI}(X_i) - (m - 1).$$

(2) *Let X be a topological space with a finite gauge compact index. If Y is a clopen proper subspace of X, then $\mathrm{GCI}(Y) < \mathrm{GCI}(X)$.*

Proof. (1) can be proved by repeating the use of Proposition 2.8.
(2) Let Y be a proper clopen subspace of X and $\mathrm{GCI}(X) = n$ be finite. For any $\delta_Y \in \Delta(Y)$, we can extend δ_Y to a $\delta \in \Delta(X)$ by letting $\delta(x) = \delta_Y(x)$ for $x \in Y$ and $\delta(x) = X - Y$ for $x \in X - Y$. Then there is a subset A of X such that $|A| < n$ and $X \prec^M_\delta A$. Then we have $Y \prec^M_{\delta_Y} Y \cap A$. Since $|A \cap Y| \leq |A| < n-1$, so $\mathrm{GCI}(Y) \leq n$. Similarly, $\mathrm{GCI}(X - Y) \leq n$. Then, using Proposition 2.8, we have $\mathrm{GCI}(X) = \mathrm{GCI}(Y \oplus (X - Y)) = \mathrm{GCI}(Y) + \mathrm{GCI}(X - Y) - 1 \geq \mathrm{GCI}(Y) + 2 - 1 > \mathrm{GCI}(Y)$, as desired. $\square$

Remark 2.10. Note that if at least one of two cardinals α, β is infinite, then $\alpha + \beta = \max\{\alpha, \beta\}$. From this, we deduce that if X and Y are disjoint spaces such that at least one of $\mathrm{GCI}(X)$ and $\mathrm{GCI}(Y)$ is infinite, then $\mathrm{GCI}(X \oplus Y) = \max\{\mathrm{GCI}(X), \mathrm{GCI}(Y)\}$.

The product of two gauge compact spaces need not be gauge compact [2, Example 10]. The example below shows that the product space need not be gauge compact even the two factor spaces have finite gauge compact indices.

Example 2.11. Let $X = Y = (\mathbb{R}, \tau_{up})$. By Example 2.4, $\mathrm{GCI}(X) = \mathrm{GCI}(Y) = 2$. Define the gauge δ on $X \times Y$ as follows:

$$\delta((x, y)) = (x - 1, +\infty) \times (y - 1, +\infty), \quad (x, y) \in X \times Y.$$

Now, for any finite number of elements $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ in $X \times Y$ with $n \geq 1$, let

$$\begin{aligned} x^* &= \min\{x_i : i = 1, 2, \cdots, n\} - 2, \\ y^* &= \max\{y_i : i = 1, 2, \cdots, n\} + 2. \end{aligned}$$

Then for every $1 \leq i \leq n$, $(x_i, y_i) \notin \delta((x^*, y^*))$ and $(x^*, y^*) \notin \delta((x_i, y_i))$. It follows that for any $A \subseteq X \times Y$ with $|A| < \aleph_0$, $X \times Y \prec^M_\delta A$ does not hold. Therefore $\mathrm{GCI}(X \times Y) \nleq \aleph_0$, hence $\mathrm{GCI}(X \times Y) \geq \aleph_1$. On the other hand, for any gauge η on the product space $X \times Y$, let $A = \{(a, a) : a \text{ is an integer}\}$. Then $|A| < \aleph_1$ and $X \times Y \prec^M_\eta A$ holds. Thus $\mathrm{GCI}(X \times Y) = \aleph_1$.

Now a natural question is: which spaces X have the property that for any space Y with a finite gauge compact index, $\mathrm{GCI}(X \times Y)$ is finite. Note that a space X is compact if and only if for any $\delta \in \Delta(X)$, there exists a finite set $A \subseteq X$ such that $X = \bigcup_{x \in A} \delta(x)$.

Proposition 2.12. *Let X be a space such that $\mathrm{GCI}(X \times Y)$ is finite for all Y with a finite gauge compact index . Then $\mathrm{GCI}(X)$ is finite and X is compact.*

Proof. Since the product of X with the trivial space $\{e\}$ with exactly one element is homeomorphic to X and $\mathrm{GCI}(\{e\}) = 2$, by the assumption it follows that $\mathrm{GCI}(X) = \mathrm{GCI}(X \times \{e\})$ is finite.

Let $Y = (\mathbb{N}, \tau)$ be the space where $\mathbb{N}$ is the set of all natural numbers excluding 0 and $\tau = \{\varnothing, \mathbb{N}\} \cup \{\uparrow n : n \in \mathbb{N}\}$. Then $\mathrm{GCI}(Y) = 2$. So $\mathrm{GCI}(X \times Y)$

is finite. For any $\delta \in \Delta(X)$, consider the $\lambda \in \Delta(X \times Y)$ defined by

$$\lambda(x, m) = \delta(x) \times \uparrow m, \quad (x, m) \in X \times Y.$$

Then there exist a finite set $A = \{(x_1, m_1), (x_2, m_2), \cdots, (x_k, m_k)\}$ such that $X \times Y \prec_\lambda^M A$. We claim that $X = \bigcup\{\delta(x_i) : i = 1, 2, \cdots, k\}$. To see this, assume that $x_0 \notin \delta(x_i)$ for any i. Then $(x_0, m_1 + \cdots + m_k + 1) \notin \lambda(x_i, m_i)$ and $(x_i, m_i) \notin \lambda(x_0, m_1 + \cdots + m_k + 1)$ for every i, which contradicts the assumption on the set A. Therefore $X = \bigcup\{\delta(x_i) : i = 1, 2, \cdots, k\}$. Hence X is compact. $\square$

3. Spaces whose gauge compact indices equal to two

We first consider the gauge compact indices of some spaces arising from intrinsic topologies on posets (partially ordered sets). Let $(P, \leq)$ be a poset. A subset U of P is called an upper set if $U = \uparrow U = \{y \in P : y \geq x \text{ for some } x \in U\}$. Dually $U \subseteq P$ is a lower set, if $U = \downarrow U = \{y \in P : y \leq x \text{ for some } x \in U\}$.

A topology on a poset P is order compatible if the closure of each point $x \in P$ with respect to this topology equals the lower part of x, that is,

$$\mathrm{cl}(\{x\}) = \downarrow x = \{y \in P : y \leq x\}.$$

The finest order compatible topology on P is the Alexandroff topology $\gamma(P)$ which consists of all upper subsets of P. The coarsest order compatible topology on P is the upper interval topology (or weak topology) $\nu(P)$, of which $\{P - \downarrow A : A \text{ is a finite subset of } P\}$ is a basis (see [12, Proposition 1.8]). Thus all open sets in an order compatible topology are upper sets.

An element a of a poset P is called a linking element if it is comparable with any element in P: for any $x \in P$, either $a \leq x$ or $x \leq a$ holds.

If $a \in P$ is a linking element and τ is an order compatible topology on poset P, then for any $\delta \in \Delta(P, \tau)$ we have $P \prec_\delta^M \{a\}$, so $\mathrm{GCI}(P, \tau) = 2$.

The following is an example of poset P which does not have a linking element but $\mathrm{GCI}(P, \nu(P)) = 2$.

Example 3.1. Let $P = \mathbb{N} \cup \{a, b\}$, where $\mathbb{N}$ is the set of all natural numbers excluding 0. Define the partial order $\preceq$ on P by

(i) $1 \preceq a, b \preceq a$.

(ii) For $m, n \in \mathbb{N}$, $m \preceq n$ iff $m \leq n$ (in the ordinary sense).

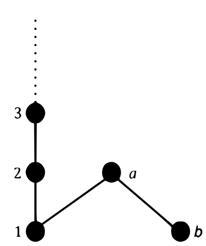

FIGURE 1. Poset $P = \mathbb{N} \cup \{a, b\}$

For any finite subset A of the poset $(P, \preceq)$, $\downarrow A$ is a finite set. As $\{P - \downarrow A : A \subseteq P \text{ is finite}\}$ is a base of the upper interval topology $\nu(P)$, by Lemma 2.3, to determine $\mathrm{GCI}(P, \nu(P))$, we only need to consider $\delta \in \Delta(P, \nu(P))$ satisfying $\delta(x) = P - \downarrow A_x$ for some finite subset $A_x \subseteq P$ with $x \notin \downarrow A_x$. Then as $\downarrow A_a \cup \downarrow A_b$ is a finite subset of P, there exists $m_0 \in \mathbb{N} - (\downarrow A_a \cup \downarrow A_b)$, implying $m_0 \in \delta(a)$ and $m_0 \in \delta(b)$. For any other $n \in \mathbb{N}$, $n \in \delta(m_0)$ if $m_0 \leq n$, and $m_0 \in \delta(n)$ if $n \leq m_0$. Therefore $P \prec_\delta^M \{m_0\}$, implying $\mathrm{GCI}(P, \nu(P)) = 2$. However, the poset $(P, \preceq)$ clearly does not have a linking element.

For the Alexandroff topology on a poset, we have a better result.

Proposition 3.2. *For any poset P, $\mathrm{GCI}(P, \gamma(P)) = 2$ if and only if P has a linking element.*

Proof. Again, we only need to prove the necessity.

Assume that $\mathrm{GCI}(P, \gamma(P)) = 2$. Define $\delta(x) = \uparrow x$ for each $x \in P$. Then δ is a gauge on $(P, \gamma(P))$. So there is an element $e \in P$ such that $P \prec_\delta^M \{e\}$, which implies immediately that e is a linking element of P. $\square$

Another order compatible topology on a poset P is the Scott topology which is the most important topological structure in domain theory. A non-empty subset D of a poset P is a directed set if every two elements in D have an upper bound in D.

A subset U of a poset P is called a Scott open set if (i) $U = \uparrow U$, and (ii) for any directed set $D \subseteq P$, $\bigvee D \in U$ implies $D \cap U \neq \varnothing$ whenever $\bigvee D$ exists. The family $\sigma(P)$ of all Scott open sets of P is indeed a topology on P [4]. The space $\Sigma P = (P, \sigma(P))$ is called the Scott space of P.

For two elements a and b in a poset P, a is way-below b, denoted by $a \ll b$, if for any directed subset D of P, if $\bigvee D$ exists and $b \leq \bigvee D$ then there exists $d \in D$ such that $a \leq d$. An element x is compact if $x \ll x$. The set of all compact elements of P is denoted by $K(P)$. For any $k \in K(P)$, $\uparrow k$ is Scott open.

A poset P is called algebraic, if for any $a \in P$, $\{x \in K(P) : x \leq a\}$ is directed and $a = \bigvee\{x \in K(P) : x \leq a\}$. For any algebraic poset P, the family $\{\uparrow u : u \in K(P)\}$ is a base of the Scott topology on P (see [4, Corollary II-1.15]).

Lemma 3.3. *An element e of an algebraic poset P is a linking element if and only if for any $x \in K(P)$, $x \leq e$ or $e \leq x$ holds.*

Proof. If e is a linking element of P, then clearly for any $x \in K(P)$, we have either $x \leq e$ or $e \leq x$.

Conversely, assume that e satisfies the condition. Let b be any element of P. Since P is algebraic, $b = \bigvee\{y \in K(P) : y \leq b\}$. If there is a $y \in K(P)$ such that $y \leq b$ and $y \not\leq e$, then $e \leq y$, which implies that $e \leq b$. Otherwise, for every $y \in K(P)$ with $y \leq b$, we have $y \leq e$. So $b = \bigvee\{y \in K(P) : y \leq b\} \leq e$. All these show that e is a linking element of P. $\square$

Proposition 3.4. *Let P be an algebraic poset. Then* $\mathrm{GCI}(\Sigma P) = 2$ *if and only if P has a linking element.*

Proof. We only need to prove the necessity. Note that for every $u \in K(P)$, $\uparrow u \in \sigma(P)$.

Assume, on the contrary, that P has no linking element. Let $P - K(P) = \{a_i : i \in I\}$. Then no a_i is a linking element. By Lemma 3.3, there is $u_i \in K(P)$ such that $a_i \not\leq u_i \not\leq a_i$. Since $a_i = \bigvee\{x \in K(P) : x \leq a_i\}$, there is $v_i \in K(P)$ such that $v_i \leq a_i$ and $v_i \not\leq u_i$. Now define a gauge δ on P (with respect to the Scott topology) as follows:

$$\delta(x) = \begin{cases} \uparrow x, & \text{if } x \in K(P); \\ \uparrow v_i, & \text{if } x = a_i. \end{cases}$$

Since $\mathrm{GCI}(\Sigma P) = 2$, there is $x_0 \in P$ such that $P \prec_\delta^M \{x_0\}$. If x_0 is a compact element, then $\delta(x_0) = \uparrow x_0$. For any $y \in K(P)$, we must have either $x_0 \in \delta(y)$ or $y \in \delta(x_0)$, implying $y \leq x_0$ or $x_0 \leq y$. By Lemma 3.3, x_0 is a linking element, contradicting our assumption.

Therefore, $x_0 = a_i$ for some $i \in I$. But then $u_i \notin \uparrow v_i = \delta(a_i)$ and $a_i \notin \uparrow u_i = \delta(u_i)$, contradicting the assumption on x_0. All these show that P must have a linking element. $\square$

Given a set X, the weakest T_1 topology on X is the co-finite topology, denoted by τ_{cof}, where $U \in \tau_{\mathrm{cof}}$ if and only if either $U = \varnothing$ or $X - U$ is a finite set.

Lemma 3.5. *For any infinite set X,* $\mathrm{GCI}(X, \tau_{\mathrm{cof}}) = 3$.

Proof. Let $\delta \in \Delta(X)$. Take any $a \in X$. If $\delta(a) = X$, then $X \prec_\delta^M \{a\}$. If $\delta(a) \neq X$, then $X - \delta(a) = \{a_1, a_2, \cdots, a_n\}$ is a finite set and $n \geq 1$. Since $X - \bigcup\{\delta(a_i) : i = 1, 2, \cdots, n\}$ is finite, we have $\bigcap\{\delta(a_i) : i = 1, 2, \cdots, n\} \neq \varnothing$. Choose any element

$$c \in \bigcap\{\delta(a_i) : i = 1, 2, \cdots, n\}.$$

For each $x \in X$, if $x \in \delta(a)$ then $x \prec_\delta^M a$. If $x \notin \delta(a)$, then $x = a_i$ for some i, so $c \in \delta(a_i) = \delta(x)$ which implies $x \prec_\delta^M c$. Hence $X \prec_\delta^M A$ where $A = \{a, c\}$. Thus $\mathrm{GCI}(X, \tau_{\mathrm{cof}}) \leq 3$.

Now, we will show that $\mathrm{GCI}(X, \tau_{\mathrm{cof}}) \neq 2$. First note that if two sets have the same cardinality, then their corresponding co-finite topological spaces are homeomorphic. Since $|X| = |X \times \{1, 2\}|$, so the two spaces (X, τ_{cof}) and $(X^*, \tau_{\mathrm{cof}})$ are homeomorphic, where $X^* = X \times \{1, 2\}$. Define the gauge δ on X^* as follows:

$$\begin{aligned} \delta((x, 1)) &= X^* - \{(x, 2)\}, \\ \delta((x, 2)) &= X^* - \{(x, 1)\}. \end{aligned}$$

For any $x \in X$, $(x, 1) \prec_\delta^M (x, 2)$ does not hold. So $X^* \prec_\delta^M \{u\}$ does not hold for any $u \in X^*$. Thus, $\mathrm{GCI}(X, \tau_{\mathrm{cof}}) = \mathrm{GCI}(X^*, \tau_{\mathrm{cof}}) \neq 2$. Hence we conclude that $\mathrm{GCI}(X, \tau_{\mathrm{cof}}) = 3$. $\square$

If a T_1 space X is finite and $\mathrm{GCI}(X) = 2$, then it is Hausdorff (every finite T_1 space is discrete), thus $|X| = 1$.

Lemma 3.6. *If X is a T_1 space such that $\mathrm{GCI}(X) = 2$ and $|X| > 1$, then X is an infinite set.*

Theorem 3.7. *For any T_1 space (X, τ), $\mathrm{GCI}(X, \tau) = 2$ if and only if $|X| = 1$.*

Proof. Trivially, if $|X| = 1$, then $\mathrm{GCI}(X) = 2$.
For the sufficiency, if $\mathrm{GCI}(X, \tau) = 2$ and $|X| \neq 1$, then by Lemma 3.6, X is an infinite set. Since (X, τ) is T_1, τ is finer than or equal to the co-finite topology τ_{cof} on X. Hence, by Remark 2.2(3), $\mathrm{GCI}(X, \tau) \geq \mathrm{GCI}(X, \tau_{\mathrm{cof}})$. However, by Lemma 3.5, $\mathrm{GCI}(X, \tau_{\mathrm{cof}}) = 3$, thus $\mathrm{GCI}(X, \tau) \geq 3$, a contradiction. Hence $|X| = 1$ holds. □

Given any space X and $x \in X$, if $Y = \mathrm{cl}(\{x\})$, it is easy to show that for any $\delta \in \Delta(Y)$, $Y \prec_\delta^M \{x\}$. Therefore, we have the following result.

Corollary 3.8. *A topological space X is T_1 if and only if for any subspace $Y \subseteq X$, $\mathrm{GCI}(Y) = 2$ implies $|Y| = 1$.*

4. Spaces whose gauge compactness index is three

In this section, we prove more results on spaces whose gauge compact indices are 3.

A topological space X is called hyperconnected [11] if no two non-empty open sets are disjoint; equivalently, if X is not the union of two proper closed sets. Every hyperconnected space is connected.

Theorem 4.1. *Let X be a T_1 space with $|X| > 2$. If $\mathrm{GCI}(X) = 3$ then X is hyperconnected.*

Proof. Let X be a T_1 space with $|X| > 2$ and $\mathrm{GCI}(X) = 3$. If X is not hyperconnected, then there are two non-empty open subsets U and V of X such that $U \cap V = \varnothing$. Choose $a \in U$ and $b \in V$. If $U = \{a\}$ and $V = \{b\}$ then $|U| = |V| = 1$. Moreover, X is T_1 so $U \cup V$ is closed. Then $X = U \cup V \cup (X - (U \cup V))$ is the union of three non-empty disjoint open sets, which then implies $\mathrm{GCI}(X) \geq 4$, by Corollary 2.9(1). Thus we can assume that $|U| > 1$ (otherwise $|V| > 1$). Since U, with the subspace topology, is a T_1 space, by Theorem 3.7, $\mathrm{GCI}(U) \geq 3$. Then there exists a gauge β on U such that for every $x \in U$ there exists $y \in U$ such that $x \prec_\beta^M y$ does not hold. Now, define the gauge δ on X as follows:

(1) if $x \in U$, let $\delta(x) = \beta(x)$;
(2) if $x \in V$, let $\delta(x) = V$;
(3) if $x \notin (U \cup V)$, choose an open set $\delta(x)$ such that $x \in \delta(x)$ and $a, b \notin \delta(x)$.

Since $\mathrm{GCI}(X) = 3$, there exists $\{x_1, x_2\} \subseteq X$ such that $X \prec_\delta^M \{x_1, x_2\}$ holds. It is easy to see that one of x_i's must be in U and another be in V (if

$a \prec_\delta^M x$ then $x \in U$, and if $b \prec_\delta^M x$ then $x \in V$). Assume that $x_1 \in U$ and $x_2 \in V$.

By the assumption on β, there exists $x \in U$ such that $x_1 \prec_\beta^M x$ does not hold, equivalently $x \prec_\beta^M x_1$ does not hold, therefore $x \prec_\delta^M x_1$ does not hold. Clearly $x \prec_\delta^M x_2$ does not hold either. This contradicts that $X \prec_\delta^M \{x_1, x_2\}$. Hence X must be hyperconnected. □

Remark 4.2. A T_1 space with only two elements is not hyperconnected but its gauge compact index is 3. So, the condition $|X| > 2$ in the above theorem is not removable.

Definition 4.3. A space X is called an F-space if for any $\delta \in \Delta(X)$ and any proper non-empty closed subset F,

$$\bigcap\{\delta(x) : x \in F\} \neq \varnothing.$$

Example 4.4.

(1) For any infinite set X, (X, τ_{cof}) is an F-space.
(2) The set $\mathbb{R}$ of real numbers with the co-countable topology is an F-space.
(3) Let $|X| = \alpha$ be a regular cardinal and τ be the topology on X such that $U \in \tau$ if and only if $U = \varnothing$ or $|X - U| < |X|$. Then (X, τ) is an F-space.
(4) The space $(P, \nu(P))$ considered in Example 3.1 is an F-space.

Proposition 4.5. *If $f : X \to Y$ is a surjective continuous mapping and X is an F-space, then Y is an F-space.*

Proof. Let $\lambda \in \Delta(Y)$ and K be a closed non-empty proper subset of Y. Define the gauge δ on X as follows:

$$\delta(x) = f^{-1}(\lambda(f(x))) \quad \text{for any } x \in X.$$

The set $F = f^{-1}(K)$ is a closed non-empty proper subset of X. Since X is an F-space, so $\bigcap_{x \in F} \delta(x) \neq \varnothing$. Take one element $c \in \bigcap_{x \in F} \delta(x) = \bigcap_{x \in F} f^{-1}(\lambda(f(x)))$. Now for every $y \in K$, since f is surjective, there is $x \in F$ such that $f(x) = y$. So, we have $f(c) \in \lambda(f(x)) = \lambda(y)$. Thus, $f(c) \in \bigcap_{y \in K} \lambda(y)$, implying $\bigcap_{y \in K} \lambda(y) \neq \varnothing$. Hence Y is an F-space. □

Proposition 4.6. *Every clopen subspace of an F-space is an F-space.*

Proof. Let X be an F-space and $A \subseteq X$ be clopen. Given any gauge δ on A and a non-empty proper closed subset K of A, we extend δ to a gauge κ on X by letting

$$\kappa(x) = \begin{cases} \delta(x), & \text{if } x \in A, \\ X - A, & \text{if } x \notin A. \end{cases}$$

Since A is clopen, K is closed in X. Moreover X is an F-space, so $\bigcap_{x\in K} \kappa(x) \neq \varnothing$. Since $K \subseteq A$, then $\bigcap_{x\in K} \kappa(x) = \bigcap_{x\in K} \delta(x) \neq \varnothing$. Thus, A is an F-space. $\square$

Remark 4.7. Let X be the set of all real numbers equipped with the co-countable topology. Then $A = \{1, 2, 3, \cdots\}$ is a closed subset of X. Let $\delta(x) = \{x\}$ for all $x \in A$. Now, $K = \{2, 4, 6, \cdots\}$ is a proper non-empty closed subset of A and $\bigcap_{x\in K} \delta(x) = \varnothing$. It follows that A is a closed subspace of the F-space X, and A is not an F-space. Hence a closed subspace of an F-space need not be an F-space.

Theorem 4.8. *If X is a T_1 F-space with $|X| \geq 2$, then* $\mathrm{GCI}(X) = 3$.

Proof. Firstly, by Theorem 3.7, we have that $\mathrm{GCI}(X) \neq 2$ because $|X| \neq 1$. It now remains to show that $\mathrm{GCI}(X) \leq 3$. If X is finite, then X is a discrete space. Hence for every $x \in X$, $\{x\}$ is clopen. Let δ be the gauge on X defined by $\delta(x) = \{x\}$. Choose one $a \in X$ and consider the proper closed set $F = X - \{a\}$. Since X is an F-space, $\bigcap\{\delta(x) : x \in F\} \neq \varnothing$. But this is true only when F has only one element. It thus follows that $|X| = 2$. Thus, $\mathrm{GCI}(X) = 3$.

Now, assume that X is infinite. Let $\delta \in \Delta(X)$ and $a \in X$ such that $\delta(a) \neq X$ (if $\delta(a) = X$, then $X \prec_\delta^M \{a\}$). Hence, $F = X - \delta(a)$ is a proper closed set. By the assumption on X, $\bigcap\{\delta(z) : z \in F\} \neq \varnothing$. Take $c \in \bigcap\{\delta(z) : z \in F\}$. For any $x \in X$, if $x \in F$ we have $c \in \delta(x)$; if $x \notin F$, $x \in \delta(a)$. Hence $X \prec_\delta^M \{a, c\}$. It follows that $\mathrm{GCI}(X) \leq 3$. Therefore $\mathrm{GCI}(X) = 3$. $\square$

Remark 4.9. The set of real numbers $\mathbb{R}$ with the upper topology is a T_0 F-space whose gauge compact index is 2. Thus Theorem 4.8 need not be true if the space is not T_1.

We still haven't been able to obtain a complete characterization of T_1 spaces whose gauge compact indices equal 3. We previously conjectured that every such space is an F-space. Unfortunately, the example below gives a negative answer.

Example 4.10. Let $\mathbb{Z}$ be the set of all integers equipped with the topology $\tau = \{\varnothing\} \cup \{A \subseteq \mathbb{Z} : \mathbb{Z} \setminus A \text{ is finite}\} \cup \{B \subseteq 2\mathbb{Z} : 2\mathbb{Z} \setminus B \text{ is finite}\}$. The topology τ is finer than the co-finite topology, so it is T_1.

Let $\delta \in \Delta(\mathbb{Z}, \tau)$. If $\delta(1) = X$, then $\mathbb{Z} \prec_\delta^M \{1\}$. If $\delta(1) \neq X$, then $X - \delta(1) = \{z_1, z_2, z_3, \cdots, z_n\}$ is a finite set and $n \geq 1$. Since $X - \bigcup_{i=1}^{n} \delta(z_i)$ is finite, $\bigcap\{\delta(x) : x \notin \delta(1)\} \neq \varnothing$. Choose one element $c \in \bigcap\{\delta(x) : x \notin \delta(1)\}$, then $\mathbb{Z} \prec_\delta^M \{1, c\}$. Hence $\mathrm{GCI}(\mathbb{Z}, \tau) \leq 3$. But as $\mathbb{Z}$ is an infinite T_1 space, by Theorem 3.7, $\mathrm{GCI}(\mathbb{Z}, \tau) \neq 2$, so $\mathrm{GCI}(\mathbb{Z}, \tau) = 3$. Now, consider the gauge $\lambda \in \Delta(\mathbb{Z}, \tau)$ given by $\lambda(x) = \mathbb{Z} - \{x - 2, x + 1\}, (x \in \mathbb{Z})$. The set $F = \mathbb{Z} - 2\mathbb{Z}$ is a proper non-empty closed set and $\bigcup\{\mathbb{Z} - \lambda(x) : x \in F\} = \mathbb{Z}$. Hence

$\bigcap\{\lambda(x) : x \in F\} = \varnothing$. Thus $(\mathbb{Z}, \tau)$ is a T_1 space with a gauge compact index of 3 but not an F-space.

5. Properties of gauge compact indices in respective to mappings

In this section we study the relationship between the gauge compact indices of X and Y where there is a certain type of surjective mapping $f : X \to Y$. As remarked at the beginning of the Section 2, all spaces considered in this paper are non-empty.

Recall from [2] that a mapping $f : X \to Y$ from a topological space X to a topological space Y is said to be M-uniformly continuous, if for every $\lambda \in \Delta(Y)$ there exists $\delta \in \Delta(X)$ such that for any $x, y \in X$, $x \prec_\delta^M y$ implies $f(x) \prec_\lambda^M f(y)$. Every continuous mapping is M-uniformly continuous. The converse is true for all X iff Y is an R_0 space [2].

Theorem 5.1. *If there is a surjective M-uniformly continuous mapping $f : X \to Y$, then* $\mathrm{GCI}(Y) \leq \mathrm{GCI}(X)$.

Proof. Let $\lambda \in \Delta(Y)$. Since f is M-uniformly continuous, there is a $\delta \in \Delta(X)$ such that $x_1 \prec_\delta^M x_2$ implies $f(x_1) \prec_\lambda^M f(x_2)$. Then there exists a subset $A \subseteq X$ with $|A| < \mathrm{GCI}(X)$ such that $X \prec_\delta^M A$. Furthermore, for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Then $x \prec_\delta^M a$ for some $a \in A$. So $y = f(x) \prec_\lambda^M f(a)$. It follows that $Y \prec_\lambda^M f(A)$. Since $|f(A)| \leq |A| < \mathrm{GCI}(X)$, we obtain $\mathrm{GCI}(Y) \leq \mathrm{GCI}(X)$. □

A mapping between topological spaces that maps open subsets to open subsets is called an open mapping [14]. Open mappings need not be continuous.

Corollary 5.2. *If there is a bijective open mapping from a space X to a space Y, then* $\mathrm{GCI}(X) \leq \mathrm{GCI}(Y)$.

Proof. Let f be a bijective open mapping from the space X to a space Y. Then the inverse mapping $f^{-1} : Y \to X$ is bijective and continuous. By Theorem 5.1, we have $\mathrm{GCI}(X) \leq \mathrm{GCI}(Y)$. □

The combination of Theorem 5.1 and Corollary 5.2 deduces the following.

Corollary 5.3. *If there is a bijective, open and M-uniformly continuous mapping $f : X \to Y$, then* $\mathrm{GCI}(X) = \mathrm{GCI}(Y)$.

The example below shows that a bijective, open and M-uniformly continuous mapping need not be a homeomorphism.

Example 5.4. Let $P = (\mathbb{R}, \leq)$ be the poset of real numbers with the ordinary order of numbers. Let $X = (P, \sigma(P))$ and $Y = (P, \gamma(P))$. Note that $\sigma(P) = \{\varnothing, \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$ and $\gamma(P) = \sigma(P) \cup \{[a, +\infty) : a \in \mathbb{R}\}$. For any gauge λ on Y and $y_1, y_2 \in Y$, if $y_1 \leq y_2$ then $y_2 \in \lambda(y_1)$, so $y_1 \prec_\lambda^M y_2$ holds for any two elements $y_1, y_2 \in Y$. It then follows immediately that every mapping $f : X \to Y$ is M-uniformly continuous. In particular, the identity mapping

$id : X \to Y$, $x \mapsto x$, is M-uniformly continuous. As $\sigma(P) \subseteq \gamma(P)$, the identity mapping is open.

Now $U = [0, +\infty) \in \gamma(P)$, but $i^{-1}(U) = U \notin \sigma(P)$. So id is not continuous, thus not a homeomorphism.

Theorem 5.5. *If* $\mathrm{GCI}(X) = 2$*, then every* M*-uniformly continuous mapping from* X *to a* T_1 *space is a constant function.*

Proof. Let $\mathrm{GCI}(X) = 2$, $f : X \to Y$ be M-uniformly continuous, and Y be T_1. By Theorem 5.1, $\mathrm{GCI}(f(X)) \leq \mathrm{GCI}(X) = 2$, implying $\mathrm{GCI}(f(X)) = 2$. The set $f(X)$, with the subspace topology, is a T_1 space. By Theorem 3.7, $|f(X)| = 1$. Thus, f is a constant function. □

One might conjecture that the converse of Theorem 5.5 also holds. Below, we give a counterexample to this conjecture.

In order to simplify the explanation, we first prove a simple lemma.

Lemma 5.6. *Let* $f : X \to Y$ *be an* M*-uniformly continuous mapping from a space* X *to a* T_1 *space* Y*. Then for any* $x_1, x_2 \in X$*,* $x_1 \in \mathrm{cl}(\{x_2\})$ *implies* $f(x_1) = f(x_2)$*.*

Proof. Assume, on the contrary, that $x_1 \in \mathrm{cl}(\{x_2\})$ and $f(x_1) \neq f(x_2)$. Chose a gauge $\lambda \in \Delta(Y)$ satisfying $\lambda(f(x_1)) = Y - \{f(x_2)\}$ and $\lambda(f(x_2)) = Y - \{f(x_1)\}$. Since f is M-uniformly continuous, there is a gauge $\delta \in \Delta(X)$ such that $u \prec_\delta^M v$ implies $f(u) \prec_\lambda^M f(v)$. From $x_1 \in \mathrm{cl}(\{x_2\})$ we have $x_2 \in \delta(x_1)$, so $x_1 \prec_\delta^M x_2$. Therefore $f(x_1) \prec_\lambda^M f(x_2)$. But this is not true by the definition of $\lambda(f(x_1))$ and $\lambda(f(x_2))$. This contradiction proves that $f(x_1) = f(x_2)$ must hold. □

Example 5.7. Let $P = \{a, b, c, d\}$ be the poset in which $b < a, b < c$ and $d < c$.

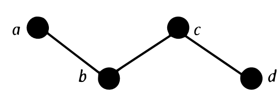

FIGURE 2. Poset $P = \{a, b, c, d\}$

For any gauge $\delta \in \Delta(P, \nu(P))$, we have $P \prec_\delta^M \{a, c\}$. So, $\mathrm{GCI}(P, \nu(P)) \leq 3$. Also for the gauge λ given by

$$\lambda(a) = P - \downarrow c, \lambda(b) = P - \downarrow d, \lambda(c) = \lambda(d) = P - \downarrow a,$$

there is no $x \in P$ satisfying $P \prec_\lambda^M \{x\}$. So, $\mathrm{GCI}(P, \nu(P)) \neq 2$ and therefore $\mathrm{GCI}(P, \nu(P)) = 3$. Also in the space $(P, \nu(P))$, it holds that $\mathrm{cl}(\{a\}) = \{a, b\}$ and $\mathrm{cl}(\{c\}) = \{b, c, d\}$. If $f : (P, \nu(P)) \to Y$ is an M-uniformly continuous mapping from $(P, \nu(P))$ to a T_1 space Y, then by Lemma 5.6, we have $f(a) = f(b)$ from $\mathrm{cl}(\{a\}) = \{a, b\}$ and $f(b) = f(c) = f(d)$ from $\mathrm{cl}(\{c\}) = \{b, c, d\}$. Hence f is a constant function.

6. Remarks on some further work

We end the paper with some remarks and possible future work.

(1) If X is a Hausdorff space such that $\mathrm{GCI}(X \times Y)$ is finite for any Y with a finite gauge compact index, then by Proposition 2.12, $\mathrm{GCI}(X)$ is finite. So, X is finite. We still do not know whether this conclusion holds for non-Hausdorff spaces, that is whether the following statement is valid:

"If X is a T_1 space such that $\mathrm{GCI}(X \times Y)$ is finite for any Y with a finite gauge compact index, then X is a finite set."

(2) We have proved that if P is an algebraic poset, then $\mathrm{GCI}(\Sigma P) = 2$ iff P has a linking element, where ΣP is the Scott space of P. We do not know whether this conclusion is still valid for other classes of posets, such as the continuous directed complete posets (see [4] on continuous posets).

(3) Quite a number of different other cardinal functions on topological spaces have been introduced and investigated, for example "weight", "density", "Lindelöf degree", "extent" (the extent $e(X)$ of X is the supremum of the cardinals of its closed discrete subsets), etc. The two cardinal functions more relevant to gauge compact index are the Lindelöf degree and extent. We still do not have any significant result on their connections with gauge compact index. We leave that exploration to our future work.

Acknowledgements. *We must thank the referee for carefully reading the original manuscript and giving us a lot of valuable suggestions that helped us to improve the paper significantly. We would also like to express our appreciation to Dr. Ho Weng Kin and Dr. Lee Peng Yee for their ideas. The first author wishes to thank LPDP, Ministry of Finance of Republic Indonesia, for providing financial support for this research.*

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